Every weak P-category is P-equivalent to a strict P-category

Miles Gould
University of Glasgow

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Abstract

It has long been known that every weak monoidal category A is equivalent via monoidal functors and monoidal natural transformations to a strict monoidal category $\mathbf{st}(A)$. We extend this result to weak P-categories, for any strongly regular (operadic) theory P.

1 Introduction

Many definitions exist of categories with some kind of "weakened" algebraic structure, in which the defining equations hold only up to coherent isomorphism. The paradigmatic example is the theory of weak monoidal categories, as presented in [9], but there are also definitions of categories with weakened versions of the structure of groups [3], Lie algebras [2], crossed monoids [1], sets acted on by a monoid [10], rigs [7], and others. A general definition of such categories-with-structure is obviously desirable, but hard in the general case. In this paper, we restrict our attention to the case of strongly regular theories (equivalently, those given by non-symmetric operads) and present possible definitions of weak P-category and weak P-functor for any non-symmetric operad P. In support of this definition, we present a generalisation of Joyal and Street's result from [5] that every weak monoidal category is monoidally equivalent to a strict monoidal category.

The idea is to consider the strict models of our theory as algebras for an operad, then to obtain the weak models as (strict) algebras for a weakened version of that operad (which will be a **Cat**-operad). We weaken the operad using a similar approach to that used in Penon's definition of n-category: see [11], or [4] for a non-rigorous summary. The weak P-categories obtained are the "unbiased" ones: for instance, if P is the terminal operad (whose strict algebras are monoids), then the weak P-categories will have composites of all arities, not just 0 and 2.

In section 2, we present our definitions of weak P-category and weak P-functor. In section 3 we extend Joyal and Street's proof (or rather, Leinster's unbiased version) to the more general case of weak P-categories. In section 4, we examine the strictification functor defined in section 3, and show that it has an interesting universal property. In section 5, we explain why our approach cannot be straightforwardly extended to theories which are not given by operads, and outline some of the approaches that could be taken to deal with this.

2 Weak P-categories

By a **plain** operad, we mean what is elsewhere called a "non-symmetric" or "non-Σ" operad, that is one with no symmetric group action defined on it. Throughout, let P be a plain operad. A **Cat-operad** is an operad enriched in **Cat**, i.e. a sequence of categories $Q(0), Q(1), \ldots$ and composition functors, satisfying the usual operad axioms, as given, for instance, in [8] section 2.2. More generally, a $\mathcal{V}$-operad is an operad enriched in $\mathcal{V}$. Since operads can be thought of as one-object multicategories, we shall refer to the objects of the categories $Q(i)$ as **1-cells** and the arrows of these categories as **2-cells** of Q.

A **strongly regular** algebraic theory is one that can be presented using equations that use the same variables in the same order on both sides, with each variable appearing only once on each side. For instance, the theory of monoids is strongly regular, as is the theory of sets acted on by a given monoid M. The theory of commutative monoids is not strongly regular (intuitively, because of the equation $a \cdot b = b \cdot a$) and the theory of groups is not strongly regular (again intuitively, because of the equation $g \cdot g^{-1} = 1$). It can be shown, for instance as in [8] section C.1, that the strongly regular theories are exactly those given by plain operads.

Plain operads are algebras for a straightforward multi-sorted algebraic theory, so there is an adjunction

$$\mathbf{Set}^{\mathbb{N}} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{Operad} \quad (\bot)$$

The left adjoint is given by taking labelled trees, as described in [8] section 3.2. Let $D : \mathbf{Operad} \to \mathbf{Cat\text{-}Operad}$ be the functor which takes discrete categories aritywise.

Definition 2.1. The **weakening of** P, $\mathrm{Wk}(P)$, is the **Cat**-operad with the same 1-cells as FUP, and the unique category structure such that the extension of the counit is a map of **Cat**-operads and is full and faithful aritywise.

