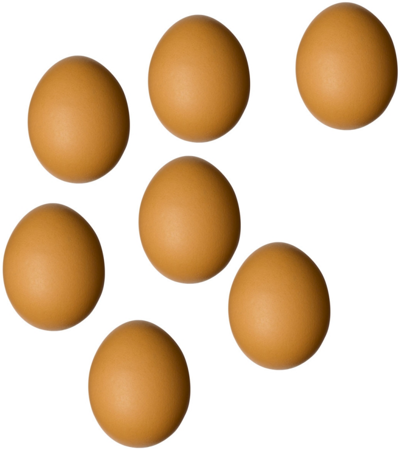
Counting, measure and metrics

Tom Leinster (Glasgow/EPSRC)

Making free use of ideas of:

John Baez (Riverside)
Andreas Blass (Michigan)
Christina Cobbold (Glasgow)
André Joyal (Montréal)
Stephen Schanuel (Buffalo)
⋮

The size of a finite set



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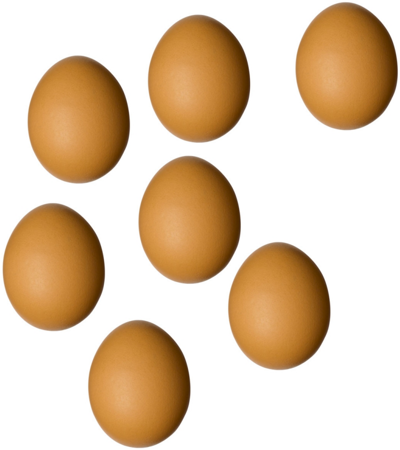
size $\left(\right.$  $\left.\right) = 7$

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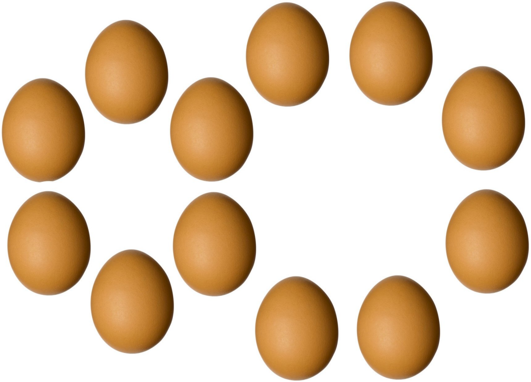

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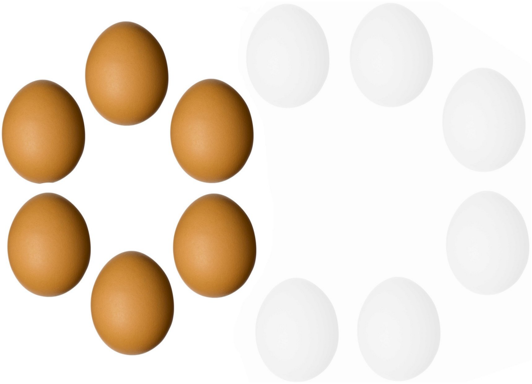

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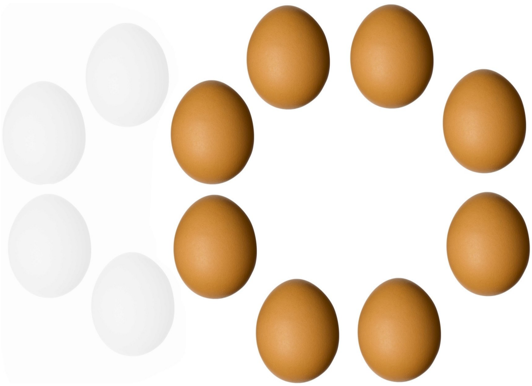

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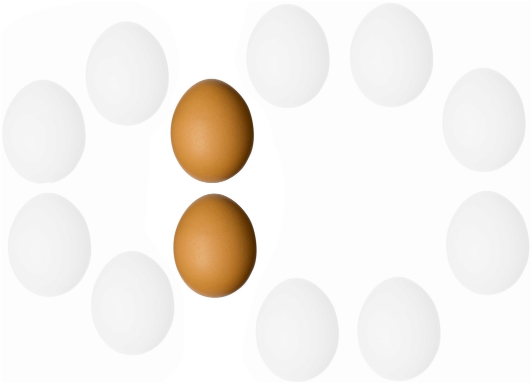

