Magnitude homology

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In celebration of the 60th birthday of Clemens Berger

Back in 2007...

A talk in Nice about the Euler characteristic of a category led to this...

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THE EULER CHARACTERISTIC OF A CATEGORY
AS THE SUM OF A DIVERGENT SERIES

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(*communicated by J. Daniel Christensen*)

Abstract

The Euler characteristic of a cell complex is often thought of as the alternating sum of the number of cells of each dimension. When the complex is infinite, the sum diverges. Nevertheless, it can sometimes be evaluated; in particular, this is possible when the complex is the nerve of a finite category. This provides an alternative definition of the Euler characteristic of a category, which is in many cases equivalent to the original one.

1. Introduction

What is the Euler characteristic of an infinite cell complex?

Since 2007, progress has been made in two directions:

- Euler characteristic for *enriched* categories, renamed as magnitude
- magnitude homology of enriched categories, categorifying magnitude.

Plan

1. Magnitude, generally

2. The magnitude of a metric space

3. Magnitude homology, generally

4. The magnitude homology of a metric space

1. Magnitude, generally

Size

For many types of mathematical object, there is a canonical notion of size.

- Sets have cardinality. It satisfies

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ |A \times B| &= |A| \times |B|. \end{aligned}$$

- Subsets of $\mathbb{R}^n$ have volume. It satisfies

$$\begin{aligned} \mathrm{vol}(A \cup B) &= \mathrm{vol}(A) + \mathrm{vol}(B) - \mathrm{vol}(A \cap B) \\ \mathrm{vol}(A \times B) &= \mathrm{vol}(A) \times \mathrm{vol}(B). \end{aligned}$$

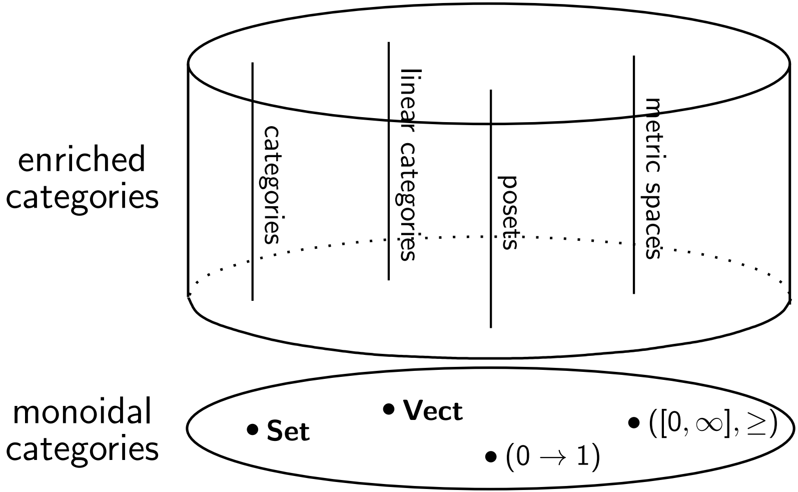
- Topological spaces have Euler characteristic. It satisfies

$$\begin{aligned} \chi(A \cup B) &= \chi(A) + \chi(B) - \chi(A \cap B) \quad \text{(under hypotheses)} \\ \chi(A \times B) &= \chi(A) \times \chi(B). \end{aligned}$$

Challenge Find a general definition of 'size', including these and other examples.

One answer The magnitude of an enriched category.

Enriched categories



Usually, our enriched categories will have *only finitely many objects*.

The magnitude of an enriched category

Let $\boldsymbol{V} = (\boldsymbol{V}, \otimes, I)$ be a monoidal category.

Suppose we have a notion of the size of each object of $\boldsymbol{V}$.

Formally: suppose we have a semiring k and a monoid homomorphism

$$|\cdot| : ((\mathrm{ob}\, \boldsymbol{V})/\cong, \otimes, I) \to (k, \cdot, 1).$$

Let $\boldsymbol{A}$ be a $\boldsymbol{V}$-enriched category with finitely many objects.