More concretely, take FUP, and, for any $A, B \in FUP(n)$, place an arrow $A \to B$ iff $\varepsilon(A) = \varepsilon(B)$. The composite of two arrows $A \to B \to C$ is the unique arrow $A \to C$. In particular, the arrows $A \to B$ and $B \to A$ are inverses. See Fig. 1.

An **algebra** for a $\mathcal{V}$-operad Q is an object $A \in \mathcal{V}$ and an arrow $h : Q \circ A \to A$ which commutes with composition in Q, where $Q \circ A$ is the coproduct $\coprod_{n \in \mathbb{N}} Q(n) \times A^n$ (this notation was introduced by Kelly in [6] for clubs). This leads us immediately to the following definition:

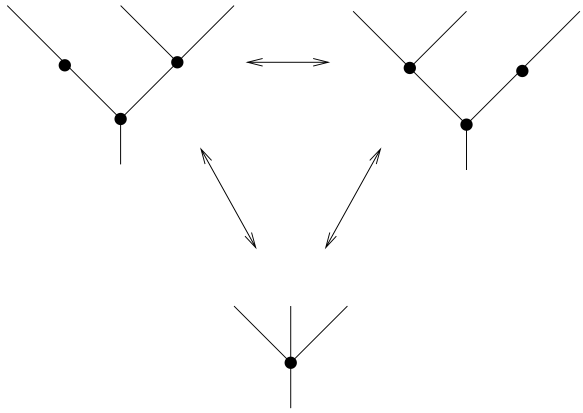


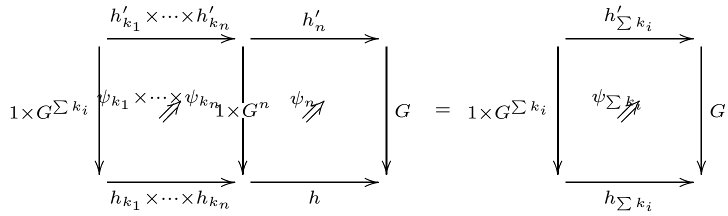
Figure 1: Part of Wk(P)(3) with $P = 1$

Definition 2.2. A **weak P-category** is an algebra for Wk(P).

In the case $P = 1$, this reduces exactly to Leinster's definition of unbiased monoidal category in [8] section 3.1. There, two 1-cells ϕ and ψ have the same image under ε iff they have the same arity, so the categories Wk(1)(i) are indiscrete. We refer to the image under h of a map $q \to q'$ in $Wk(P)$ as $\delta_{q,q'}$. This is clearly a natural transformation $h(q, \_) \to h(q', \_)$. As a special case, we write δ_q for $\delta_{q,\varepsilon(q)}$.

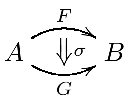
Definition 2.3. A **strict P-category** is an algebra for DP, or equivalently a weak P-category in which every component of δ is an identity arrow.

Definition 2.4. Let (A, h') and (B, h) be weak P-categories. A **weak P-functor** from (A, h') to (B, h) is a pair (G, ψ), where $G : A \to B$ is a functor and ψ is a sequence of natural transformations $\psi_i : h_i(1 \times G^i) \to Gh'_i$, satisfying the following:

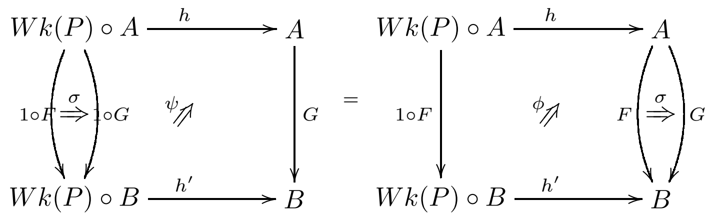


(1)

Definition 2.5. Let (F, ϕ) and (G, ψ) be weak P-functors $(A, h) \to (B, h')$. A **P-transformation** $\sigma : (F, \phi) \to (G, \psi)$ is a natural transformation



such that



(2)

Note that there is only one possible level of strictness here. There is a 2-category, **Wk-P-Cat**, whose objects are weak P-categories, whose 1-cells are weak P-functors, and whose 2-cells are P-transformations. Similarly, there is a 2-category **Str-P-Cat** of strict P-categories, strict P-functors, and P-transformations, which can be considered a sub-2-category of **Wk-P-Cat**.