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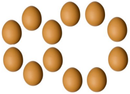
- size $\left(\right.$  $\left.\right)$

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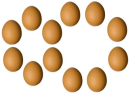
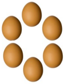
- size $\Bigg($  $\Bigg) =$ size $\Bigg($  $\Bigg)$

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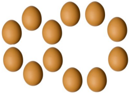
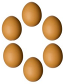
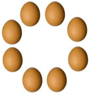
- size () $=$ size () $+$ size ()

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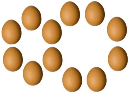
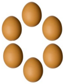
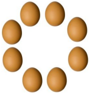
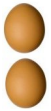
- size () $=$ size () $+$ size () $-$ size ()

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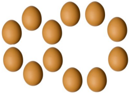
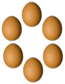
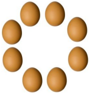
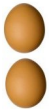
- size () $=$ size () $+$ size () $-$ size ()

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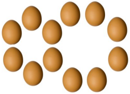
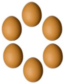
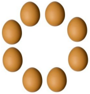
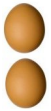
- size () $=$ size () $+$ size () $-$ size ()

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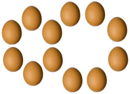
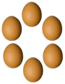
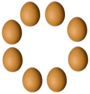
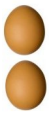
- size () $=$ size () $+$ size () $-$ size ()

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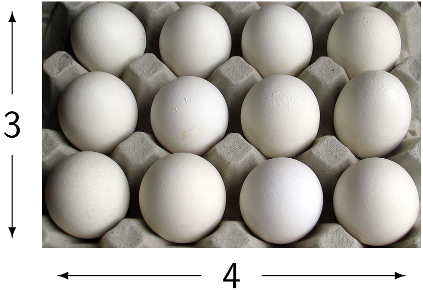

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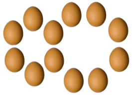
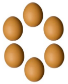
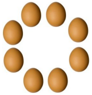
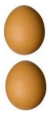
- size $\left(\right.$  $\left.\right) =$ size $\left(\right.$  $\left.\right) +$ size $\left(\right.$  $\left.\right) -$ size $\left(\right.$  $\left.\right)$

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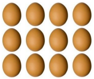
- size $\left(\right.$  $\left.\right)$

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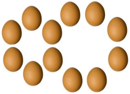
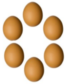
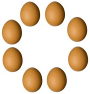
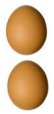
- size () $=$ size () $+$ size () $-$ size ()

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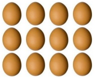
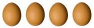
- size () $=$ size ()

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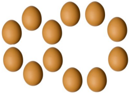
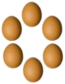
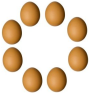
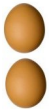
- size () = size () + size () − size ()

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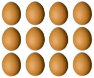
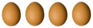
- size () = size () × size ()

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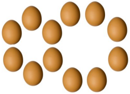
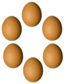
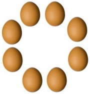
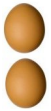
- size () = size () + size () − size ()

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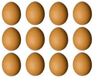
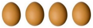
- size () = size () × size ()

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The size of a topological space

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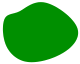
What counts as ‘1’? Let’s declare: size $\Bigg($  $\Bigg) = 1$.

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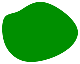
What counts as ‘1’? Let’s declare: size $\Bigg($  $\Bigg) =$ size $\Bigg($  $\Bigg) = 1.$

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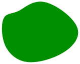
What counts as '1'? Let's declare: size $\Bigg($  $\Bigg) =$ size $\Bigg($  $\Bigg) = 1.$

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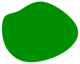
What counts as '1'? Let's declare: size () = size () $= 1$.

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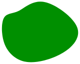
What counts as '1'? Let's declare: size $\Bigg($  $\Bigg) =$ size $\Bigg($  $\Bigg) = 1.$

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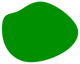
What counts as '1'? Let's declare: size $\Bigg($  $\Bigg) =$ size $\Bigg($  $\Bigg) = 1.$

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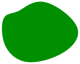
What counts as '1'? Let's declare: size () = size () = 1.

