Bicategorical Quantum Mechanics: beyond quantaloids

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Why enrichement?

1 ***A general statement of intentions***

a plausible categorical approach/interpetation of QM *should have to consider quantum mechanical systems as* structured *and even more importantly as* varying *objects*

2 ***Enrichment is the narural environment to treat structured objects, particularly so when these objects are categories of categories***

- provides typing of the homobjects of the enriched categories.
- sustains the organisation of quantum mechanical systems into 2-categories of their enriched 'abstraction' (like the 2-category $\mathcal{V}$-**Cat** of the enriched categories and their functors or the bicategory $\mathcal{V}$-**Mod** of the $\mathcal{V}$-enriched categories and the *modules* (also known as distributors or profunctors) among them. We call these 2-categorical structures of the enriched quantum mechanical systems as ***meta-systems***.
- relates intrinsically to *variation*, as has been elaborated for instance in a series of papers by Betti, Carboni, Street, Walters and others (see for instance the *Variation through enrichement* or the *Axiomatics of bicategories of modules*).

Why bicategorical enrichement?

Well, there are some very strong conceptual and formal reasons that enrichement should be done not just on a monoidal category but rather on a bicategory:

On the conceptual side:

3 *bicategories are profound in abstracting higher order structures*

Carboni, Street and Sabadini in *Bicategories of processes:*

> “...are bicategories rather than categories necessary for modelling processes? While the objects of any *abstract* category can be thought as states, and the arrows as processes, the kind of processes discussed above have internal stucture and can be compared. As these comparisons naturally arise as 2-cells, the above question must be answered in the affirmative. In most systems is the internal structure that is interesting...Bicategories play an essential role in modelling these structures.”

Lawvere also points out the distinctive role of bicategories

> "Just as category theory can explicitly encapsulate much more mathematics than pure set theory, while yet remaining universal, so bicategories contain qualitatively more information than pure categories. On the other hand, the notion of tricategory (and even ∞-category) which has been proved useful in homotopy theory, has the striking feature that even there the concept of bicategory is central, since it is the structure relating *any two* levels" (emphasis in the original, [1], p.181-182)

From a more formal point of view:

4 *enrichement over a monoidal category:*

- does *not* provide *typing* for objects in the enriched systems-categories
- does *not* suffice to represent *variation* when the varying structure is a category itself, which is really what we would like at least the *enriched* quantum mechanical systems to be. In particular *sheaves*-which we regard as the canonical notion of a varying structure-are actually (equivalent to) enriched categories over an appropriate bicategory even in the case of varying sets. Their 2-dimensional generalisation in the context of variable categories (eg. ***stacks***) is naturally bicategorical.
- usually presupposes symmetry of the background category (which in general is *not* the case for bicategories)
- is anyway *a special case* of bicategorical enrichement, since monoidal categories are in fact (equivalent to) *bicategories with a single object*.

Why beyond quantaloids?

5 *What is a quantaloid?*

A quantale Q is a (non-commutative) complete suplattice endowed with a monoidal structure. A quantaloid $\mathcal{Q}$ is its bicategorical generalisation, namely:

> *a bicategory enriched in the monoidal category of complete suplattices* **SL**, *ie. a locally small cocomplete bicategory whose hom-categories are complete suplattices.*[a]

[a] Viewed as a monoidal category (ie. as a monoid object in **SL**) a quantale is a single-object quantaloid.

6 *Quantaloids have certain advantages:*

- to be a $\mathcal{Q}$-enriched category is a property of $\mathcal{Q}$ being a quantaloid rather than extra structure upon it (see for instance [7]).
- this is reflected in the simplicity of the resulting calculus for distributors, idempotents etc. 2-cells are just inequalities and every diagram involving 2-cells automatically commutes.
- quite explicit quantum mechanical connotations (recall that a quantale, which is a key algebraic structure underlying QM, is the single-object version of a quantaloid).
- sustain to certain extent a generalisation of the notion of sheaf both in the *commutative* (ordinary set-valued sheaves are symmetric $\mathcal{Q}$-enriched categories for an appropriate quantaloid of relations) as well as the much more involved *non-commutative* case (eg. sheaves over a proper quantale) in the form of appropriately defined "$\mathcal{Q}$-orders".

