'Think how primitive a machine your Personal Computer is, yet it completely outwits you on simple mathematical problems like deciding if a ten digit number is an exact square.

[. . .]

These computers are really very stupid things,' she said.

—David Ruelle, Conversations on mathematics with a visitor from outer space, *Mathematics: Frontiers and Perspectives*, AMS, 2000.

In general we seem to make up for inadequacies of the human mind [. . .] by a search for 'order' or 'meaning' often pushed to absurd limits. The unceasing, obsessional search for regularities is certainly fundamental to human intelligence.

—David Ruelle

The use of geometric intuition has no logical necessity in mathematics [. . .]. If one had to construct a mathematical brain, one would probably use resources more efficiently than creating a visual system. But the system is there already, it is used to great advantage by human mathematicians, and it gives a special flavor to human mathematics.

—David Ruelle

Ahlfors told me that in his youth, his thesis director [...] had made him read the memoirs of Fatou and Julia [...]. Ahlfors told me that at the time, they struck him as 'the pits of complex analysis.' Further, he said that he only understood what Fatou and Julia had been getting at when he saw the pictures Mandelbrot and I were showing.

Today, it is hard to imagine how incomprehensible the suject must have been before computer pictures (and even harder to imagine how Fatou and Julia managed to write their papers).

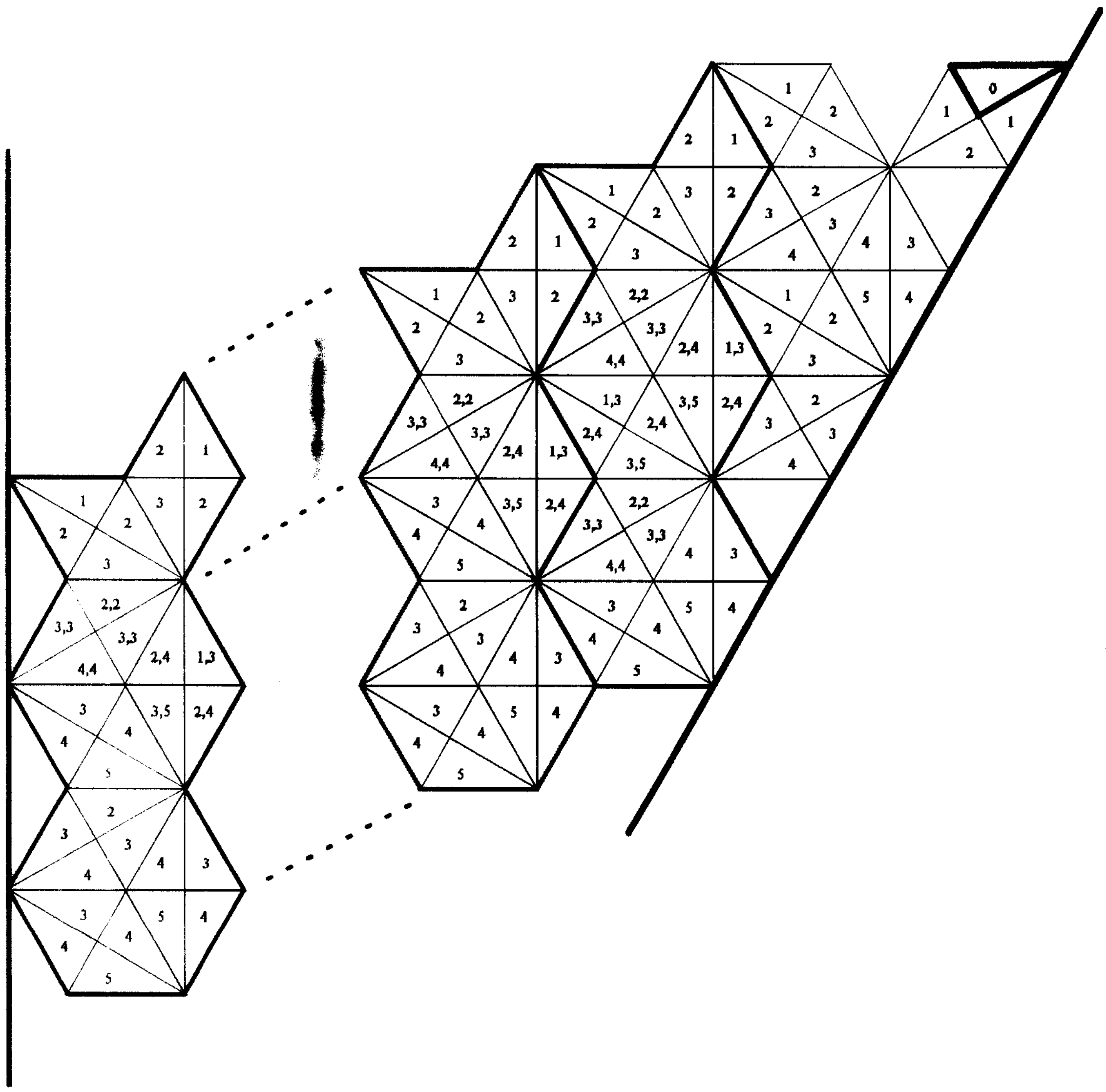
—John Hubbard

Mathematics is a part of physics.

—V.I. Arnold

It is only possible to understand the commutativity of multiplication by counting and re-counting soldiers by ranks and files or by calculating the area of a rectangle in the two ways.

—V.I. Arnold



(Stroppel)

$$\frac{1}{(1,-1,1,-1,1,-1,\ldots)} = (1,1,0,0,0,0,\ldots)$$

$$\frac{1}{(1,-1,1,-1,1,-1,\ldots)} = (1,1,0,0,0,0,\ldots)$$

$$\frac{1}{1-x+x^2-x^3+x^4-x^5+\cdots} = 1+x$$

Names like ‘Prone’ and ‘Supine’ correspond [. . .] to only one of the many aspects of the concept of cartesian and [. . .] cocartesian.

I am also against the habit [of naming] a new mathematical concept with words that have a precise meaning in everyday language (as prone, supine, etc).

Precisely, I do not know what does it mean exactly ‘Cartesian’ (has something to do with Descartes . . .), but I know precisely what is a ‘Cartesian arrow’ (in mathematics).

—Eduardo Dubuc

That Hermite was not used to thinking in the concrete is certain. He had a kind of positive hatred for geometry and once curiously reproached me with having made a geometrical memoir.

[. . .]