There is an $\mathrm{ob}\, \boldsymbol{A} \times \mathrm{ob}\, \boldsymbol{A}$ matrix $Z_{\boldsymbol{A}}$ with (a, b)-entry

$$Z_{\boldsymbol{A}}(a, b) = |\boldsymbol{A}(a, b)|.$$

Definition When $Z_{\boldsymbol{A}}^{-1}$ exists, the magnitude of $\boldsymbol{A}$ is

$$|\boldsymbol{A}| = \sum_{a,b \in \boldsymbol{A}} Z_{\boldsymbol{A}}^{-1}(a, b) \in k.$$

(The definition can be extended to many cases where $Z_{\boldsymbol{A}}^{-1}$ does *not* exist.)

The magnitude of an ordinary category

Take $\boldsymbol{V} = \textbf{FinSet}$ and define $|\cdot| : \mathrm{ob}(\textbf{FinSet}) \to \mathbb{Q}$ to be cardinality.

We obtain a definition of the magnitude (or Euler characteristic) of a finite category. It is a rational number.

Example Let $\boldsymbol{A} = (\bullet \rightrightarrows \bullet)$.

Then

$$Z_{\boldsymbol{A}} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad \text{so} \qquad Z_{\boldsymbol{A}}^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

and so

$$|\boldsymbol{A}| = 1 + (-2) + 0 + 1 = 0.$$

Generally, let $\boldsymbol{A}$ be a finite category, with classifying space $B\boldsymbol{A}$.

Theorem $|\boldsymbol{A}| = \chi(B\boldsymbol{A})$, *under finiteness conditions ensuring that* $\chi(B\boldsymbol{A})$ *is well-defined.*

The magnitude of a poset

The magnitude of a poset is better known as its Euler characteristic (1960s).

Example Let M be a triangulated manifold.

Write A for the poset of simplices in the triangulation, ordered by inclusion.

Then

$$|A| = \chi(M).$$

The magnitude of a linear category

. . . is related to the Euler form in commutative algebra.

Metric spaces

A metric space can be seen as an enriched category, as follows.

View the ordered set $([0, \infty), \geq)$ as a monoidal category under $+$.

A metric space can be seen as a category enriched in $[0, \infty)$:

- the objects are the points
- the hom-objects are the distances $d(a, b) \in [0, \infty)$
- composition is the triangle inequality $d(a, b) + d(b, c) \geq d(a, c)$,

etc.

The magnitude of a metric space

Metric spaces are categories enriched in $[0, \infty)$.

But to talk about the *magnitude* of a finite metric space, we need a 'size function': a monoid homomorphism

$$|\cdot| : ([0, \infty), +, 0) \to (k, \cdot, 1)$$

for some semiring k.

Take $k = \mathbb{R}$.

Essentially the only choice of $|\cdot|$ is $|x| = C^x$ for some constant $C \geq 0$.

Up to rescaling of the metric, the only nontrivial possibilities are $C = e^{\pm 1}$.

Take $C = e^{-1}$, so that $|x| = e^{-x}$.

Outcome: we get a definition of the magnitude of a finite metric space. It's a real number.

But what does it *mean*?

2. The magnitude of a metric space

— done explicitly —

The magnitude of a finite metric space

Let A be a finite metric space.

Write Z_A for the $A \times A$ matrix with entries

$$Z_A(a, b) = e^{-d(a,b)}$$

$(a, b \in A)$.

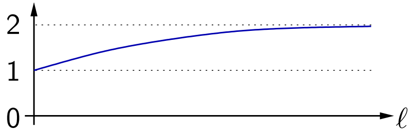
If Z_A is invertible (which it is if $A \subseteq \mathbb{R}^n$), the magnitude of A is

$$|A| = \sum_{a,b \in A} Z_A^{-1}(a, b) \ \in \mathbb{R}$$

—the sum of all the entries of the inverse matrix of Z_A.

