$$X \nearrow A\text{, ring} \qquad X \searrow \mathcal{C}\text{, category}$$

HH_*: diff'l forms $H^0(X, \Omega^p)$, $H^q(X, \Omega^p)$

HH^*: polyvector fields $H^0(X, \Lambda^p T_X)$, $H^q(X, \Lambda^p T_X)$

(e.g. deformations are controlled by $H^1(X, T_X) \to HH^2(X)$)

① HH_*, HH^* of algebras.

A assoc. alg. w. 1 / k.

M A-bimodule $= A \otimes A^{op}$-module

$HH_*(M) = \mathrm{Tor}_*^{A\otimes A^{op}}(A, M)$ – k v.s. $\quad (A \otimes_{A\otimes A^{op}} -)$

$HH^*(M) = \mathrm{Ext}^*_{A\otimes A^{op}}(A, M)$ – k v.s.

$* = 0$: $\quad HH_0(M) = A \otimes_{A\otimes A^{op}} M = M/[M, A]$ $\quad$ $ma - am$

$* = 0$: $\quad HH^0(M) = \mathrm{Hom}_{A\otimes A^{op}}(A, M) = Z_A(M)$ $\quad$ $= \{m : am = ma \ \forall a \in A\}$

<u>Bar complex</u>: $\quad \dots \to A^{\otimes 3} \to A \otimes A \xrightarrow{m} A \to 0$

$a|b \mapsto ab$

$a|b|c \mapsto ab|c - a|bc$

Exercise A comm.

$\Rightarrow$ $HH_1(A) \cong \Omega^1_A = \{ a \cdot db : a, b \in A \} \big/ {d(b_1 b_2) = b_1 db_2 + b_2 db_1}$

$HH^1(A) \cong Der(A) = \{ g \in Hom_k(A, A) : g(ab) = a g(b) + b g(a) \}$

($X = Spec\, A$ $\Omega^1_A = H^0(X, \Omega^1_X)$

$Der(A) = H^0(X, T_X)$)

Remark A, B are Morita equivalen

(this means: A-mod $\cong$ B-mod

$\Rightarrow$ $HH_*(A) \cong HH_*(B)$

$HH^*(A) \cong HH^*(B)$

A $B = M_n(A)$

② . HH of varieties

$X/k \qquad \dim X = n \qquad$ smooth $\qquad X \overset{\Delta}{\hookrightarrow} X \times X$