7 *However they are also a bit restrictive:*

For instance, Ross Street has long ago observed that:

> "To be relevant to cohomology, enriched category theory must be developed over a base bicategory which does not necessarily have posetal homs." ([4], p.5)

Here we would like though to bring out a more intuitive aspect of the restrictiveness of quantaloids observing that:

> what appears to be advantageous in quantaloidal enrichement proves also to be its limitation. The rather rudimentary posetal structure of $\mathcal{Q}$ which is, moreover, inherited (or induced) at the level of the 2-categorical structures built upon it (ie. what we have called the *metasystems*) has the effect that only structures of limited complexity can be represented by means of such an enrichement.

The structure of homarrows

To make more comprehensible the restrictive nature of quantaloidal enrichement we need to be a bit more explicit about the structure of bicategory-enriched categories, especially in terms of their homarrows.

Consider, for instance, what does it really mean to enrich in a bicategorical rather than a monoidal environment:

- objects in the enriched categories are typed by the objects of the bicategory $\mathcal{W}$.
- as a result, homobjects become homarrows, ie. objects in the hom-categories $\mathcal{W}(U, V)$ determined by the corresponding types.
- the structure of the underlying categories and the relations between homarrows become more involved.

8 *Underlying category of an enriched category: the monoidal case*

Let $\mathcal{A}$ be a $\mathcal{V}$-category. We call its *underlying* category $\mathcal{A}_0$ the ordinary category $\mathcal{A}_0 = (\mathcal{V}\text{-}\mathbf{Cat})(\mathcal{I}, \mathcal{A})$ in $\mathcal{V}$-**Cat**, where $\mathcal{I}$ is the *unit* $\mathcal{V}$-category, namely the $\mathcal{V}$-category with a single object $\{*\}$ and with the single homobject $\mathcal{I}(*, *) = I$, ie. the unit object of the monoidal structure of $\mathcal{V}$.
A $\mathcal{V}$-functor $A\colon \mathcal{I} \to \mathcal{A}$ is then thought of as an object of $\mathcal{A}$, while a natural transformation between two such functors $f\colon A \Rightarrow B$ has the single component $f\colon I \to \mathcal{A}(A, B)$, ie. a map in $\mathcal{A}_0$.

This enables us effectively to speak *indirectly* as if it were about "morphisms" in the enriched category, in the way that has been pointed out by Kelly already in his seminal *Bacic Concepts of Enriched Category Theory*:

> "Finally, since strictly speaking there are no "morphisms" in the $\mathcal{V}$-category $\mathcal{A}$, it is harmless to call a map $f\colon A \to B$ in $\mathcal{A}_0$ which is an element $f\colon I \to \mathcal{A}(A, B)$ of $\mathcal{A}(A, B)$, a "map $f\colon A \to B$ in $\mathcal{A}$"" ([2], p.11)

It is then worth considering how much of this framework 'carries' over in the context of bicategorical enrichement.

In particular, if A is a category enriched over the bicategory $\mathcal{W}$, can we still pick out 'individual arrows' (ie. elements) in the homarrow $\mathsf{A}(A, B)$ and how exactly do homarrows relate with each other?

It is not hard to see that a key difference from the monoidal case is that *since the objects of the* $\mathcal{W}$*-enriched categories are now typed* by the objects of $\mathcal{W}$, *we have to examine two distinct cases depending on whether the objects in the enriched category* A *have the* same or different *types*.

9 ***the objects A, B are of the same type, eg. U***

- in any bicategory $\mathcal{W}$, the hom-category $\mathcal{W}(U, U)$ is a monoidal category, the monoidal structure being provided by the composition of $\mathsf{1\text{-}cells}$.

  The unit category in $\mathcal{W}(U, U)$-$\mathbf{Cat}$, denoted as $*_U$ (or sometimes as $\widehat{U}$), is obviously the $\mathcal{W}(U, U)$-category with a single-object $\{*_U\}$ *of type U* and a single homobject $*_U(*, *) = 1_U$, ie. the identity $\mathsf{1\text{-}cell}$ on U, which is the unit of the monoidal category $\mathcal{W}(U, U)$.

