A universal Banach space

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In this talk, **Banach space** will mean real Banach space (although it probably doesn't matter). A **map** $\alpha : X \longrightarrow Y$ of Banach spaces will mean a linear map that is contractive, or distance-decreasing: $\|\alpha(x)\| \leq \|x\|$ for all x. By $X \oplus Y$ I mean the direct sum with norm $\|(x,y)\| = \frac{1}{2}(\|x\| + \|y\|)$.

Let $\mathcal{C}$ be the category whose objects are triples (X, ξ, u) where

- X is a Banach space
- $\xi : X \oplus X \longrightarrow X$
- $u \in X$ with $\|u\| \leq 1$ (or equivalently, $u : \mathbb{R} \longrightarrow X$)

and $\xi(u,u) = u$. Maps in $\mathcal{C}$ are defined in the obvious way, i.e. they're maps of Banach spaces respecting the two pieces of structure.

Example $(\mathbb{R}, \text{mean}, 1) \in \mathcal{C}$.

Question What is the initial object of $\mathcal{C}$? (It does have one.)

— *Long pause to allow audience to think and make suggestions* —

Theorem *The initial object of* $\mathcal{C}$ *is* $(L^1[0,1], \gamma, 1)$, *where* 1 *is the function with constant value* 1 *and* γ *is 'juxtapose and squeeze':*

$$(\gamma(f_1, f_2))(t) = \begin{cases} f_1(2t) & \text{if } t \in [0, \frac{1}{2}) \\ f_2(2t-1) & \text{if } t \in (\frac{1}{2}, 1]. \end{cases}$$

The proof is straightforward. It can be done directly or by using one of several results on initial algebras, as e.g. in the book of Barr and Wells. For instance, one such result both tells us that $\mathcal{C}$ has an initial object and constructs it for us.

The rough idea is as follows. Given $(X, \xi, u) \in \mathcal{C}$, we have to show that there is a unique θ making the following diagram commute:

$$\begin{tikzcd} L^1[0,1] \oplus L^1[0,1] \arrow[r, "\gamma"] \arrow[d, dotted, "\theta \oplus \theta"'] & L^1[0,1] \arrow[d, dotted, "\theta"] & \mathbb{R} \arrow[l, "1"'] \arrow[d, equal] \\ X \oplus X \arrow[r, "\xi"'] & X & \mathbb{R}. \arrow[l, "u"] \end{tikzcd}$$

The right-hand square determines θ on the constant function 1, hence on all constant functions. The left-hand square then determines θ on all functions that are almost everywhere constant on each half of the interval. Carrying on like this, we get θ on all step functions whose breakpoints are dyadic rationals. Continuity then determines θ on all of L^1.

Why is this theorem good? Because the definition of Banach space can be motivated in all sorts of ways, most of which have nothing to do with integration. The theorem shows that once we have the concept of Banach space, then with a tiny amount more input (the definition of $\mathcal{C}$), the concept of integrability just pops out.

The usual definition of L^1 is quite complicated:

- first we say what a null set is, and so what it means for a sequence of functions to converge almost everywhere
- a step function is a function that is piecewise constant
- $\mathcal{L}^{\text{inc}}$ is the set of functions that are almost everywhere limits of an increasing sequence of step functions
- $\mathcal{L}^1$ is the set of functions that can be expressed as the difference of two elements of $\mathcal{L}^{\text{inc}}$
- L^1 is $\mathcal{L}^1$ quotiented out by almost everywhere equality.

Somehow, we've managed to leap over all of this.

How is that possible? Essentially it's because the 'tiny amount more input' that we've added to the concept of Banach space is the concept of mean, and integration on $[0, 1]$ is a continuous version of mean.

So we've got the notion of integrability. Better still, we've got integration itself: $\int_0^1$ is the unique map $(L^1[0,1], \gamma, 1) \longrightarrow (\mathbb{R}, \text{mean}, 1)$. That $\int_0^1$ is a map in $\mathcal{C}$ says:

$$\int_0^1 f(x)\,dx = \frac{1}{2}\left\{\int_0^1 f\left(\frac{x}{2}\right)\,dx + \int_0^1 f\left(\frac{x+1}{2}\right)\,dx\right\} \tag{1}$$

$$\int_0^1 1\,dx = 1. \tag{2}$$

Corollary *$\int_0^1$ is the unique bounded linear functional on $L^1[0,1]$ satisfying (1) and (2).* □

At first sight, we need not just 'bounded' but 'of norm ≤ 1'. However, we're only using the uniqueness half of the definition of initial object, and in the proof of the Theorem, that half only uses boundedness.

Notice that the Corollary makes no reference to categories. It's an entirely elementary characterization of integral. A direct proof is not hard, of course, but the fact itself does not seem very widely known.

(In question time after the talk, Jiří Adámek mentioned that this characterization of integral is used in the textbook *Calculus*, Leonard Gillman and Robert McDowell, W.W. Norton & Co., New York, 1978.)

(I've also had pointed out to me the work of Alex Simpson and Matthias Schröder, which seems to be closely related. See for instance the notes from the talk 'Probabilistic observations and valuations' at Simpson's web site.)