Methods always seem to be born in his mind in some mysterious way. In his lectures [. . .] he liked to begin his argument by: 'Let us start from the identity. . . ' and here he was writing a formula the accuracy of which was certain, but whose origin in his brain and way of discovery he did not explain and we could not guess.

—Jacques Hadamard

roduction

main general theorems on Lie Algebras are covered, roughly the content urbaki's Chapter I.

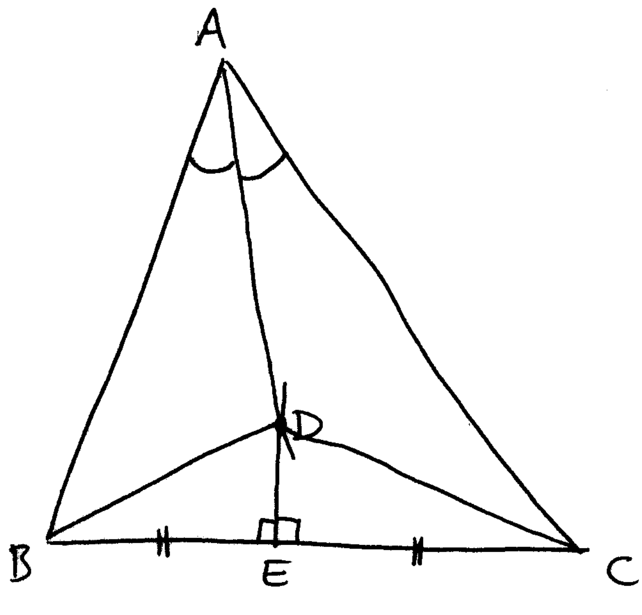
I have added some results on free Lie algebras, which are useful, both ie's theory itself (Campbell-Hausdorff formula) and for applications to p-groups.

Lack of time prevented me from including the more precise theory of imple Lie algebras (roots, weights, etc.); but, at least, I have given, as a Chapter, the typical case of $\mathfrak{sl}_n$.

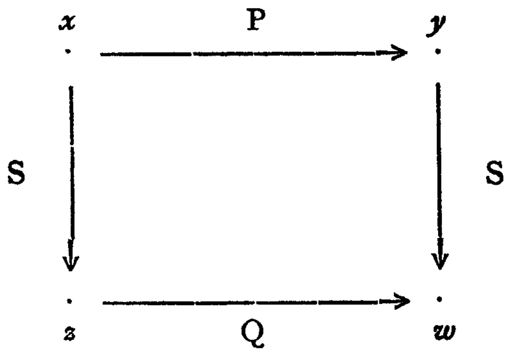
This part has been written with the help of F. Raggi and J. Tate. I want ank them, and also Sue Golan, who did the typing for both parts.

Jean-Pierre Serre

ard, Fall 1964


A
D
B
E
C

to another, the correlate of the one has the relation Q to the correlate of the other, and *vice versa*. A figure will make this clearer. Let x and y be two terms having the relation P. Then there are to be two terms z, w, such that x has the relation S to z, y has the relation S to w, and z has the relation Q to w. If this happens with every pair of terms such as x and y, and if the converse happens with every pair of terms such as z and w, it is clear that for every instance in which the relation P holds there is a corresponding instance in which the relation Q holds, and *vice versa*; and this is what we desire to secure by our definition. We can eliminate some redundancies in the above sketch of a definition, by observing that, when the above conditions are realised, the relation P is the same as the relative product of S and Q and the converse of S, *i.e.* the P-step from x to y may be replaced by the succession of the S-step from x to z, the Q-step from z to w, and the backward S-step from w to y. Thus we may set up the following definitions :—



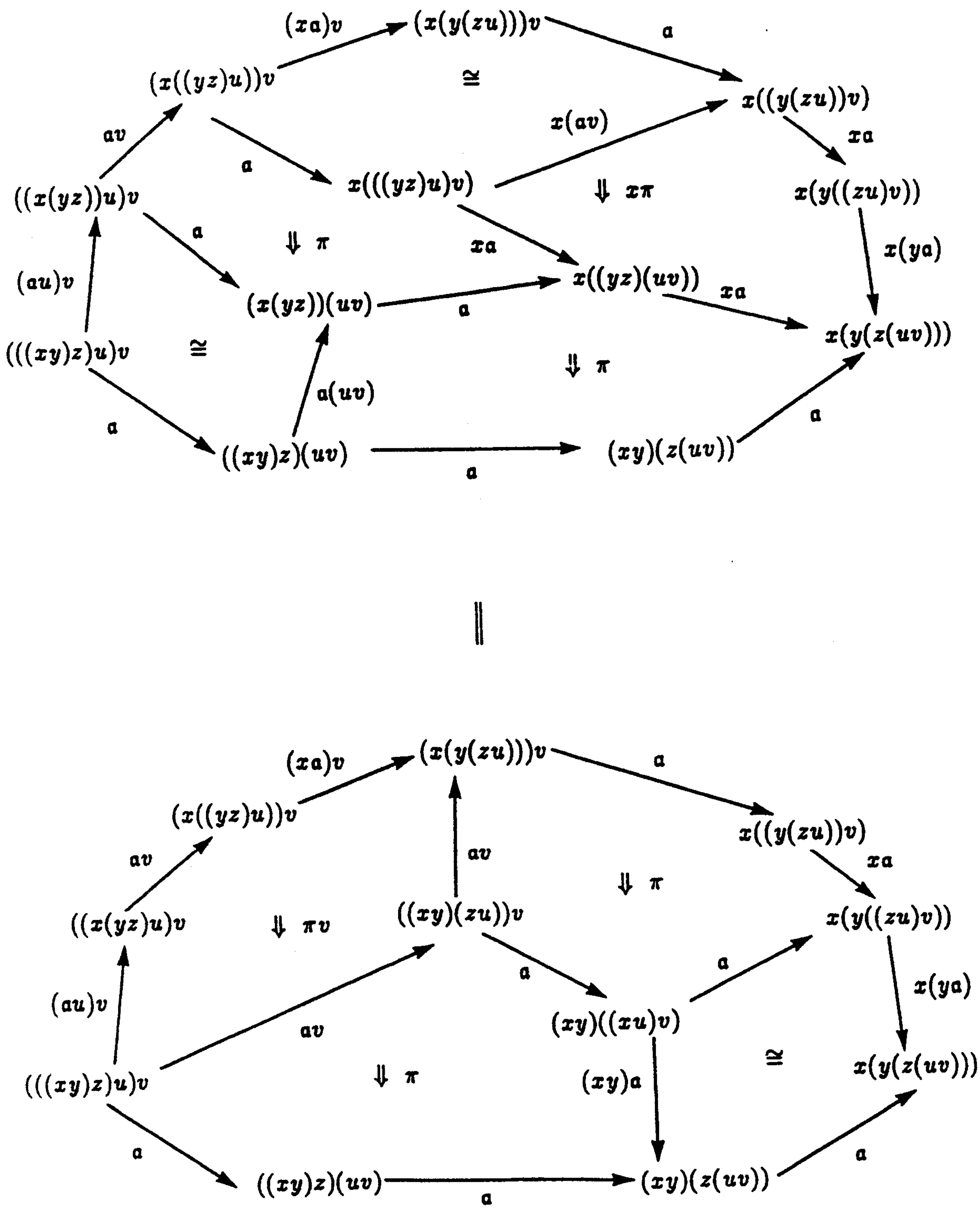
A relation S is said to be a "correlator" or an "ordinal correlator" of two relations P and Q if S is one-one, has the field of Q for its converse domain, and is such that P is the relative product of S and Q and the converse of S.