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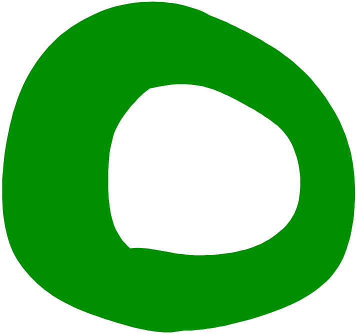

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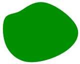
What counts as '1'? Let's declare: size () = size () $= 1$.

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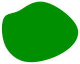
The size of a topological space

What counts as '1'? Let's declare: size () = size () $= 1$.

Now use the same rules as before! Some consequences:

- size () = size () + size () − size (   )
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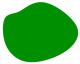
What counts as '1'? Let's declare: size () = size () = 1.

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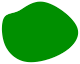
What counts as '1'? Let's declare: size () $=$ size () $= 1$.

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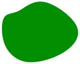
What counts as '1'? Let's declare: size () $=$ size () $= 1$.

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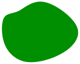
What counts as '1'? Let's declare: size $\Bigg($  $\Bigg) =$ size $\Bigg($  $\Bigg) = 1.$

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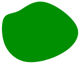
hollow

The size of a topological space

What counts as '1'? Let's declare: size $\Bigg($  $\Bigg) =$ size $\Bigg($  $\Bigg) = 1$.

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- size $\Bigg($  $\Bigg) =$ size $\Bigg($  $\Bigg) +$ size $\Bigg($  $\Bigg) -$ size $\Bigg($  $\Bigg)$

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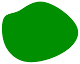
What counts as '1'? Let's declare: size () = size () = 1.

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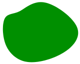
What counts as '1'? Let's declare: size $\Bigg($  $\Bigg)$ $=$ size $\Bigg($  $\Bigg)$ $= 1.$

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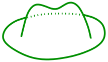
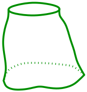
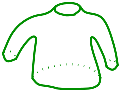
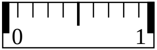
- By similar calculations, size $\Bigg($  $\Bigg)$ $= 1$, size $\Bigg($  $\Bigg)$ $= 0$,

  size $\Bigg($  $\Bigg)$ $= -1$, size $\Bigg($  $\Bigg)$ $= -2$.

The size of a subset of $\mathbb{R}^n$

The size of a subset of $\mathbb{R}^n$

We'll need a ruler, say of length $1\,\mathrm{cm}$:  .

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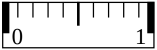
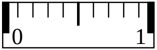
We'll need a ruler, say of length $1\,\mathrm{cm}$: 

.

Let's use a real interval: 0 1 .

But should we include the endpoints, 0 and 1?

The size of a subset of $\mathbb{R}^n$

We'll need a ruler, say of length $1\,\mathrm{cm}$: 

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Let's use a real interval: 

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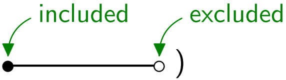
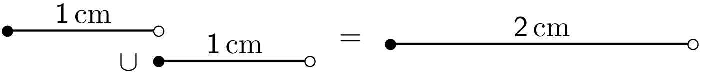
Including just one endpoint (

) gives a good ruler:



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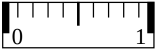
We'll need a ruler, say of length $1\,\mathrm{cm}$: 

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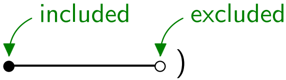
Including just one endpoint (

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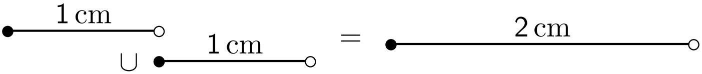
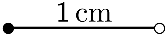
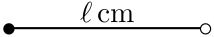


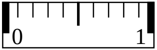
So we declare: size(

) $= 1\,\mathrm{cm}$, and more generally,

size(

) $= \ell\,\mathrm{cm}$.