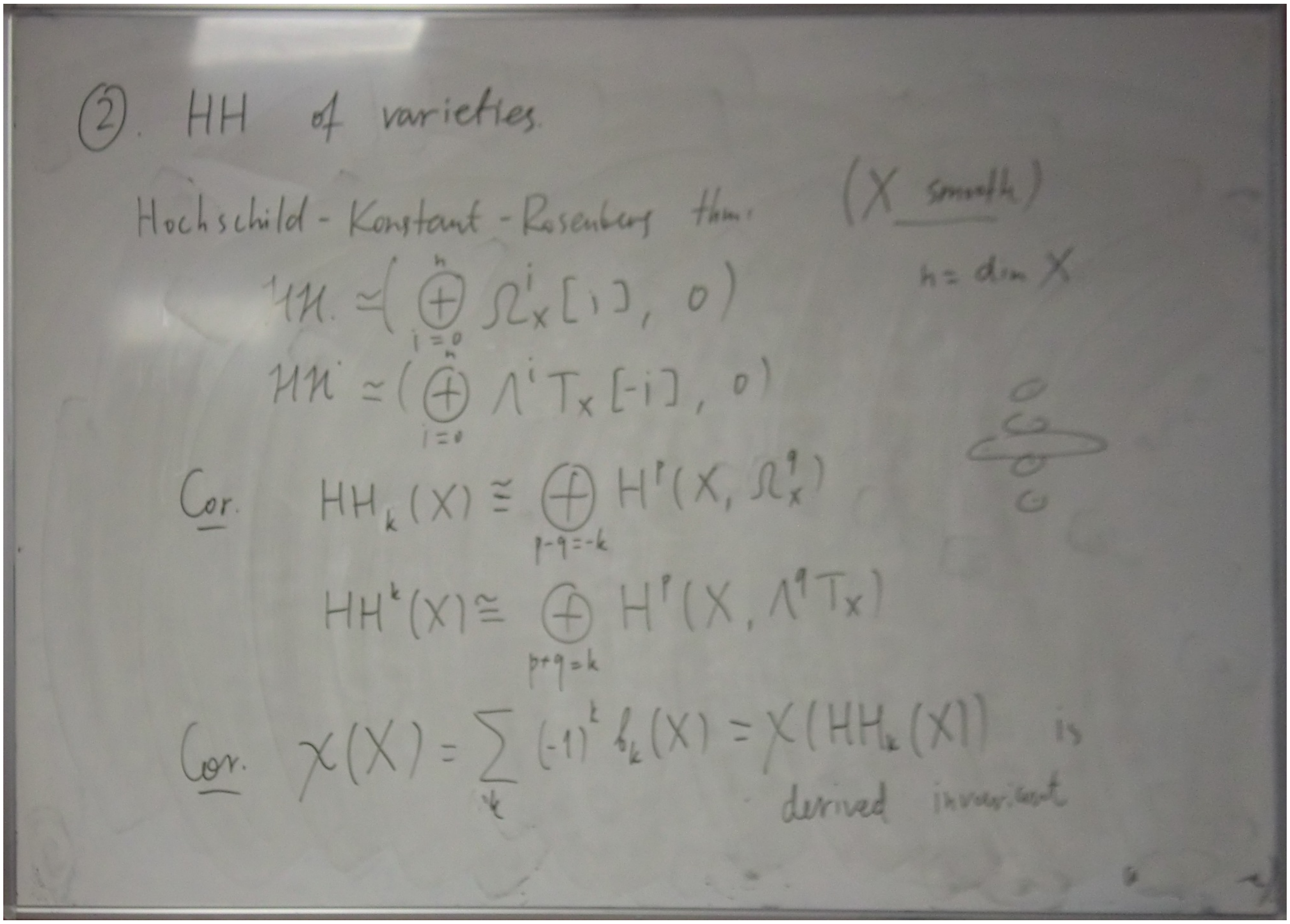
k v.s. $\left(\begin{array}{l} HH_k(X) := \mathrm{Tor}_k^{X\times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) = R\Gamma^{-k}(X, \mathcal{HH}_\cdot) \\ HH^k(X) := \mathrm{Ext}^k_{X\times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) = R\Gamma^{k}(X, \mathcal{HH}^\cdot) \end{array} \right.$ $\qquad \mathcal{O}_\Delta = \Delta_* \mathcal{O}$, coherent sheaf of $\mathcal{O}_{X\times X}$-mod