First examples

- $|\emptyset| = 0$.
- $|\bullet| = 1$.
- $\left|\bullet\!\overset{\leftarrow\ \ell\ \rightarrow}{}\!\bullet\right| = \text{sum of entries of } \begin{pmatrix} e^{-0} & e^{-\ell} \\ e^{-\ell} & e^{-0} \end{pmatrix}^{-1} = \dfrac{2}{1+e^{-\ell}}$

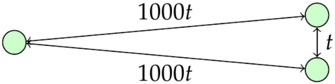


- If $d(a,b) = \infty$ for all $a \neq b$ then $|A| = \text{cardinality}(A)$.

Slogan: Magnitude is the 'effective number of points'.

Example: a 3-point space (Simon Willerton)

Take the 3-point space

$A =$ 

- When t is small, A looks like a 1-point space.
- When t is moderate, A looks like a 2-point space.
- When t is large, A looks like a 3-point space.

Magnitude functions

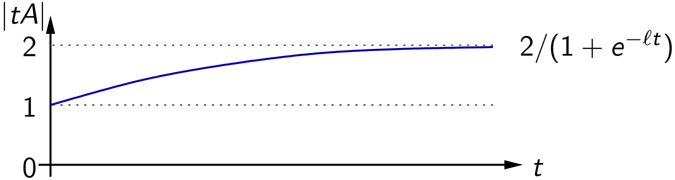
Magnitude assigns to each metric space not just a *number*, but a *function*.

For $t > 0$, write tA for A scaled up by a factor of t.

The magnitude function of a metric space A is the partially-defined function

$$\begin{array}{ccc} (0, \infty) & \to & \mathbb{R} \\ t & \mapsto & |tA| . \end{array}$$

E.g.: the magnitude function of $A = (\bullet \xleftarrow{} \ell \xrightarrow{} \bullet)$ is



A magnitude function has only finitely many singularities (none if $A \subseteq \mathbb{R}^n$).

It is increasing for $t \gg 0$, and $\lim_{t \to \infty} |tA| = \text{cardinality}(A)$.

The magnitude of a compact metric space

A metric space M is positive definite if for every finite $B \subseteq M$, the matrix Z_B is positive definite.

E.g.: $\mathbb{R}^n$ with Euclidean or ℓ^1 metric; sphere with geodesic metric; hyperbolic space; any ultrametric space.

Theorem (Mark Meckes)
All sensible ways of extending the definition of magnitude from finite metric spaces to compact positive definite spaces are equivalent.

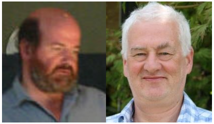
For a compact positive definite space X,

$$|A| = \sup\{|B| \ : \ \text{finite } B \subseteq A\}.$$

Magnitude encodes geometric information

Theorem (Juan-Antonio Barceló & Tony Carbery)

For compact $A \subseteq \mathbb{R}^n$,



$$\mathrm{vol}_n(A) = C_n \lim_{t \to \infty} \frac{|tA|}{t^n}$$

where C_n is a known constant.

Theorem (Heiko Gimperlein & Magnus Goffeng)

Assume n is odd. For 'nice' compact $A \subseteq \mathbb{R}^n$,

$$|tA| = c_n \,\mathrm{vol}_n(A) t^n + c_{n-1} \,\mathrm{vol}_{n-1}(\partial A) t^{n-1} + O(t^{n-2})$$

as $t \to \infty$, where c_n and c_{n-1} are known constants.

The magnitude function knows the volume and the surface area.

Magnitude encodes geometric information

Magnitude satisfies an asymptotic inclusion-exclusion principle:

Theorem (Gimperlein & Goffeng)

Assume n is odd. Let $A, B \subseteq \mathbb{R}^n$ with A, B and $A \cap B$ nice. Then

$$|t(A \cup B)| + |t(A \cap B)| - |tA| - |tB| \to 0$$

as $t \to \infty$.