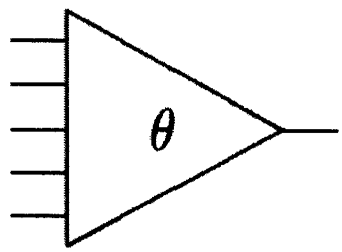
Two relations P and Q are said to be "similar," or to have "likeness," when there is at least one correlator of P and Q.

These definitions will be found to yield what we above decided to be necessary.

It will be found that, when two relations are similar, they share all properties which do not depend upon the actual terms in their fields. For instance, if one implies diversity, so does the other; if one is transitive, so is the other; if one is connected, so is the other. Hence if one is serial, so is the other. Again, if one is one-many or one-one, the other is one-many

(TA1) **(nonabelian 4-cocycle condition)** for all $(x,y,z,u,v) \in T(s,t) \times T(r,s) \times T(q,r) \times T(p,q) \times T(o,p)$, the equation

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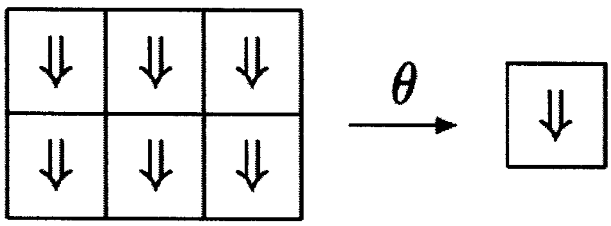
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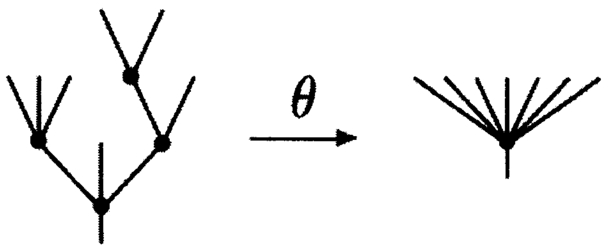
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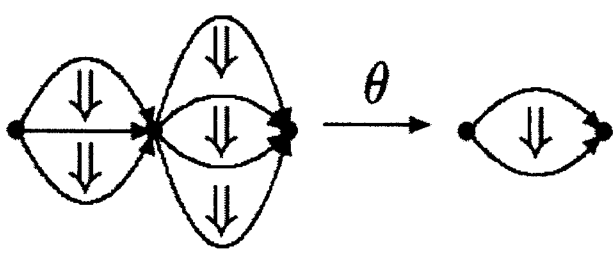
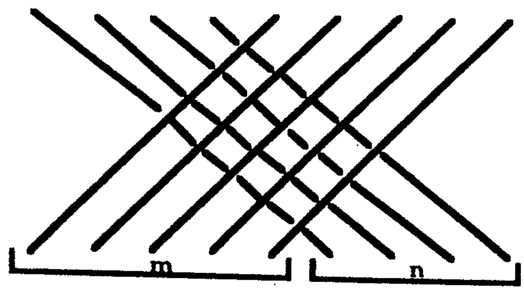
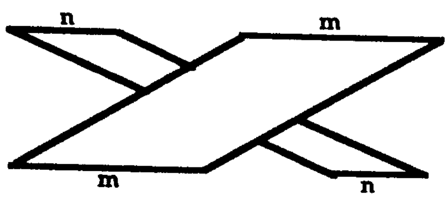

θ

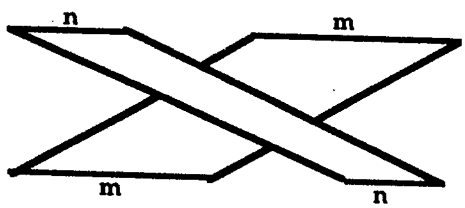
illustrated by the following diagram:



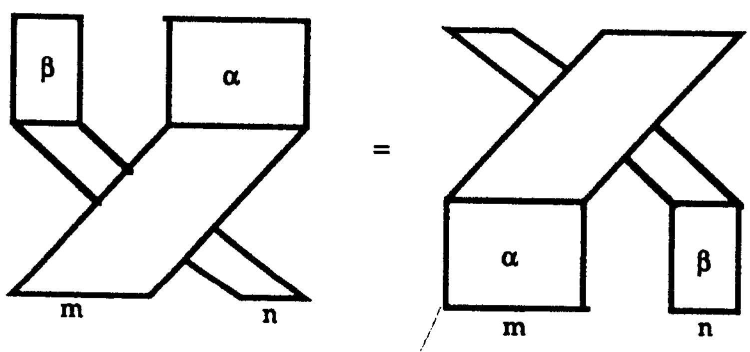
This can also be depicted as follows:



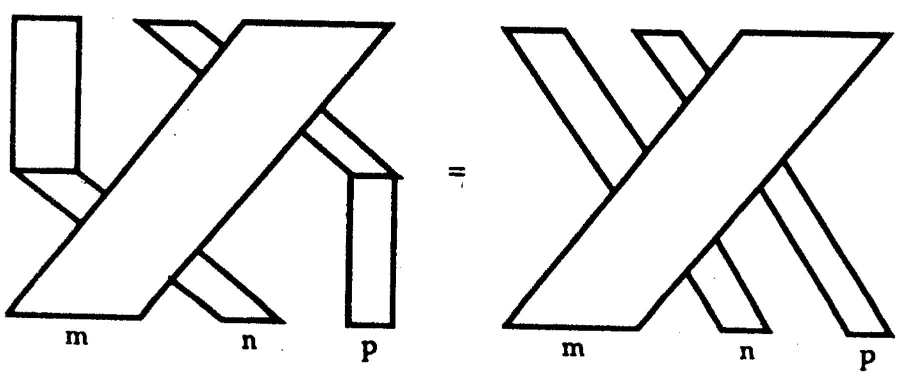
The picture for $c'_{m,n} = (c_{n,m})^{-1}$ is then as follows:



Naturalness of $c_{m,n}$ is proved pictorially by the equality



for all $\alpha \in \mathfrak{B}_m$, $\beta \in \mathfrak{B}_n$. Axiom (B2) is proved pictorially by the equality



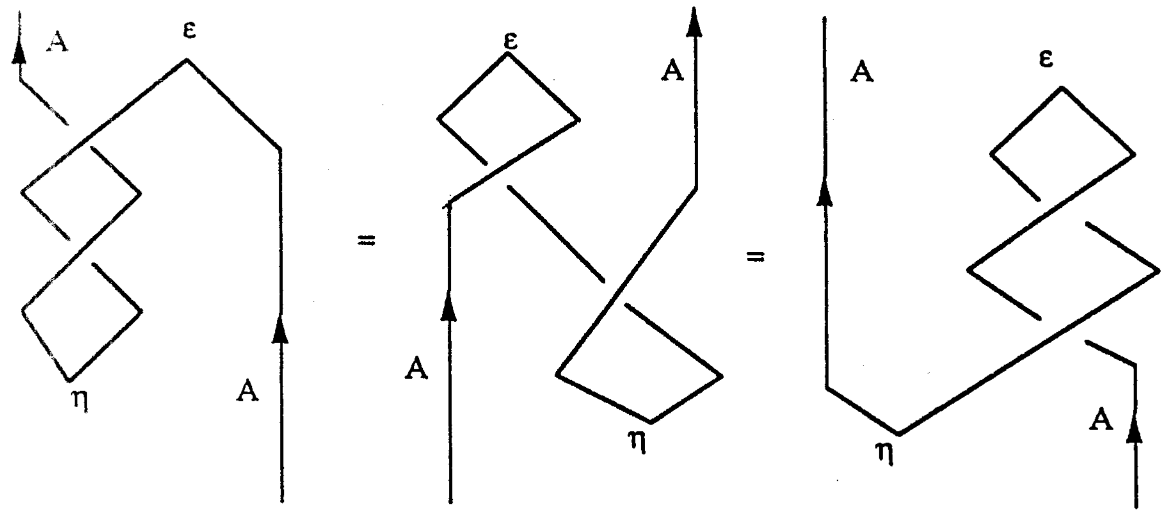
(Joyal, Street)

$$\psi_A = (1\otimes\varepsilon)(c_{A^*,A}\otimes 1)(c_{A,A^*}\otimes 1)(\eta\otimes 1) : A \xrightarrow{\sim} A$$

which is natural in those objects A which have duals; the inverse is given by

$$\psi_A{}^{-1} = (1\otimes\varepsilon)(1\otimes c_{A^*,A}{}^{-1})(1\otimes c_{A,A^*}{}^{-1})(\eta\otimes 1) : A \xrightarrow{\sim} A.$$

Using the geometry of braided tensor categories [2; Chapter 3], we can represent ψ_A by the left-hand 3D diagram below. The other two diagrams are deformations, and so give the same value ψ_A.



This gives the following two formulas for ψ_A which, of course, can also be verified algebraically :

$$\psi_A = (\varepsilon\otimes 1)(c_{A,A^*}\otimes 1)(1\otimes c_{A,A^*})(1\otimes\eta) = (1\otimes\varepsilon)(1\otimes c_{A,A^*})(1\otimes c_{A^*,A})(\eta\otimes 1).$$

The corresponding three diagrams for $\psi_A{}^{-1}$ are obtained from the above three by changing all the crossings. We also have the formula

$$\psi_A{}^{-1} = (1\otimes\varepsilon)(1\otimes c_{A^*,A}{}^{-1})(c_{A,A}\otimes 1)(1\otimes\eta),$$

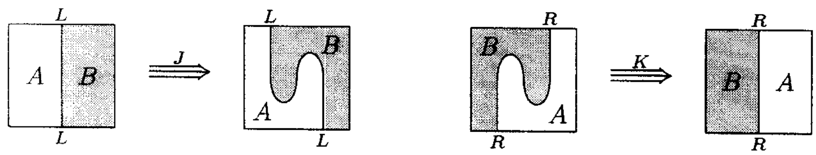
as represented by the following diagram which can be seen to be a deformation of the other diagrams for $\psi_A{}^{-1}$.

(Lauda)

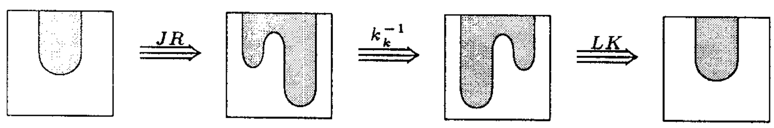
object in pAdj, namely B. Hence, by the universal property of the Eilenberg-Moore completion, we get a **Gray**-functor $\bar{\Lambda}: \mathbf{EM}_{\mathbb{T}}(\mathcal{K}) \to \mathrm{pAdj}$ that preserves Eilenberg-Moore objects. It is easy to see that Λ and $\bar{\Lambda}$ define an isomorphism of **Gray**-categories. □

3.4 The walking pseudo ambijunction

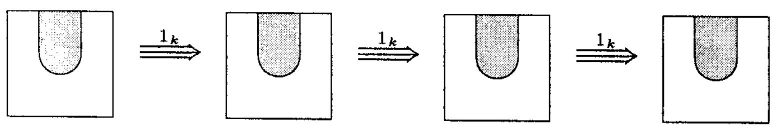
An ambidextrous pseudoadjunction, or pseudo ambijunction for short, is a 2-sided pseudoadjunction. This means that we have the additional 2-morphisms $j: 1_B \Rightarrow RL$ and $k: LR \to 1_A$ and the additional 3-morphisms