  A category A_U *enriched* over $\mathcal{W}(U, U)$ is a category whose homobjects are objects of $\mathcal{W}(U, U)$, ie. endomorphisms on U. Just like in ordinary monoidal enrichement, its objects will correspond to $\mathcal{W}(U, U)$-functors $*_U \to \mathsf{A}_U$, whereas a $\mathcal{W}(U, U)$-natural transformation between two such functors A, B, ie. a $\mathsf{2\text{-}cell}$ $\sigma \colon A \Rightarrow B \colon *_U \to \mathsf{A}$ in $\mathcal{W}(U, U)$-$\mathbf{Cat}$ will have a single component corresponding to a global element of the homobject $\mathsf{A}_U(A, B)$.

- This category can also be seen, however, as a $\mathcal{W}$-category, whose objects are all of the *same* type U, while its homarrows, being endomorphisms on U are just $\mathsf{1\text{-}cells}$ in the bicategory $\mathcal{W}$. It is the category of the *objects of* A *of type U*.

  It seems that in the case of bicategorical enrichement, *we cannot define an underlying category* for a $\mathcal{W}$-category A *as a whole* but rather only for the categories A_U *for each type U*. In the context of *such a* category and its underlying category $(\mathsf{A}_U)_0$, we may consider *individual* 'morphisms' of the homarrow $\mathsf{A}(A, B)$ to be the $\mathsf{2\text{-}cells}$ $\sigma \colon 1_U \Rightarrow \mathsf{A}(A, B)$ in $\mathcal{W}$.

***Recall that a category A enriched over the bicategory $\mathcal{W}$ is defined as*:**

- For each object U in $\mathcal{W}$, a *set* A_U of *objects over or of type U*. We write $U = tA$ to denote that the object $A \in \mathsf{A}$ is of type U.
- for objects A, B over U, V respectively, a $\mathsf{1\text{-}cell}$ $\mathsf{A}(B, A) \colon U \to V$ in $\mathcal{W}$
- for objects A, B, C over U, V, W respectively, *identity* and *composability* $\mathsf{2\text{-}cells}$ j_A and μ in $\mathcal{W}$ defined by:

$$\begin{tikzcd} U \arrow[r, bend left=50, "1_U"] \arrow[r, bend right=50, "\mathsf{A}(A,A)"'] \arrow[r, phantom, "\Downarrow j_A"] & U \end{tikzcd} \qquad \begin{tikzcd} & V \arrow[dr, "\mathsf{A}(C,B)"] \arrow[d, phantom, "\Downarrow \mu"] & \\ U \arrow[ur, "\mathsf{A}(B,A)"] \arrow[rr, "\mathsf{A}(C,A)"'] & {} & W \end{tikzcd}$$

  and satisfying appropriate axioms for the left and right identities as well as associativity.

10 *C is an object of a different type, eg. V*

Suppose we are given objects A, B in A with $tA = tB = U$ and C with $tC = V \neq U$. Let us call a homarrow determined by two objects in A of *different* type a *mixed* homarrow. It is not difficult to see that we *cannot* directly 'identify' morphisms in a mixed homarrow like $\mathsf{A}(C, A)$, simply because *there can be no $\mathcal{W}$-natural* transformation σ: $C \Rightarrow A$ between the $\mathcal{W}$-functors C: $*_V \to \mathsf{A}$ and A: $*_U \to \mathsf{A}$ picking out the objects C and A respectively. These functors *act now on different $\mathcal{W}$-categories*.
However, we can still define (for instance as in Street's [4], p.7) what we may call *comparison*[a] 2-cells $\mathsf{A}(C, f)$: $\mathsf{A}(C, A) \Rightarrow \mathsf{A}(C, B)$ (and dually $\mathsf{A}(f, C)$: $\mathsf{A}(B, C) \Rightarrow \mathsf{A}(A, C)$) corresponding to the composites: $\mu \bullet (\mathsf{A}(C, A) \circ f)$ and $\mu \bullet (f \circ \mathsf{A}(B, C))$ and given, eg. for $\mathsf{A}(C, f)$, diagramatically by[b]:

$$\begin{tikzcd}[column sep=large] U \arrow[r, "\mathsf{A}(A,B)"'] \arrow[r, bend left=70, "1_U"] \arrow[r, phantom, shift left=3ex, "\Downarrow f"] \arrow[rr, bend right=70, "\mathsf{A}(C,B)"'] \arrow[rr, phantom, shift right=5ex, "\Downarrow \mu"] & U \arrow[r, "\mathsf{A}(C,A)"'] \arrow[r, bend left=70, "\mathsf{A}(C,A)"] \arrow[r, phantom, shift left=3ex, "\Downarrow \mathrm{id}"] & V \end{tikzcd} \quad = \quad \begin{tikzcd}[column sep=huge] U \arrow[r, bend left=60, "\mathsf{A}(C,A)"] \arrow[r, bend right=60, "\mathsf{A}(C,B)"'] \arrow[r, phantom, "\Downarrow \mathsf{A}(C,f)"] & V \end{tikzcd}$$

[a]the term comparison is used to put an emphasis on the specific case when such 2-cells 'compare' mixed type homarrows, although the definition holds of course for the objects A, B, C being of *any* type.

[b]Notice that by abuse of notation we identify $\mathsf{A}(C, A)$ with the identity 2-cell on itself $\mathrm{id}_{\mathsf{A}(C,A)}$. We also denote by $\bullet$ the *vertical* and by $\circ$ the *horizontal* composition of 2-cells.

From a more conceptual standpoint

Let us see what these formalities may imply for a perspective bicategorical interpetation/approach to QM. A declaration is appropriate here:

11 ***Objectives of BiQM***

Bicategorical QM intends to provide mainly an abstraction for:

- quantum mechanical systems as enriched categories unfolded in various levels of structural complexity (features which reflect the inherently *implicative* and *contextual* nature of quantum mechanical processes)
- the distinctive role of measurement processes and observables
- the non-trivial nature of *composite* quantum mechanical systems (ie. quantum mechanical entaglement)

A point of departure towards the elaboration of the formal framework of BiQM seems to be Ross Street's work on *Cosmoi*, appropriately adapted to notions of *metasystems* of enriched/variable quantum mechanical systems.

12 *So what is restrictive in quantaloidal enrichement after all?*

- A basic intuition in our approach is that a homarrow $\mathsf{A}(A, B)$ can be interpeted as representing a ***collective transition amplitude*** for the corresponding processes $f\colon A \to B$ in the enriched system A. This is a generalisation of the quite standard view of homsets as *sets of paths* between the domain and codomain objects. More abstractly, a homarrow represents a *higher order* or *implicated* process, which *enfolds* the collection of processes $A \to B$ in A to a single ***meta-transition*** in the hom-category $\mathcal{W}(U, V)$.
- For a generic bicategory $\mathcal{W}$, our analysis suggests that (when $tA = tB = U$) *the more the* 2-cells $f\colon 1_U \Rightarrow \mathsf{A}(A, B)$ in the underlying hom-category $\mathcal{W}(U, U)$, *the more the individual morphisms* that can be 'picked out' or 'identified' in the collective transition $\mathsf{A}(A, B)$ and hence so much *the more the comparison 2-cells* $\mathsf{A}(C, f)$ amongst (mixed type) homarrows, since the latter are in fact *contextualised* or induced by the former.
- If $\mathcal{W}$ is a quantaloid 2-cells are expressed as inequalities and therefore there can be *at most one* 'identifiable' morphism $f\colon B \to A$ in the homarrow $\mathsf{A}(A, B)$ just in case when $1_U \leq \mathsf{A}(A, B)$, and consequently *at most one* comparison 2-cell $\mathsf{A}(C, f)\colon \mathsf{A}(C, A) \Rightarrow \mathsf{A}(C, B)$ precisely when the inequality $\mathsf{A}(C, A) \leq \mathsf{A}(C, B)$ holds.
- If the interpetation of a homarrow as a transition amplitude seems quite abstract in the setting of a generic bicategory, it looks almost self-suggestive when $\mathcal{W}$ is a quantaloid. The suplattice structure of the hom-categories indicates us to take a further step and interpet homarrows *not just as abstract transition amplitudes* but in fact as *probabilities* for the occurence of transitions $f\colon A \to B$, naturally ordered by means of the partial order of the 1-cells in each hom-category, which provides thus a natural notion of *comparability* of probability amplitudes for collective transitions.
- This is a subtle point though. Because a more thoughtful inspection shows that this 'suggestive' probabilistic interpetation is rather *void of physical meaning*. Indeed, let us make the reasonable assumption that ordering the probability amplitudes/homarrows is physically grounded somehow on an appropriate notion of *density* (or *intensity*) of the corresponding transitions. In other words, *the more the transitions* $f\colon A \to B$ *the greater the probability* assigned to the homarrow $\mathsf{A}(A, B)$.
- However this is exactly what *we are not able to know* when $\mathcal{W}$ is a quantaloid, because in this case *there are not enough* 2-cells available to pick out 'individual' transitions $A \to B$ so much less to afford a representation of their *density*! All we can know is whether there can be *some* transition $A \to B$ or a comparison 2-cell $\mathsf{A}(C, A) \leq \mathsf{A}(C, B)$ with respect to a third object C.[a]