But not all results are asymptotic! Write B^n for the unit ball in $\mathbb{R}^n$.

Theorem (Barceló & Carbery; Willerton)

Assume n is odd. Then $|tB^n|$ is a known rational function of t over $\mathbb{Z}$.

Examples

- $\left|tB^1\right| = |[-t, t]| = 1 + t$
- $\left|tB^3\right| = 1 + 2t + t^2 + \frac{1}{3!}t^3$
- $\left|tB^5\right| = \dfrac{24 + 72t^2 + 35t^3 + 9t^4 + t^5}{8(3 + t)} + \dfrac{t^5}{5!}$.

A different point of view

Write $q = e^{-1}$ and treat it as a formal symbol.

A formal calculation shows that the magnitude of a finite metric space A is

$$\sum_{\ell \in [0,\infty)} \left[\sum_{n \in \mathbb{N}} (-1)^n \big| \{ (a_0, \ldots, a_n) \ : \ a_0 \neq \cdots \neq a_n, \ d(a_0, \ldots, a_n) = \ell \} \big| \right] q^\ell,$$

where $d(a_0, \ldots, a_n)$ means $d(a_0, a_1) + \cdots + d(a_{n-1}, a_n)$.

Treat this as a formal expression, like a power series but with real exponents.

"Well-behaved" expressions $\sum_{\ell \in [0,\infty)} c_\ell q^\ell$ ($c_\ell \in \mathbb{Z}$) are called Novikov series, and form a ring.

The ring of Novikov series is a subset of $\mathbb{Z}^{[0,\infty)}$ (consider $\ell \mapsto c_\ell$).

E.g. q^ℓ corresponds to $\delta_\ell \colon [0, \infty) \to \mathbb{Z}$, where

$$\delta_\ell(k) = \begin{cases} 1 & \text{if } k = \ell \\ 0 & \text{otherwise.} \end{cases}$$

Multiplication of Novikov series corresponds to convolution of functions $[0, \infty) \to \mathbb{Z}$.

3. Magnitude homology, generally

Richard Hepworth and Simon Willerton, Categorifying the magnitude of a graph, 2017.

Tom Leinster and Michael Shulman, Magnitude homology of enriched categories and metric spaces, 2021.

Two perspectives on Euler characteristic

So far: Euler characteristic has been treated as an analogue of cardinality.

Alternatively: Given any homology theory H_* of any kind of object X, we can define

$$\chi(X) = \sum_{n=0}^{\infty} (-1)^n \operatorname{rank} H_n(X).$$

- $\chi(X)$ is a *number*
- $H_*(X)$ is an *algebraic structure*, and functorial in X.

That is, H_* is a categorification of χ.

So, homology categorifies Euler characteristic.

Warm-up: homology of an ordinary category

Any *ordinary* category $\boldsymbol{A}$ gives rise to a chain complex $C_*(\boldsymbol{A})$:

$$C_n(\boldsymbol{A}) = \coprod_{a_0,\ldots,a_n \in \boldsymbol{A}} \mathbb{Z} \cdot \big(\boldsymbol{A}(a_0, a_1) \times \cdots \times \boldsymbol{A}(a_{n-1}, a_n)\big)$$

where $\mathbb{Z} \cdot -: \mathbf{Set} \to \mathbf{Ab}$ is the free abelian group functor.

The homology $H_*(\boldsymbol{A})$ of $\boldsymbol{A}$ is the homology of $C_*(\boldsymbol{A})$.

Theorem $H_*(\boldsymbol{A}) = H_*(B\boldsymbol{A})$.

Key features of the definition of homology of a category:

- $(\mathbf{Set}, \times, 1)$ is a monoidal category, whose unit object 1 is terminal.
- $\mathbf{Ab}$ is both abelian and monoidal.
- $\mathbb{Z} \cdot -$ is a strong monoidal functor.