The size of a subset of $\mathbb{R}^n$

We'll need a ruler, say of length $1\,\mathrm{cm}$: 

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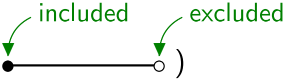
Including just one endpoint (

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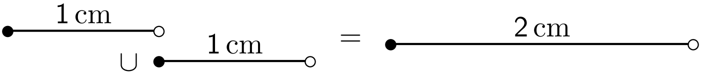
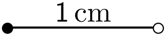


So we declare: size(

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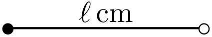
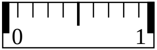
size(

) $= \ell\,\mathrm{cm}$.

We also declare: size($\bullet$) $=$ 1 point

The size of a subset of $\mathbb{R}^n$

We'll need a ruler, say of length $1\,\mathrm{cm}$: 

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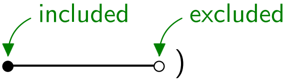
Including just one endpoint (

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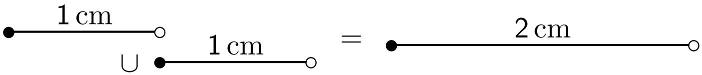
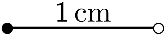


So we declare: size(

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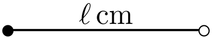
size(

) $= \ell\,\mathrm{cm}$.

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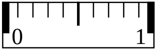
The size of a subset of $\mathbb{R}^n$

We'll need a ruler, say of length $1\,\mathrm{cm}$: 

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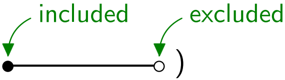
Including just one endpoint (

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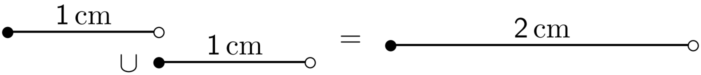
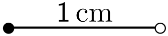


So we declare: size(

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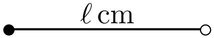
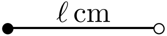
size(

) $= \ell\,\mathrm{cm}$.

We also declare: size($\bullet$) $= 1\,\text{point} = 1\,\mathrm{cm}^0 = 1$.

The size of a subset of $\mathbb{R}^n$

We declared: size(

) $= \ell\,\mathrm{cm}$ and size($\bullet$) $= 1$.

Now let's use the same rules as before, and calculate some sizes.

The size of a subset of $\mathbb{R}^n$

We declared: size($\overset{\ell\,\mathrm{cm}}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\circ}$) $= \ell\,\mathrm{cm}$ and size($\bullet$) $= 1$.

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Usually we like to include endpoints/boundaries of figures.

The size of a subset of $\mathbb{R}^n$

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- size($\overset{\ell\,\mathrm{cm}}{\bullet\!\!-\!\!-\!\!-\!\!-\!\!\bullet}$)

The size of a subset of $\mathbb{R}^n$

We declared: size($\bullet\!\overset{\ell\,\mathrm{cm}}{\rule[0.5ex]{4em}{0.4pt}}\!\circ$) $= \ell\,\mathrm{cm}$ and size($\bullet$) $= 1$.

Now let's use the same rules as before, and calculate some sizes.

Usually we like to include endpoints/boundaries of figures.

- size($\bullet\!\overset{\ell\,\mathrm{cm}}{\rule[0.5ex]{4em}{0.4pt}}\!\bullet$) $=$ size($\bullet\!\overset{\ell\,\mathrm{cm}}{\rule[0.5ex]{4em}{0.4pt}}\!\circ$) $+$ size($\bullet$)

The size of a subset of $\mathbb{R}^n$

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Now let's use the same rules as before, and calculate some sizes.

Usually we like to include endpoints/boundaries of figures.

- size($\overset{\ell\,\mathrm{cm}}{\bullet\!\!\longrightarrow\!\!\!\!\!\!\!\longrightarrow\!\!\bullet}$) $=$ size($\overset{\ell\,\mathrm{cm}}{\bullet\!\!\longrightarrow\!\!\!\!\!\!\!\longrightarrow\!\!\circ}$) $+$ size($\bullet$) $= \ell\,\mathrm{cm} + 1$.

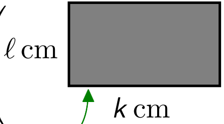
The size of a subset of $\mathbb{R}^n$

We declared: size$(\bullet\overset{\ell\,\mathrm{cm}}{\text{————}}\circ) = \ell\,\mathrm{cm}$ and size$(\bullet) = 1$.

