Category Theory 6

The Yoneda Lemma

This is to accompany the reading of 7 November–14 November. Please report mistakes and obscurities to T.Leinster@maths.gla.ac.uk.

I encourage you to do *all* the questions, and remind you that the exam questions are likely to bear a strong resemblance to the questions here.

1. State and prove the Yoneda Lemma.

2. Let $\mathcal{A}$ be a locally small category. Define the **Yoneda embedding** $H_\bullet : \mathcal{A} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$. Prove each of the following directly (without using the Yoneda Lemma):
   (a) $H_\bullet$ is faithful
   (b) $H_\bullet$ is full
   (c) if $X : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$, $A \in \mathcal{A}$, and $X(A)$ has an element u that is 'universal' in a sense that you should make precise, then $X \cong H_A$.

3. Prove Lemma 3.3.8.

4. Let $\mathcal{B}$ be a category and $J : \mathcal{C} \longrightarrow \mathcal{D}$ a functor. There is an induced functor

$$J \circ - : [\mathcal{B}, \mathcal{C}] \longrightarrow [\mathcal{B}, \mathcal{D}]$$

   defined by composition with J. *(If you can't see how J is defined on maps, look back at Remark 1.3.14.)*
   (a) Show that if J is full and faithful then so is $J \circ -$. *(Typical category theory question. It's straightforward in the sense that nothing sneaky's involved: you just follow your nose. On the other hand, it may take you a while to get oriented. Remain calm.)*
   (b) Deduce that if J is full and faithful and $G, G' : \mathcal{B} \longrightarrow \mathcal{C}$ with $J \circ G \cong J \circ G'$ then $G \cong G'$.
   (c) Now deduce that right adjoints are unique: if $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G, G' : \mathcal{B} \longrightarrow \mathcal{A}$ with $F \dashv G$ and $F \dashv G'$ then $G \cong G'$. *(Hint: the Yoneda embedding is full and faithful.)*