Lemma 2.6. *A P-transformation $\sigma : (F, \phi) \to (G, \psi)$ is invertible as a P-transformation if and only if it is invertible as a natural transformation.*

Proof. "Only if" is obvious: we concentrate on "if". It's enough to show that σ^{-1} is a P-transformation, which is to say that

$$\begin{array}{ccc} h(q, Ga_\bullet) & \xrightarrow{\psi} & Gh(q, a_\bullet) \\ {\scriptstyle h(q,\sigma^{-1}_{a_\bullet})}\downarrow & & \downarrow{\scriptstyle \sigma^{-1}_{h(q,a_\bullet)}} \\ h(q, Fa_\bullet) & \xrightarrow{\phi} & Fh(q, a_\bullet) \end{array} \tag{3}$$

commutes for all $(q, a_\bullet) \in Wk(P) \circ A$, and this follows from the fact that $\sigma_{h(q,a_\bullet)} \circ \phi = \psi \circ h(q, \sigma_{a_\bullet})$.

□

3 Main Theorem

Let P be a plain operad, and $Q = \mathrm{Wk}(P)$, with $\pi : Q \to P$ the projection map. We write composition in P as $p \circ (p_1 \ldots p_n)$, and composition in Q as $q\langle q_1 \ldots q_n\rangle$. We also adopt the $\bullet$ notation from chain complexes and write, for instance, $p_\bullet$ for a sequence of objects in P and $p^\bullet_\bullet$ for a double sequence. Let $Q \circ A \xrightarrow{h} A$ be a weak P-category. We construct a strict P-category $\mathbf{st}(A)$ and a weak P-functor $(F, \phi) : \mathbf{st}(A) \to A$ whose underlying functor F is full, faithful and essentially surjective on objects, and hence an equivalence of weak P-categories.

In fact, **st** is functorial, and is left adjoint to the forgetful functor **Str-P-Cat** $\to$ **Wk-P-Cat**. The theorem then says that the unit of this adjunction is pseudo-invertible, and that the strict P-categories and strict P-functors form a weakly coreflective sub-2-category of **Wk-P-Cat**.

Definition 3.1. Let P, Q, A be as above. The **strictification of** A, $\mathbf{st}(A)$, is defined as follows:

- An object of $\mathbf{st}(A)$ is an object of $P(i) \times A^i$ for some $i \in \mathbb{N}$.
- An arrow $(p, a_\bullet) \to (p', a'_\bullet)$ in $\mathbf{st}(A)$ is an arrow $h(p, a_\bullet) \to h(p', a'_\bullet)$ in A. Composition and identities are as in A.

We define an action h' of Q on $\mathbf{st}(A)$ as follows:

- On objects, h' acts by $h'(q, (p, a_\bullet)^\bullet) = (\pi(q\langle p^\bullet\rangle), a^\bullet_\bullet)$.
- Let $f_i : (p_i, a_i) \to (p'_i, a'_i)$ for $i = 0 \ldots n$. Then $h'(p, f_\bullet)$ is the composite

$$\begin{aligned} h(p \circ (p_\bullet), a_\bullet) &\xrightarrow{\delta^{-1}_{p\langle p_\bullet\rangle}} h(p\langle p_\bullet\rangle, a_\bullet) = h(p, h(p_0, a_0), \ldots, h(p_n, a_n)) \\ &\xrightarrow{h(p,f_\bullet)} h(p, h(p'_0, a'_0), \ldots, h(p'_n, a'_n)) = h(p\langle p'_\bullet\rangle, a'_\bullet) \\ &\xrightarrow{\delta_{p\langle p'_\bullet\rangle}} h(p \circ (p'_\bullet), a'_\bullet). \end{aligned}$$