Now let's use the same rules as before, and calculate some sizes.

Usually we like to include endpoints/boundaries of figures.

- size$(\bullet\overset{\ell\,\mathrm{cm}}{\text{————}}\bullet) =$ size$(\bullet\overset{\ell\,\mathrm{cm}}{\text{————}}\circ) +$ size$(\bullet) = \ell\,\mathrm{cm} + 1$.

- size $\left(\vphantom{\Big|} \right.$  $\left. \vphantom{\Big|} \right)$

boundaries included

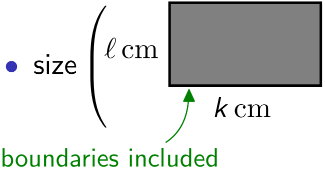
The size of a subset of $\mathbb{R}^n$

We declared: size$\left(\overset{\ell\,\mathrm{cm}}{\bullet\!\!\rule[0.5ex]{4em}{0.4pt}\!\!\circ}\right) = \ell\,\mathrm{cm}$ and size$(\bullet) = 1$.

Now let's use the same rules as before, and calculate some sizes.

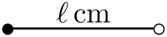
Usually we like to include endpoints/boundaries of figures.

- size$\left(\overset{\ell\,\mathrm{cm}}{\bullet\!\!\rule[0.5ex]{4em}{0.4pt}\!\!\bullet}\right) =$ size$\left(\overset{\ell\,\mathrm{cm}}{\bullet\!\!\rule[0.5ex]{4em}{0.4pt}\!\!\circ}\right) +$ size$(\bullet) = \ell\,\mathrm{cm} + 1$.

- size $\Bigg($ 

$\Bigg) =$ size$\left(\overset{k\,\mathrm{cm}}{\bullet\!\!\rule[0.5ex]{4em}{0.4pt}\!\!\bullet}\right) \times$ size $\left(\ell\,\mathrm{cm}\ \begin{matrix}\bullet\\ \Big|\\ \bullet\end{matrix}\right)$

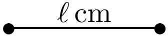
The size of a subset of $\mathbb{R}^n$

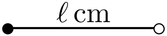
We declared: size(

) $= \ell\,\mathrm{cm}$ and size($\bullet$) $= 1$.

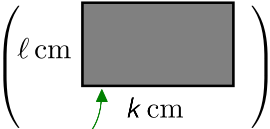
Now let's use the same rules as before, and calculate some sizes.

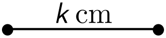
Usually we like to include endpoints/boundaries of figures.

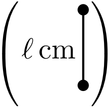
- size(

) $=$ size(

) $+$ size($\bullet$) $= \ell\,\mathrm{cm} + 1$.

- size 

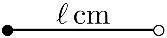
$=$ size(

) $\times$ size 

boundaries included

$$= (k\,\mathrm{cm} + 1)(\ell\,\mathrm{cm} + 1)$$

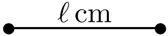
The size of a subset of $\mathbb{R}^n$

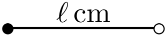
We declared: size(

) $= \ell\,\mathrm{cm}$ and size($\bullet$) $= 1$.

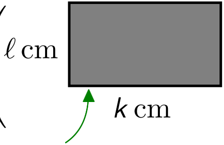
Now let's use the same rules as before, and calculate some sizes.

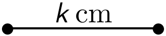
Usually we like to include endpoints/boundaries of figures.

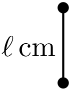
- size(

) = size(

) + size($\bullet$) $= \ell\,\mathrm{cm} + 1$.

- size $\Bigg($ 

$\Bigg)$ = size(

) $\times$ size $\Bigg($ 

$\Bigg)$

boundaries included

$$= (k\,\mathrm{cm} + 1)(\ell\,\mathrm{cm} + 1) = k\ell\,\mathrm{cm}^2 + (k + \ell)\,\mathrm{cm} + 1.$$

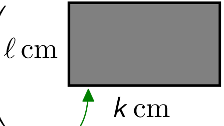
The size of a subset of $\mathbb{R}^n$

We declared: size($\overset{\ell\,\mathrm{cm}}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\circ}$) $= \ell\,\mathrm{cm}$ and size($\bullet$) $= 1$.