[a] Comparison 2-cells worth further study. They seem more informative in the case of mixed homarrows, but even when A, B, C are all of the same type U and $\mathcal{W}$ is a quantaloid, comparison cells actually determine the *underlying order* of the category A_U (see for instance [7], pp. 11). It is interesting to see how such a notion of *order* induced by comparison 2-cells may apply in the case of a generic bicategory or how may be interpeted in the case of mixed type homarrows.

- In a sense, the naturally provided structure for the probability amplitudes, namely the homarrows, at the level of system transitions, ie. of the 1-cells in $\mathcal{W}$, *does not suffice to convey* (or *does not support*) some visible physical meaning for the transitions at the level of the enriched system A *itself*. The internal structure of the homarrows is completely *enfolded* or *suppressed*, as if it were, by the posetal structure of the quantaloid.

- Is it possible to sustain such a physical interpetation of the homarrows *as transition amplitudes* in terms of some appropriate notion of *density* or, more generally, in terms of their *internal structure*? Our analysis thus far indicates that an affirmative answer requires the underlying bicategory $\mathcal{W}$ to be structurally richer than a quantaloid or even any posetal bicategory.

  This rise in complexity does not have necessarily to be made at the expense of abolishing the convenient probability structure explicit in a quantaloid. It may mean though that our notion of probability has to be much refined and elaborated in order to reflect the more subtle and complex relations between objects, both at the level of the system as well as of the metasystem.

- The analysis of the plausibility of a physical interpetation of *homarrows-as-transition-amplitudes* in the case of $\mathcal{W}$ being a quantaloid is quite reminiscent of the so-called *standard* or *statistical* interpetation of QM. According to that interpetation, the aim of QM is to discover the probabilities related with various quantum mechanical transitions. These probabilities are calculated by the mathematical apparatus of the theory on a *collective* basis (eg. as outcomes of the measurement of a physical observable for a large number of identically prepared systems) *without* any reference to *individual* processes or any assumption of some *ontological* background that would account for these probabilities in terms of *real* properties of *real* quantum mechanical systems or objects. The detailed dynamics and development of quantum mechanical systems 'collapse' once more to some coarse meaning of probability as synomymous to the lack of knowledge of their deeper structure.

- What we suggest here is that in order to make proper physical sense of such probabilities, we have to go beyond this quantaloidal representation of the collective transition amplitudes and introduce a calculus based on more complex and richer bicategorical structures. This can be adapted to a more realistic interpetation of QM, where subtler or *implicated* quantum mechanical systems and processes can be probed by means of *fine tuning or fine graining their representation, namely by varying or making richer the background bicategory* $\mathcal{W}$. This kind of dynamical contextuality could provide for instance an explanation of the probabilistic apparatus of the standard interpetation of QM in terms of a 'deeper' or underlying *subquantum* level of the reality, much along the line of thought introduced by David Bohm in his *ontological interpetation* of QM.

A 15 minutes course in QM

The basic formalism and its intended physical meaning

- The main mathematical tool in QM is the *Hilbert space* $\mathcal{H}$. Its elements (vectors) represent the *states* of a quantum mechanical system. Using the Dirac notation we denote a state as $|\psi\rangle$ (a *ket*). We may take them to be normalised, ie. $|\psi| = 1$. The dual of a Hilbert space, $\mathcal{H}^*$, is the space of the linear functionals $\mathcal{H} \to \mathbb{C}$. Its elements are the *bras*, $\langle\psi|$.