Lemma 3.2. $\mathbf{st}$*(A) is a strict P-category.*

Proof. The associativity of the action on objects is obvious, as are the identity and strictness conditions. We must show that the action on arrows is associative. Let $f_i^j : (p_i^j, a_{i\bullet}^j) \to (q_i^j, b_{i\bullet}^j)$, $\sigma \in Q(n)$, and $\tau_i \in Q(k_i)$ for $j = 1 \ldots k_i$ and $i = 1 \ldots n$. We wish to show that $h'(\sigma \circ (\tau_\bullet), f^\bullet_\bullet) = h'(\sigma, h'(\tau_1, f_1^\bullet), \ldots, h'(\tau_n, f_n^\bullet))$

The LHS is

$$\begin{aligned} h(\sigma \circ (\tau_\bullet) \circ (p^\bullet_\bullet), a^\bullet_\bullet) &\xrightarrow{\delta^{-1}_{\sigma\circ(\tau_i)\langle p^\bullet_\bullet\rangle}} h(\sigma \circ (\tau_\bullet), h(p_1^1, a_{1\bullet}^1), \ldots, h(p_n^{k_n}, a_{n\bullet}^{k_n})) \\ &\xrightarrow{h(\sigma\circ(\tau_\bullet), f^\bullet_\bullet)} h(\sigma \circ (\tau_\bullet), h(q_1^1, b_{1\bullet}^1), \ldots, h(q_n^{k'_n}, b_{n\bullet}^{k'_n})) \\ &\xrightarrow{\delta_{\sigma\circ(\tau_\bullet)\langle q^\bullet_\bullet\rangle}} h(\sigma \circ (\tau_\bullet) \circ (q^\bullet_\bullet), b^\bullet_\bullet). \end{aligned}$$

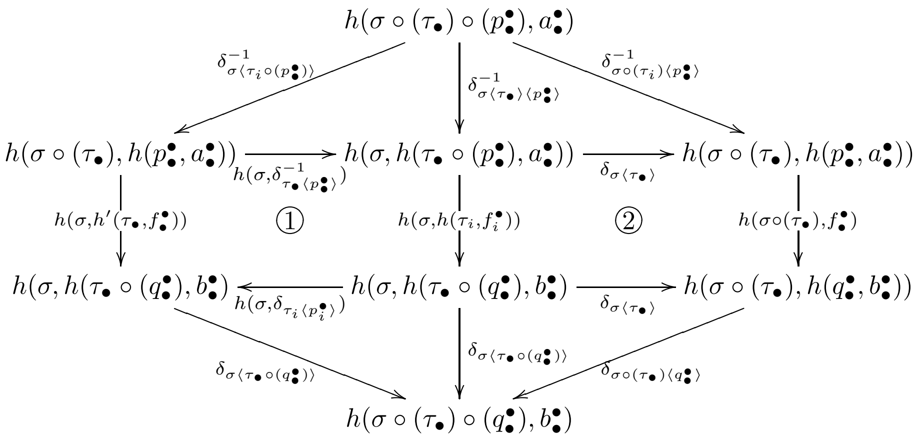
The RHS is

$$\begin{aligned} h(\sigma \circ (\tau_\bullet) \circ (p^\bullet_\bullet), a^\bullet_\bullet) &\xrightarrow{\delta^{-1}_{\sigma\langle\tau_i\circ(p^\bullet_\bullet)\rangle}} h(\sigma, h(\tau_1 \circ (p_1^\bullet), a_{1\bullet}^\bullet), \ldots, h(\tau_n \circ (p_n^\bullet), a_{n\bullet}^\bullet)) \\ &\xrightarrow{h(\sigma, h'(\tau_\bullet, f^\bullet_\bullet))} h(\sigma, h(\tau_1 \circ (q_1^\bullet), b_{1\bullet}^\bullet), \ldots h(\tau_n \circ (q_n^\bullet), b_{n\bullet}^\bullet)) \\ &\xrightarrow{\delta_{\sigma\langle\tau_i\circ(p^\bullet_\bullet)\rangle}} h(\sigma \circ (\tau_\bullet) \circ (q^\bullet_\bullet), b^\bullet_\bullet), \end{aligned}$$