The magnitude homology of an enriched category: setup

Imitating the unenriched case, the context for magnitude homology is:

- a monoidal category $\boldsymbol{V}$ whose unit object is terminal (generalizing **Set**)
- a monoidal abelian category $\boldsymbol{K}$ (generalizing **Ab**)
- a strong monoidal functor $\Sigma\colon \boldsymbol{V} \to \boldsymbol{K}$ (generalizing $\mathbb{Z} \cdot -$).

Analogy This is a categorification of the context for magnitude, which was:

- a monoidal category $\boldsymbol{V}$
- a semiring k
- a monoid homomorphism $(\mathrm{ob}\, \boldsymbol{V})/\cong\, \to k$.

The magnitude homology of an enriched category: definition

We start with:

- a monoidal category $\boldsymbol{V}$ whose unit object is terminal
- a monoidal abelian category $\boldsymbol{K}$
- a strong monoidal functor $\Sigma \colon \boldsymbol{V} \to \boldsymbol{K}$.

Let $\boldsymbol{A}$ be a $\boldsymbol{V}$-category.

Define a chain complex $C_*^{\Sigma}(\boldsymbol{A}) = C_*(\boldsymbol{A})$ by

$$C_n(\boldsymbol{A}) = \coprod_{a_0,\ldots,a_n} \Sigma\big(\boldsymbol{A}(a_0, a_1) \otimes \cdots \otimes \boldsymbol{A}(a_{n-1}, a_n)\big).$$

It has differential $\partial = \sum_{i=0}^{n} (-1)^i \partial_i$, where ∂_i either composes at a_i or forgets the first/last factor.

Definition The magnitude homology $MH_*(\boldsymbol{A})$ of $\boldsymbol{A}$ is the homology of $C_*(\boldsymbol{A})$.

Magnitude homology categorifies magnitude. . . we'll come back to this.

4. The magnitude homology of a metric space

Setup

We will define the magnitude homology of a metric space, treating it as a category enriched in the monoidal category $([0, \infty), +, 0)$.

We need a monoidal abelian category $\boldsymbol{K}$ and a strong monoidal functor $\Sigma \colon [0, \infty) \to \boldsymbol{K}$.

What should we take $\boldsymbol{K}$ and Σ to be?

Recall: for the *magnitude* of metric spaces, we can view the size function $|\cdot|$ as taking values in a ring $\subseteq \mathbb{Z}^{[0,\infty)}$ of formal expressions $\sum_{\ell \in [0,\infty)} c_\ell q^\ell$.

Then $|\ell| = \delta_\ell \colon [0, \infty) \to \mathbb{Z}$ and multiplication is by convolution.

Categorifying this idea, take $\boldsymbol{K} = \mathbf{Ab}^{[0,\infty)}$ with convolution as $\otimes$, and define

$$\begin{array}{cccc} \Sigma \colon & [0,\infty) & \to & \mathbf{Ab}^{[0,\infty)} \\ & \ell & \mapsto & \delta_\ell \end{array}$$

where

$$\delta_\ell(k) = \begin{cases} \mathbb{Z} & \text{if } k = \ell \\ 0 & \text{otherwise} \end{cases} \qquad (k, \ell \in [0, \infty)).$$

The magnitude homology of a metric space, explicitly

Let A be a metric space. Use the coefficients functor Σ above.

The chain complex $C_*(A)$ in $\mathbf{Ab}^{[0,\infty)}$ is given by

$$C_n^\ell(A) = \mathbb{Z} \cdot \{(a_0, \ldots, a_n) \ : \ d(a_0, \ldots, a_n) = \ell\}$$

$(n \in \mathbb{N}, \ell \in [0, \infty))$.