Now let's use the same rules as before, and calculate some sizes.

Usually we like to include endpoints/boundaries of figures.

- size($\overset{\ell\,\mathrm{cm}}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\bullet}$) $=$ size($\overset{\ell\,\mathrm{cm}}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\circ}$) $+$ size($\bullet$) $= \ell\,\mathrm{cm} + 1$.

- size $\left(\right.$$\left. \right)$ $=$ size($\overset{k\,\mathrm{cm}}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\bullet}$) $\times$ size $\left(\ell\,\mathrm{cm}\ \begin{matrix}\bullet\\ |\\ \bullet\end{matrix} \right)$

boundaries included

$\frac{1}{2}\times$perimeter

area

Euler characteristic

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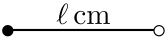
We declared: size(

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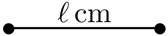
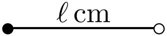
- size(

) = size(

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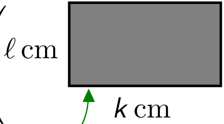
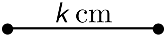
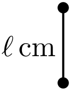
- size 

= size(

) $\times$ size 

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area $\quad \frac{1}{2}\times$perimeter $\quad$ Euler characteristic

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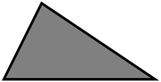
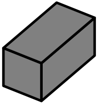
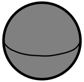
- Similarly, can compute sizes of  ,  ,  , ...

The size of a type

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These are ‘datatypes’, or ‘types’.

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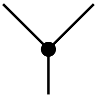

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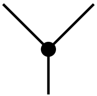

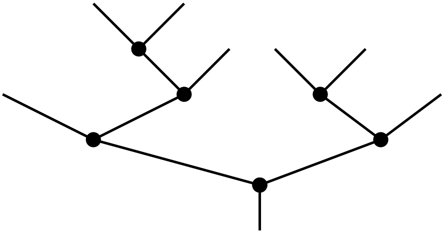

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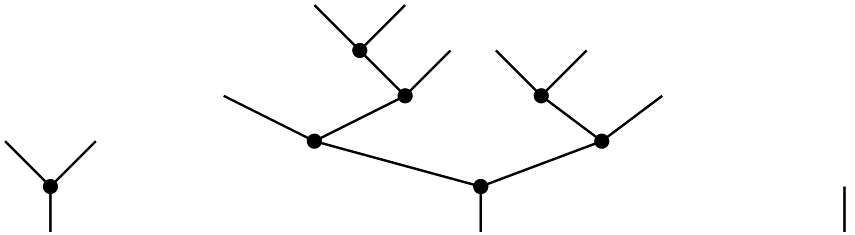

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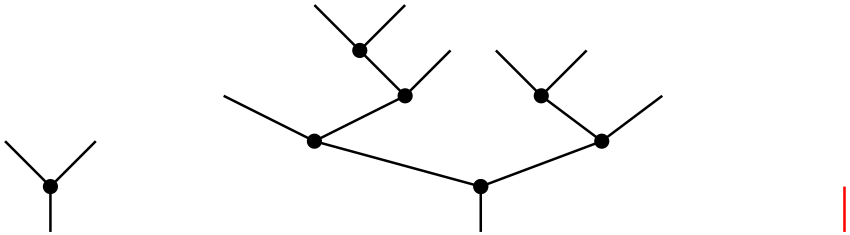

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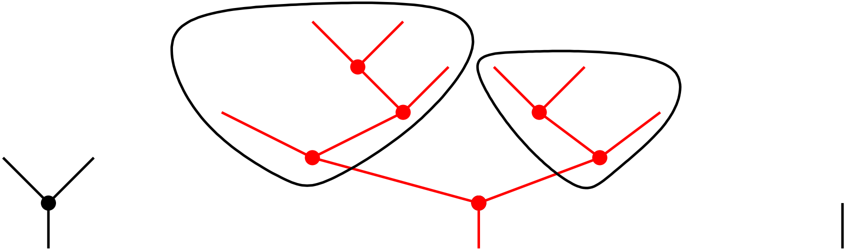

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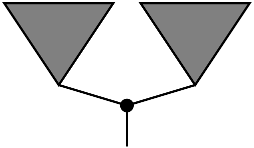
or two trees joined together:  .