- Physical observables are represented by *self-adjoint* (hermitean) operators on $\mathcal{H}$, ie. operators $\mathsf{M} : \mathcal{H} \to \mathcal{H}$ satisfying: $\mathsf{M}^\dagger = \mathsf{M}$. Adjointness has an obvious categorical connotation since for a self-adjoint operator it holds:

$$\langle\psi|\mathsf{M}\phi\rangle = \langle\mathsf{M}\psi|\phi\rangle = \overline{\langle\phi|\mathsf{M}\psi\rangle}$$

  ($\overline{\langle\rangle}$ denotes the complex conjugate value)

- Evolution of quantum mechanical systems occurs in two forms:

  1. *Unitary*, ie. dictated by the action of *unitary* operators $\mathsf{U}^\dagger = \mathsf{U}^{-1}$ which are basically isomorphisms on the Hilbert space preserving the internal product: $\langle\mathsf{U}\psi|\mathsf{U}\phi\rangle = \langle\psi|\phi\rangle$. They describe *determininstic, reversible* evolution of quantum mechanical systems.
  2. *projection or collapse*. ie. the actual process of measurement which is *non-determininstic* and *irreversible*. It is described by special operators, the *projectors*, namely *idempotent* self-adjoint operators: $\mathsf{P}^2 = \mathsf{P} = \mathsf{P}^\dagger$.

- *The Spectral Theorem*: The eigenvectors of a self-adjoint operator M form an orthonormal basis $\{m_i\}$ for (the finite-dimensional) $\mathcal{H}$. As a result, a state ψ can be decomposed like:

$$\psi = \sum_i \langle m_i|\psi\rangle|m_i\rangle$$

  Consequently, M itself can be decomposed to its projector components P_{m_i} corresponding to these eigenvectors.

- According to the fundamental ***Projection Postulate***, every measurement projects (or *reduces* or *collapses*) a state ψ to one of the eigenvectors $|m_i\rangle$ with ***probability*** given by:

$$p_i = \langle m_i|\psi\rangle$$

  these probabilities all sum up to the unit: $\sum_i |p_i^2| = 1$.

- the tensor structure accounts to great extent for the non-classical features of QM, in particular *entaglement* and *non-locality*. A compound system in a composite state $\psi \otimes \phi$ in the tensor Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, *cannot* always decompose into its components.

- The formalism of QM was associated (mostly due to the work of von Neumann) with the lattice $\mathcal{L}(\mathcal{H})$ of the orthogonal projectors corresponding to disjunctive outcomes of measurements represented by the *Projection Postulate*. This latiice is essentially a *quantale* and its *non-commutativity* is the main source of perplexities both to physicists as well as category theorists!

References

[1] F.W.Lawvere, *Adjoints in and among Bicategories*, in,*Logic and Algebra, Lecture Notes in Pure and Applied Algebra* volume 180, pages 181–189. Marcel Dekker, Inc.,

[2] G.M. Kelly, *Basic Concepts of Enriched Category Theory*, London Mathematical Society Lecture Notes Series, 64, Cambridge University Press, 1982.

[3] P. Katis, N. Sabadini, RFC Walters, *Bicategories of Processes*, Journal of Pure and Applied Algebra, vol. 115. pp. 141-178, 1997

[4] R. Street, *Enriched Categories and Cohomology*, Reprints on Theory and Applications of Categories, 14 (2005), pp. 1-18

[5] R. Street, *Cauchy characterization of enriched categories*, Reprints in Theory and Applications of Categories 4 (2004), p pp. 1-16.

[6] D. Bohm, *Wholeness and the Implicate Order*, Routledge Kegan Paul, London, 1980

[7] I.Stubbe, *Categories enriched over quantaloids:categories, distributors and functors*, Theory and Applications of Categories, vol. 14, pp. 1-45, 2005

[8] RFC Walters: *Sheaves on Sites as Cauchy-complete categories*, JPAA 24(1), 95-102, 1982

[9] R. Betti, A. Carboni, R. Street, R. Walters: *Variation through Enrichment*, JPAA 29, 109-127, 1983

[10] A. Carboni, S. Kasagnian, R. Walters: *An Axiomatics for Bicategories of modules*, JPAA (45), 127-141, 1987