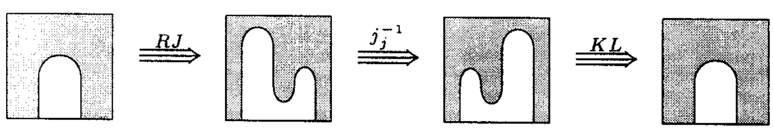
These 3-morphisms must satisfy the coherence conditions that the composite:



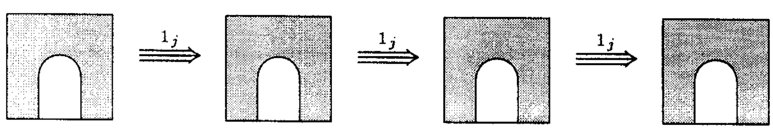
is equal to the composite:



and that the composite:



is equal to the composite:



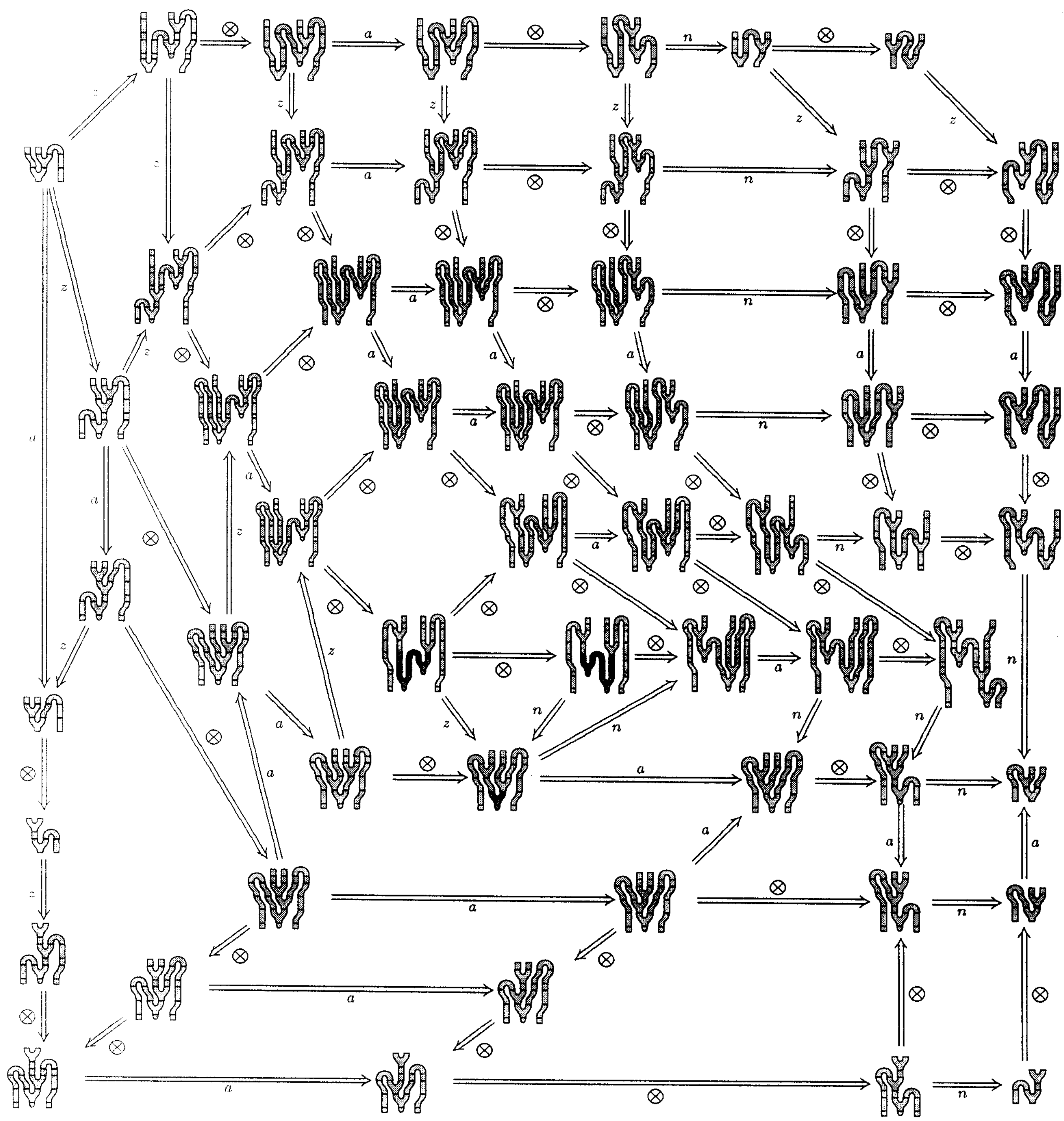
Definition 23 *The walking ambidextrous pseudoadjunction* pAmbi *is semistrict 3-category freely generated by a pseudo ambijunction.*

We now define a pseudo Frobenius algebra by categorifying the relationship between Frobenius algebras and adjunctions.

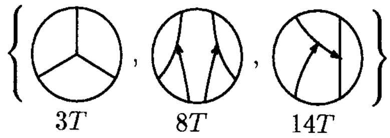
18

(Lauda)

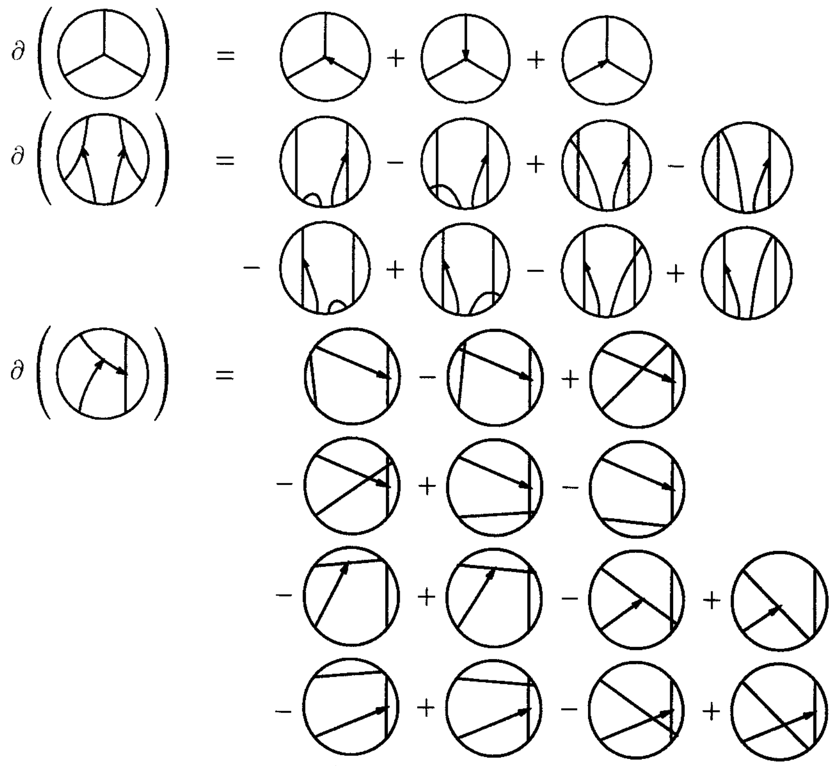
Figure 2: Proof that pFrob satisfies the relations of the $\mathcal{F}_2$.