where each $h'(\tau_i, f_i^\bullet)$ is

$$\begin{aligned} h(\tau_i \circ (p_i^\bullet), a_i^\bullet)) &\xrightarrow{\delta^{-1}_{\tau_i\langle p_i^\bullet\rangle}} h(\tau_i, h(p_i^1, a_{i\bullet}^1), \ldots h(p_i^{k_i}, a_{i\bullet}^{k_i})) \\ &\xrightarrow{h(\tau_i, f_i^\bullet)} h(\tau_i, h(q_i^1, b_{i\bullet}^1), \ldots h(q_i^{k_i}, b_{i\bullet}^{k_i}) \\ &\xrightarrow{\delta_{\tau_i\langle p_i^\bullet\rangle}}) \; h(\tau_i \circ (q_i^\bullet), b_i^\bullet). \end{aligned}$$

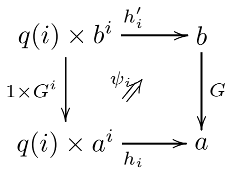
So the equation holds if the following diagram commutes:



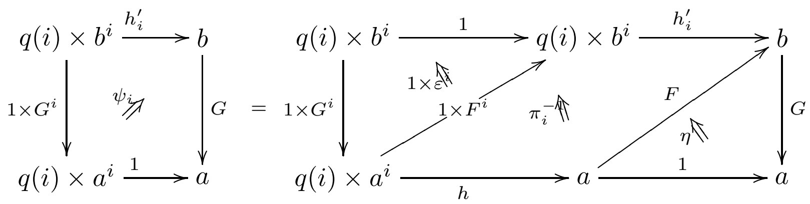
The triangles all commute because all δs are images of arrows in Q, and there is at most one 2-cell between any two 1-cells in Q. ① commutes by the definition of $h'(\tau_i, f_i^\bullet)$, and ② commutes by naturality of δ. □

Lemma 3.3. *Let $Q \circ A \xrightarrow{h} A$ and $Q \circ B \xrightarrow{h'} B$ be weak P-categories, $(F, \pi) : A \to B$ be a weak P-functor, and $(F, G, \eta, \varepsilon)$ be an adjoint equivalence. Then G naturally carries the structure of a weak P-functor, and $(F, G, \eta, \varepsilon)$ is an adjoint equivalence in **Wk**-P-**Cat**.*