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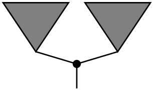
A tree is either the trivial tree | or two trees joined together:  .

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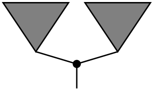
A tree is either the trivial tree | or two trees joined together:  .

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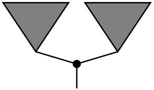
A tree is either the trivial tree | or two trees joined together:  .

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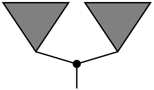
A tree is either the trivial tree $\Big|$ or two trees joined together:  .

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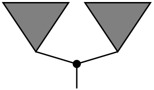
A tree is either the trivial tree | or two trees joined together:  .

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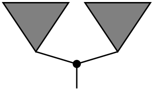
A tree is either the trivial tree $\Big|$ or two trees joined together:  .

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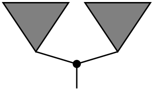
A tree is either the trivial tree $\big|$ or two trees joined together:  .

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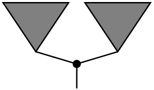
A tree is either the trivial tree $\Big|$ or two trees joined together:  .

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...which really is true! A tree is 'the same as' a 7-tuple of trees.

The size of a metric space

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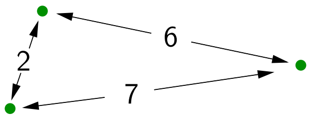
Consider finite metric spaces.



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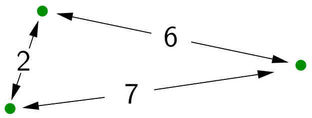

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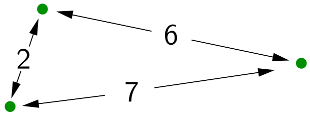

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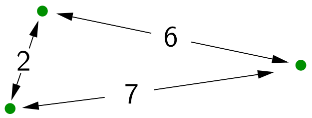

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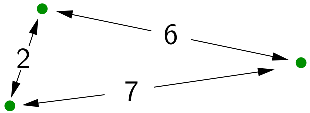

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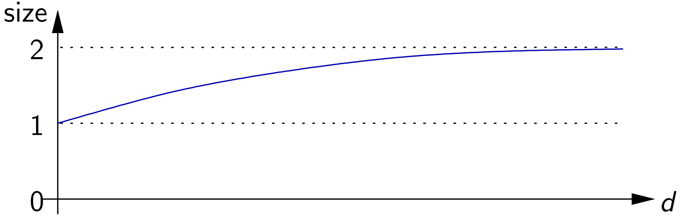

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$$\mathrm{size}(E) = 1/p_1^{p_1} p_2^{p_2} \cdots p_n^{p_n}.$$

(This is the exponential of the Shannon entropy.)

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The size is greatest when $p_1 = p_2 = \cdots = p_n = 1/n$ (uniform distribution): then $\mathrm{size}(E) = n$.

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> 'Measuring biological diversity',
> Andrew Solow (Marine Policy Center, Woods Hole),
> Stephen Polasky (Agricultural and Resource Economics, Oregon State),
> *Environmental and Ecological Statistics* 1 (1994), 95–107.

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Example of a measure taking only similarity into account

‘Effective number of species’ = size of the metric space of species

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Notes and references

These slides are available at www.maths.gla.ac.uk/$\sim$tl

Sets: The basic idea of this talk is to take the elementary rules of arithmetic and use them in contexts where they do not obviously apply. This idea has doubtless been explored by many people; I have learned most about it from

> John Baez, The mysteries of counting: Euler characteristic versus homotopy cardinality, http://math.ucr.edu/home/baez/counting

and

> Daniel A. Klain, Gian-Carlo Rota, *Introduction to Geometric Probability*, Lezioni Lincee, Cambridge University Press, 1997

and the papers of Schanuel cited below.