$\mathcal{D}_m^2 = \text{span}$



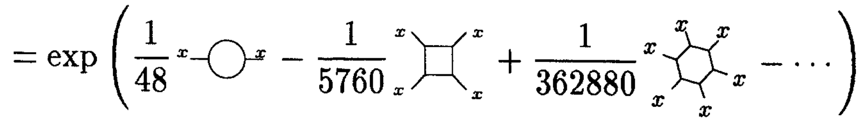
where



(Bar-Natan)

We can now write a complete formula for $j^{\frac{1}{2}}$ in terms of graphs:

$$\begin{aligned} j^{\frac{1}{2}}(x) &= \det{}^{\frac{1}{2}} \frac{\sinh \frac{\operatorname{ad} x}{2}}{\frac{\operatorname{ad} x}{2}} \\ &= \exp(\frac{1}{2}\operatorname{tr}(\log \frac{\sinh \frac{\operatorname{ad} x}{2}}{\frac{\operatorname{ad} x}{2}})) \\ &= \exp(\frac{1}{2}\operatorname{tr}(\sum_{n=0}^{\infty} b_{2n}(\operatorname{ad} x)^{2n})) \\ &= \exp(\sum_{n=0}^{\infty} b_{2n}\omega_{2n}(x)) \end{aligned}$$

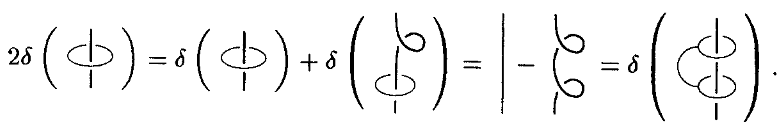


The b_{2n} were defined in Equation 1.2. The ω_{2n} are as in Equation 1.3, with x placed on the legs.

Relation 3 (Single lasso). Let C be a circus with a single lasso and C' be the same circus with a the single lasso replaced by a double lasso whose two loops are parallel to the old single lasso. Then

$$2\,\mathrm{Diag}(C) = \mathrm{Diag}(C').$$

Proof.

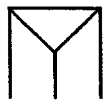


□

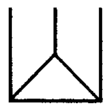
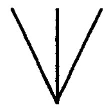
(D. Thurston)

(Boos)

Die Graphen der Springer'schen Formel aufgefasst in $\mathrm{Mor}(\emptyset, \{x, y, z\})$ ergeben

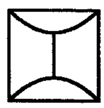
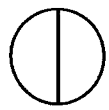
(5a)  $= (d-4)\ \Box$ 

und auch

(5b)  $= (d-4)\ \Box$  .

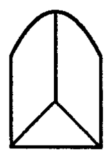
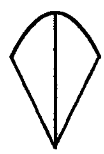
wenn man sie als Elemente von $\mathrm{Mor}_\Lambda(\{x, y, z\}, \emptyset)$ auffasst.

Kleben wir die linke Seite von (5a) mit der linken von (5b) und die beiden rechten Seiten zusammen, finden wir

(6)  $= (d-4)^2\Box$  $= -d\Box(d-1)\Box(d-4)^2.$

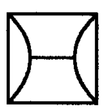
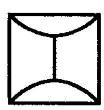
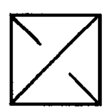
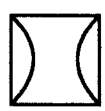
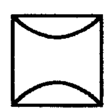
Hier ist $(d-4)^2$ als $(d-4)\Box(d-4)$ zu verstehen ist. Die zweite Gleichung folgt aus (3).

Wir zeichnen auch noch (5b) $\circ\, \gamma$, was

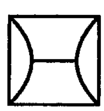
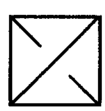
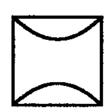
(7)  $= (d-4)\ \Box$ 

ergibt.

Nun betrachten wir noch eine weitere Konsequenz von (R4), aus der wir schliesslich und endlich die gewünschte Identität folgern können. Wir bilden den Graphen $\alpha \circ (\gamma\Box\gamma) \circ (I\Box(R4)\Box I) \circ (\gamma^t\Box\gamma^t) \circ \alpha^t$.

 +  $= 2\cdot$  $-$  $-$  .

Die beiden Graphen ganz links und die beiden Graphen ganz rechts sind isomorph. Also erhalten wir

(8)  $=$  $-$  ,

nachdem wir durch 2 geteilt haben.

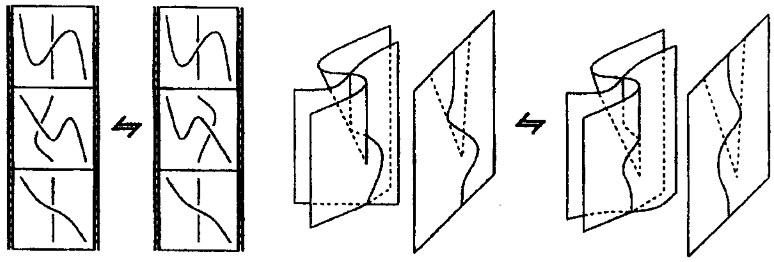


FIG. 27. A double arc passes over a fold line near a cusp.

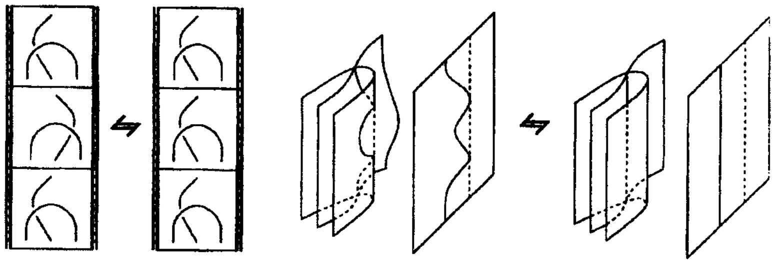


FIG. 28. Removing redundant double points crossing the fold lines.