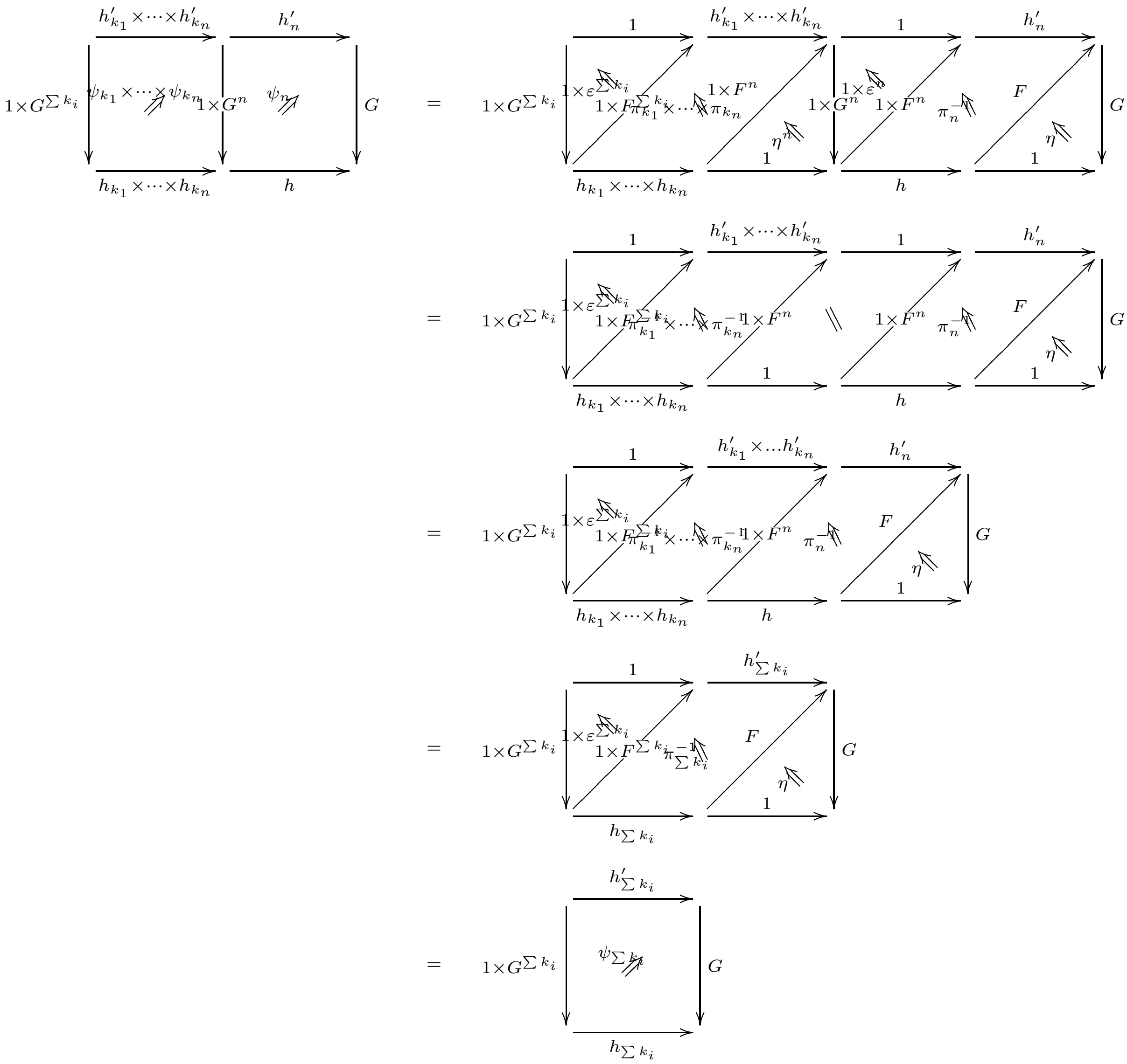
Proof. We want a sequence $(\psi_\bullet)$ of natural transformations:



let ψ_i be given by $Gh'(1 \times \varepsilon^i) \circ G\pi_i^{-1} \circ \eta_{h'}$, i.e.



We must check that ψ satisfies eq. 1:



To see that $(F, G, \eta, \varepsilon)$ is a P-equivalence, it is now enough to show that η and ε are P-transformations, since they satisfy the triangle identities by hypothesis.

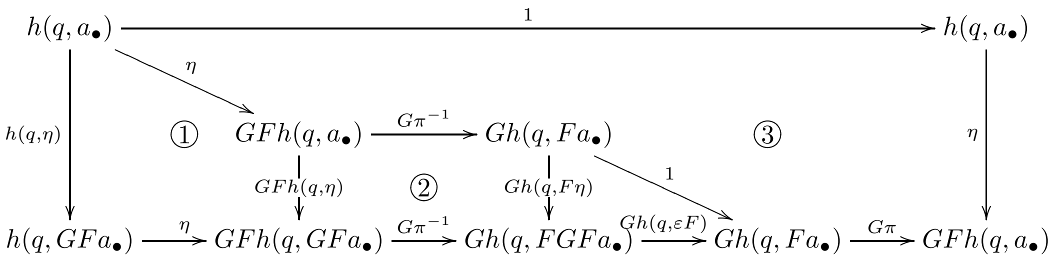
Write $(GF, \chi) = (G, \psi) \circ (F, \pi)$. Then each $\chi_{q,a_\bullet}$ is the composite

$$h(q, GFa_\bullet) \xrightarrow{\psi} Gh(q, Fa_\bullet) \xrightarrow{G\pi} GFh(q, a_\bullet)$$