Equivalently, we can replace $C_*(A)$ by a normalized version, $\hat{C}_*(A)$:

$$\hat{C}_n^\ell(A) = \mathbb{Z} \cdot \{(a_0, \ldots, a_n) \ : \ a_0 \neq \cdots \neq a_n, \, d(a_0, \ldots, a_n) = \ell\}.$$

The differential $\partial \colon \hat{C}_n(A) \to \hat{C}_{n-1}(A)$ is $\partial = \sum_{0<i<n} (-1)^i \partial_i$, where

$$\partial_i(a_0, \ldots, a_n) = \begin{cases} (a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) & \text{if } a_i \text{ is between } a_{i-1} \text{ and } a_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

Then $MH_*(A)$ is the homology of the chain complex $\hat{C}_*(A)$ in $\mathbf{Ab}^{[0,\infty)}$.

Magnitude homology is graded!

Magnitude homology of a metric space is a $[0, \infty)$-*graded* homology theory.

That is, when A is a metric space and n is a natural number, $MH_n(A)$ is not just an abelian group, but an object of $\mathbf{Ab}^{[0,\infty)}$ — a family

$$\left(MH_n^{\ell}(A)\right)_{\ell \in [0,\infty)}$$

of abelian groups.

(Compare Khovanov homology. . .)

Sample results

- 1st magnitude homology detects convexity:

Theorem *Let A be a closed subset of $\mathbb{R}^n$. Then*

$$A \text{ is convex} \iff MH_1^\ell(A) = 0 \text{ for all } \ell > 0.$$

- Work of Kyonori Gomi substantiates the slogan:

  The more geodesics are unique, the more magnitude homology is trivial.

- Ordinary homology detects the *existence* of holes.

Magnitude homology detects the *size* of holes.

Example (Ryuki Kaneta & Masahiko Yoshinaga) Let $r > 0$ and

$$A = \{x \in \mathbb{R}^N \ : \ \|x\| \geq r\}.$$

Then $H_n^\ell(A) = 0 \iff \ell/n > 2r$.

Magnitude homology categorifies magnitude, in finite case

That is, its Euler characteristic is magnitude, in the following sense.

Since magnitude homology is a $[0, \infty)$-graded theory, there is one Euler characteristic for each $\ell \in [0, \infty)$:

$$\chi_\ell(A) = \sum_{n \in \mathbb{N}} (-1)^n \operatorname{rank}(H_n^\ell(A)).$$

Then

$$\begin{aligned}
\chi_\ell(A) &= \sum_{n \in \mathbb{N}} (-1)^n \operatorname{rank}(\hat{C}_n^\ell(A)) \\
&= \sum_{n \in \mathbb{N}} (-1)^n \Big| \big\{ (a_0, \ldots, a_n) \ : \ a_0 \neq \cdots \neq a_n, d(a_0, \ldots, a_n) = \ell \big\} \Big|.
\end{aligned}$$

Put all the $\chi_\ell(A)$s into a single expression: $\chi(A) = \sum_{\ell \in [0,\infty)} \chi_\ell(A) q^\ell$. Then

$$\chi(A) = \sum_{\ell \in [0,\infty)} \sum_{n \in \mathbb{N}} (-1)^n \left| \{ (a_0, \ldots, a_n) \ : \ \ldots \} \right| q^\ell = |A|,$$

where the last step follows from earlier. So $\chi(A) = |A|$.

Some bad news — and the big open question

For metric spaces, magnitude homology categorifies magnitude. . . but only when the space is finite.

For infinite spaces, something goes wrong.

Theorem (Kaneta and Yoshinaga) *Whenever $A \subseteq \mathbb{R}^N$ is convex, $MH_n^\ell(A) = 0$ for all $n, \ell > 0$.*

Since different convex sets have different magnitudes, it seems impossible to recover magnitude from magnitude homology as its Euler characteristic (or in any other way).

Question *How can the definitions be fixed so that magnitude homology categorifies magnitude for infinite spaces too?*