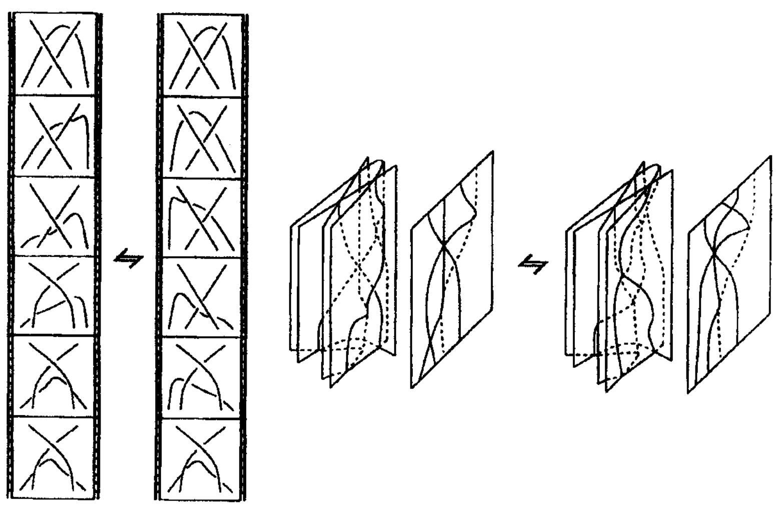
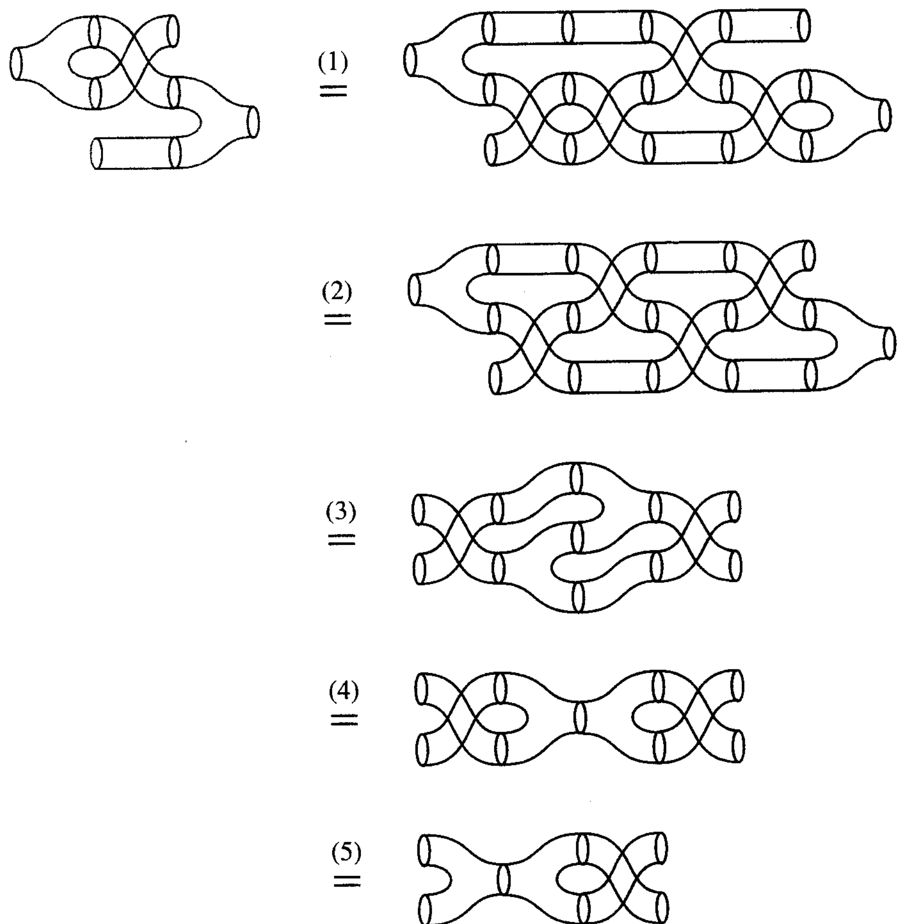
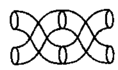


FIG. 29. A triple point near a fold line.

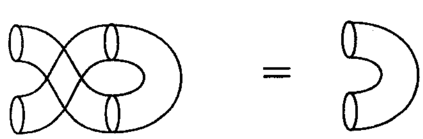
(Carter, Rieger, Saito)



(J. Kock)

: first step was to insert three new twist maps: the two maps  just give identity, and the third map, inserted just before , is justified by commu- ʼity. In step (2) we changed the order of the three rightmost twist maps, ording to the 'symmetric-group-relation' – it is an instance of the naturality he twist map. Equation (3) expressed another two instances of naturality, time with respect to comultiplication and multiplication – the twist maps he middle move outwards past comult and mult. Step (4) is the Frobenius ıtion, and finally in (5) we used commutativity of the multiplication map k again. □

.30 Symmetric Frobenius algebras. If (A, β) is a Frobenius algebra then can also picture the condition of being a symmetric one (that β satisfies the :e condition):