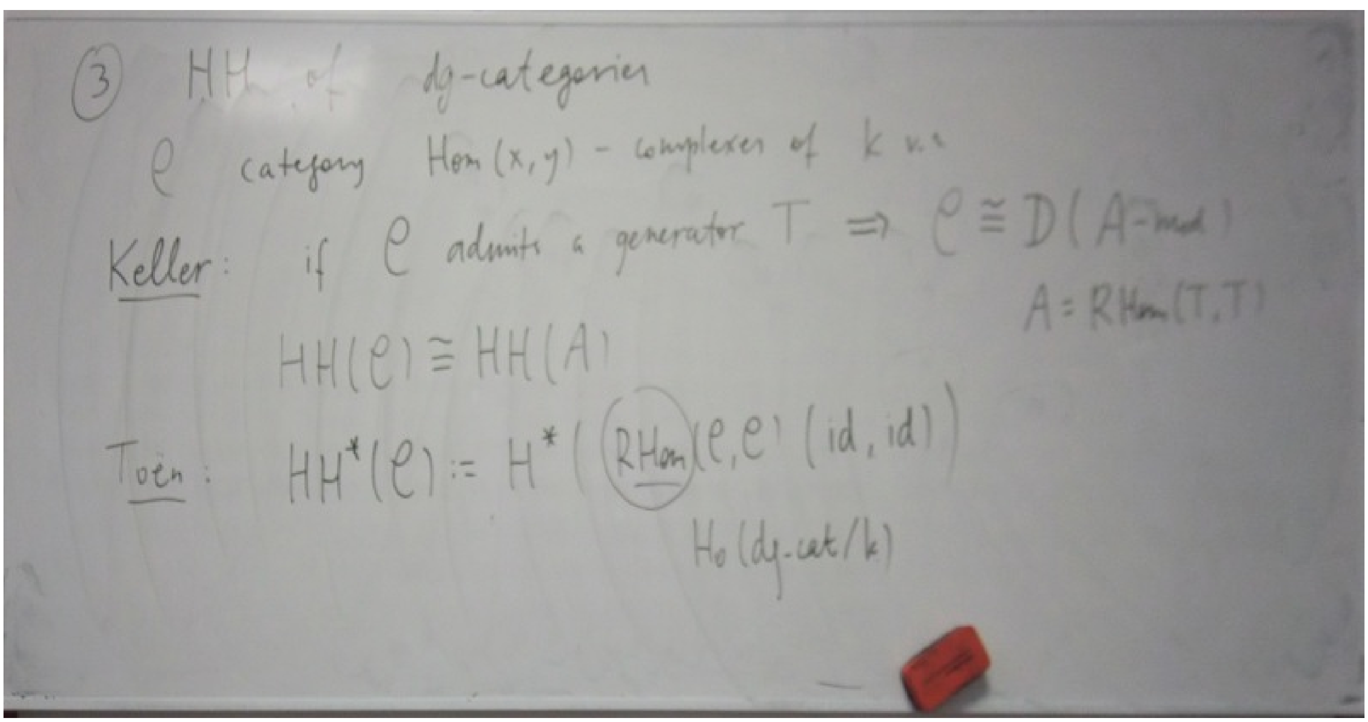
$$\mathcal{HH}_\cdot = L\Delta^* \Delta_* \mathcal{O}_X = p_{1*}(\mathcal{O}_\Delta \overset{L}{\otimes} \mathcal{O}_\Delta) \in D^b(\mathrm{Coh}\, X)$$

$$\mathcal{HH}^\cdot = p_{1*} \underline{R\mathcal{H}om}(\Delta_* \mathcal{O}_X, \Delta_* \mathcal{O}_X)$$

<u>Thm</u>

1) HH_* is functorial for Fourier-Mukai transforms
2) HH^* is functorial for Fourier-Mukai equivalences
3) HH_*, HH^* are derived invariant: X, Y $D^b(\mathrm{Coh}\, X) \cong D^b(\mathrm{Coh}\, Y) \Rightarrow HH_*, HH^*$ are the same


(2). HH of varieties
Hochschild-Konstant-Rosenberg thm: (X smooth)
$\mathcal{HH}_\cdot \simeq (\bigoplus_{i=0}^{n} \Omega^i_X[i], 0)$
n = dim X
$\mathcal{HH}^\cdot \simeq (\bigoplus_{i=0}^{n} \Lambda^i T_X[-i], 0)$
Cor. $HH_k(X) \cong \bigoplus_{p-q=-k} H^p(X, \Omega^q_X)$
$HH^k(X) \cong \bigoplus_{p+q=k} H^p(X, \Lambda^q T_X)$
Cor. $\chi(X) = \sum_k (-1)^k b_k(X) = \chi(HH_*(X))$ is derived invariant


(3) HH of dg-categories
$\mathcal{C}$ category Hom(x, y) - complexes of k v.s
Keller: if $\mathcal{C}$ admits a generator T $\Rightarrow$ $\mathcal{C} \cong D(A\text{-mod})$
$A = RHom(T,T)$
$HH(\mathcal{C}) \cong HH(A)$
Toen: $HH^*(\mathcal{C}) := H^*(RHom(\mathcal{C},\mathcal{C})(id, id))$
Ho(dg-cat/k)