Applying the definition of ψ, this is

$$h(q, GFa_\bullet) \xrightarrow{\eta} GFh(q, GFa_\bullet) \xrightarrow{G\pi^{-1}} Gh(q, FGFa_\bullet) \xrightarrow{Gh_q\varepsilon F} Gh(q, Fa_\bullet) \xrightarrow{G\pi} GFh(q, a_\bullet)$$

The axiom on η is the outside of the commutative diagram



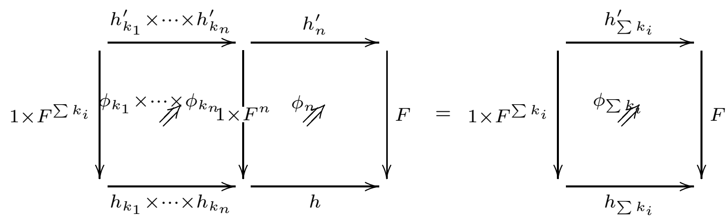
① commutes by naturality of η, ② commutes by naturality of π^{-1}, and ③ commutes since $G\pi \circ G\pi^{-1} = G(\pi \circ \pi^{-1}) = G1 = 1G$. The triangle commutes by the triangle identities. So the whole diagram commutes, and η is a P-transformation. By Lemma 2.6, η^{-1} is also a P-transformation. Similarly, ε and ε^{-1} are P-transformations.

□

Theorem 3.4. *Let $Q \circ A \xrightarrow{h} A$ be a weak P-category. Then A is equivalent to* $\mathbf{st}$*(A) via weak P-functors and P-transformations.*

Proof. Let $F : \mathbf{st}(A) \to A$ be given by $F(p, a_\bullet) = h(p, a_\bullet)$ and identification of maps. This is certainly full and faithful, and it's essentially surjective on objects because $\delta_{1_q}^{-1} : h(1_p, a) \to a$ is an isomorphism. Claim it's a weak P-functor.

We must find a sequence $(\phi_i : h_i(1 \times F^i) \to Fh')$ of natural transformations satisfying equation 1. We can take $(\phi_i)_{q,a_\bullet} = (\delta_q)_{a_\bullet}$. We must show that



(4)

All 2-cells in this diagram are instances of δ. Since there is at most one arrow between two 1-cells in Q, they are equal. So (F, ϕ) is a weak P-functor.

By Lemma 3.3, A is equivalent to $\mathbf{st}(A)$ via weak P-functors and P-transformations.

□

4 Significance of st

Theorem 4.1. *Let U' be the forgetful functor **Str**-P-**Cat** $\to$ **Wk**-P-**Cat** (considering both of these as 1-categories). Then* **st** *is left adjoint to U'.*

Proof. We exhibit $A \xrightarrow{(F',\psi)} \mathbf{st}(A)$ as an initial object of $(A \downarrow U')$, thus showing that $\mathbf{st}$ is functorial and that $\mathbf{st} \dashv U'$ (and that (F',ψ) is the component of the unit at A). Let $h'' : Q \circ B \to B$ be a strict P-category, and $(G,\gamma) : A \to U'B$ be a weak P-functor. We must show that there is a unique strict P-functor H making the following diagram commute:

$$\begin{array}{ccc} & A & \\ {\scriptstyle (F',\psi)} \swarrow & & \searrow {\scriptstyle (G,\gamma)} \\ U'\,\mathbf{st}(A) & \xrightarrow{(H,\mathrm{id})} & U'B \end{array} \tag{5}$$

(F',ψ) is given as follows:

- If $a \in A$, then $F'(a) = (1,a)$.
- If $f : a \to a'$ in A then $F'f$ is the lifting of $h(1,f)$ with source $(1,a)$ and target $(1,a')$.
- $\psi_{(p,a_\bullet)}$ is the lifting of $(\delta_|)_{h(p,a_\bullet)} : h(p,a_\bullet) \to h(1,h(p,a_\bullet))$ to a morphism $h'(p,F'(a)_\bullet) = (p,a_\bullet) \to (1,h(p,a_\bullet)) = F'(h(p,a_\bullet))$.