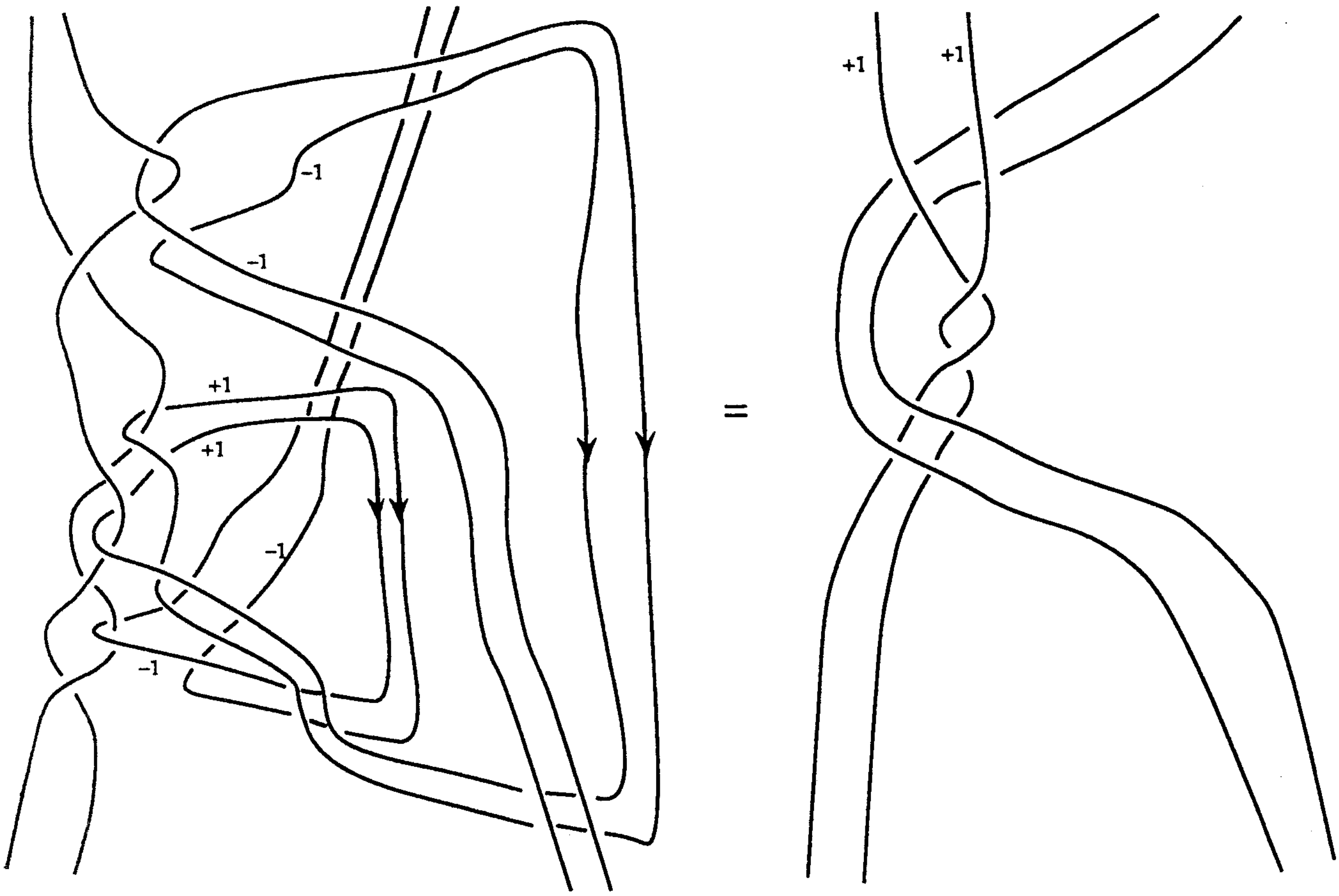
Tensor calculus (linear algebra in coordinates)

til now we have carefully avoided coordinates. In this subsection we will

(Joyal, Street, Verity)

$$
\begin{array}{ccc}
(X,U) \otimes' (X',U') & \xrightarrow{c_{(X,U),(X',U')}} & (X',U') \otimes' (X,U) \\
\Big\downarrow \theta_{(X,U)\otimes'(X',U')} & & \Big\downarrow \theta_{(X',U')} \otimes' \theta_{(X,U)} \\
(X,U) \otimes' (X',U') & \xleftarrow{c_{(X',U'),(X,U)}} & (X',U') \otimes' (X,U)
\end{array}
$$

commutes. This amounts to the following equality which, again, we leave to the reader.



We now turn our attention to the duality conditions. Commutativity of the triangle

$$
\begin{array}{ccc}
(U,X) & \xrightarrow{1 \otimes' \eta} & (U,X) \otimes' (X,U) \otimes' (U,X) \\
& \searrow^{1} & \Big\downarrow \varepsilon \otimes' 1 \\
& & (U,X)
\end{array}
$$

is proved by the following equality, and the other duality triangle is left to the reader.

$$\{\text{humanly-interesting provable statements}\}$$

$$\cap$$

$$\{\text{provable statements}\}$$

$$\cap$$

$$\{\text{true statements}\}$$

$$\{\text{humanly-interesting humanly-provable statements}\}$$

$$\cap$$

$$\{\text{humanly-interesting provable statements}\}$$

$$\cap$$

$$\{\text{provable statements}\}$$

$$\cap$$

$$\{\text{true statements}\}$$

Teaching ideals to students who have never seen a hypocycloid is as ridiculous as teaching addition of fractions to children who have never cut (at least mentally) a cake or an apple into equal parts.

—V.I. Arnold

Grothendieck changed the landscape of mathematics with a viewpoint that is 'cosmically general', in the words of Hyman Bass [. . .]. This viewpoint has been so thoroughly absorbed into mathematics that nowadays it is difficult for newcomers to imagine that the field was not always this way.

—Allyn Jackson

It is thus hard to exclude that some very interesting results are inherently off limits to the unaided human brain, and might only become accessible when sufficiently intelligent computers take over. As long, however, as humans use their own brains to do mathematics, some areas will be privileged.

—David Ruelle

Theorem 4.4.2. *(The Pfaff-Saalschütz identity)*

$$\sum_k \frac{(a+k)!(b+k)!(c-a-b+n-1-k)!}{(k+1)!(n-k)!(c+k)!} =$$
$$\frac{(a-1)!(b-1)!(c-a-b-1)!(c-a+n)!(c-b+n)!}{(c-a-1)!(c-b-1)!(n+1)!(c+n)!}.$$

Proof: Take

$$R(n,k) = -\frac{(b+k)(a+k)}{(c-b+n+1)(c-a+n+1)}.$$

■

(Wilf)

Instead of wasting our time proving things [. . .], let's dedicate our time to programming the computer to prove things. [. . .] Let's take pride, for some time, in our programming skills. Soon, of course, computers will also beat us there, and once they learn how to program, the race would be lost, since they would be able to teach themselves how to program to program. Then we would be able to wean ourselves of our competitive spirit, and do proofs (or programs) just for fun, like working out, or doing crossword puzzles.

—Doron Zeilberger

Our brains have evolved to get us out of the rain, find where the berries are, and keep us from getting killed. [. . .] I've had this image of a creature, in another galaxy perhaps, a child creature, and he's playing a game with his friends. For a moment he's distracted. He just thinks about numbers, primes, a simple proof of the twin-prime conjecture, and much more. Then he loses interest and returns to his game.

—Ron Graham

If I were a medical man, I should prescribe a holiday to any patient who considered his work important.

—Bertrand Russell