Notes and references

Topological spaces: Stephen Schanuel seems to have been the first person to have really pushed the thought that Euler characteristic is to topological spaces as cardinality is to sets. Indeed, much of the theory of Euler characteristic (with compact support) follows from the simple axioms on 'size' above. See

> Stephen H. Schanuel, Negative sets have Euler characteristic and dimension, *Category Theory (Como, 1990)*, 379–385, Lecture Notes in Mathematics 1488, Springer, 1991

and

> Stephen H. Schanuel, What is the length of a potato? An introduction to geometric measure theory, in *Categories in Continuum Physics*, Lecture Notes in Mathematics 1174, Springer, 1986

as well as Klain and Rota (*op. cit.*).

Subsets of $\mathbb{R}^n$: This section draws very heavily on Schanuel's 'What is the length of a potato?' See also Klain and Rota.

Notes and references

Types: The isomorphism $T^7 \cong T$ was first established in

> Andreas Blass, Seven trees in one, *Journal of Pure and Applied Algebra* 103 (1995), 1–21, arXiv:math.LO/9405205,

following a remark of Lawvere. There is a precise sense in which $T^7 \cong T$ but $T^n \not\cong T$ for any other value of $n > 1$, except of course $13, 19, 25, \ldots$.

To state this correctly is a little delicate. Of course, the *set* of trees is countably infinite, so there is trivially a bijection $T^n \cong T$ for any $n \geq 1$; but that is not what is meant. Blass's result is, roughly, that there is an algorithm giving a one-to-one correspondence between 7-tuples of trees and single trees, and which *only explores each tree to finite depth*.

In this situation, we have $\text{size}(T)^7 = \text{size}(T)$ and, in fact, $T^7 \cong T$. This raises the question: is there some general principle allowing one to deduce the latter from the former? The answer is yes, as shown in

Notes and references

Marcelo Fiore, Tom Leinster, Objects of categories as complex numbers, *Advances in Mathematics* 190 (2005), 264–277, arXiv:math.CT/0212377.

Metric spaces: The slide refers to a notion of the 'size' of a metric space. For mathematicians this seems to be a new concept (but see the notes below on Ecosystems); it has yet to be written up formally. The existing sources are

Tom Leinster, The cardinality of a metric space, post at *The n-Category Café*, 9 February 2008, http://golem.ph.utexas.edu/category/2008/02/metric_spaces.html

(which is detailed but contains some mistakes) and

Tom Leinster, The cardinality of a metric space, talk at CT08, Calais, www.maths.gla.ac.uk/∼tl/calais

Notes and references

First one defines the cardinality (size) of a finite metric space; then, by using an approximating sequence of finite subspaces, one defines the cardinality of a compact metric space. This appears to coincide with the notion of 'size' described in the section on subsets of $\mathbb{R}^n$.

All of the notions of size discussed so far, except perhaps that for types, are closely related to the notion of the size (or cardinality, or Euler characteristic) of a category, introduced in

> Tom Leinster, The Euler characteristic of a category, *Documenta Mathematica* 13 (2008), 21–49, www.math.uni–bielefeld.de/documenta/vol–13/02.html

For an overview, see

> Tom Leinster, New perspectives on Euler characteristic, talk at British Mathematical Colloquium 2007, Swansea, www.maths.gla.ac.uk/$\sim$tl/swansea

Notes and references

Ecosystems: For this section I am very grateful to André Joyal, who pointed out to me that the exponential of entropy behaves much like cardinality, and to Christina Cobbold, who then suggested that there might be a relation between diversity measures in ecology and the cardinality of metric spaces — as indeed there is.

The Shannon entropy of the system E is $-\sum_i p_i \log(p_i)$. Often the logarithm is taken to base 2 rather than base e.

For an introduction to measures of biodiversity, see, for instance, Chapter 7 of

Russell Lande, Steinar Engen, Bernt-Erik Sæther, *Stochastic Population Dynamics in Ecology and Conservation*, Oxford University Press, 2003,

or

Lou Jost, Entropy and diversity, *Oikos* 113, No. 2 (2006), 363–375.

Notes and references

The paper of Solow and Polasky cited in the talk introduces the notion of 'effective number of species', which was later rediscovered under the name 'cardinality of a metric space' (see the notes on Metric spaces). Naturally, they only consider situations in which there is a finite number of species, which corresponds to considering only finite metric spaces.