For commutativity of (5), we must have $H(1,a) = G(a)$, and for strictness of H, we must have $H(p,a_\bullet) = h''(p,H(1,a)_\bullet)$. These two conditions completely define H on objects.

Now, take a morphism $f : (p,a_\bullet) \to (p',a'_\bullet)$, which is a lifting of a morphism $g : h(p,a_\bullet) \to h(p',a'_\bullet)$ in A. Hf is a morphism $h''(p,Ga_\bullet) \to h''(p',Ga'_\bullet)$: the obvious thing for it to be is the composite

$$h''(p,Ga_\bullet) \xrightarrow{\gamma} Gh''(p,a_\bullet) \xrightarrow{Gg} Gh''(p',a'_\bullet) \xrightarrow{\gamma^{-1}} h''(p',Ga'_\bullet)$$

and we shall show that this is in fact the only possibility. Consider the composite

$$(1,h(p,a_\bullet)) \xrightarrow{\psi^{-1}} (p,a_\bullet) \xrightarrow{f} (p',a_\bullet) \xrightarrow{\psi} (1,h(p',a'_\bullet))$$

in $\mathbf{st}(A)$. Composition in $\mathbf{st}(A)$ is given by composition in A, so this is equal to the lifting of $\delta_| \circ g \circ \delta_|^{-1} = h(1,g)$ to a morphism $(1,h(p,a_\bullet)) \to (1,h(p',a'_\bullet))$, namely $F'g$. So $f = \psi^{-1} \circ F'g \circ \psi$, and $Hf = H\psi^{-1} \circ HF'g \circ H\psi$. By commutativity of (5), $HF' = G$ and $H\psi = \gamma$, so $Hf = \gamma^{-1} \circ Gg \circ \gamma$ as required.

This completely defines H. So we have constructed a unique H which makes (5) commute and which is strict. Hence $(F',\psi) : A \to U'\,\mathbf{st}(A)$ is initial in $(A \downarrow U')$, and so $\mathbf{st} \dashv U'$. □

The P-functor $(F, \phi) : \mathbf{st}(A) \to A$ constructed in Theorem 3.4 is pseudo-inverse to (F', ψ), which we have just shown to be the A-component of the unit of the adjunction $\mathbf{st} \dashv U'$. We can therefore say that **Str-P-Cat** is a weakly coreflective sub-2-category of **Wk-P-Cat**. Note that the counit is *not* pseudo-invertible, so this is not a 2-equivalence.

5 Further Work

Very few interesting theories are strongly regular, so this definition is not of much interest on its own. It can be straightforwardly extended to theories given by symmetric operads, but to deal with the interesting cases of groups, rings, Lie algebras, etc, we must either abandon operads and move to a more expressive formalism (for instance that of Lawvere theories), or extend the notion of an operad until it is sufficiently expressive. I have taken the latter approach: by allowing any function of finite sets, and not just permutations, to act on the sets $P(i)$, we obtain a notion of operad that is equivalent in power to clones or Lawvere theories (as was proved by Tronin in [12]).

However, naïvely extending definition 2.1 to these "distributive" operads doesn't work, as weakening the theory of commutative monoids gives the theory of strictly symmetric weak monoidal categories, rather than that of symmetric weak monoidal categories as desired. I have been working off and on on various other approaches, mainly concerned with constructing $\mathrm{Wk}(P)$ using some universal property, and have obtained some interesting early results.

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