Supplement to 'Rethinking set theory'

Tom Leinster, 6 December 2012

We prove that the ten axioms in 'Rethinking set theory' are equivalent to the condition that sets and functions form a well-pointed topos with natural numbers object and choice. This in turn is well-known to be equivalent to Lawvere's original Elementary Theory of the Category of Sets [2, 3].

Let us recall the definitions. A **topos** is an elementary topos, that is, a cartesian closed category with finite limits and a subobject classifier. A topos $\mathcal{E}$ is **well-pointed** if the terminal object 1 is a generator and if $\mathcal{E}$ is not equivalent to the terminal category. A **natural numbers object** in $\mathcal{E}$ is a triple $(N \in \mathcal{E}, z\colon 1 \longrightarrow N, s\colon N \longrightarrow N)$ that is initial among all triples of this type. A topos $\mathcal{E}$ satisfies the **axiom of choice**, or **has choice**, if every epimorphism in $\mathcal{E}$ splits.

Axiom 1 states that sets, functions, composition of functions and identities determine a category, so what we have to prove is the following.

Theorem *Let $\mathcal{E}$ be a category. Then $\mathcal{E}$ is a well-pointed topos with natural numbers object and choice if and only if $\mathcal{E}$ satisfies Axioms 2–10, where in those axioms, 'set' is interpreted as 'object of $\mathcal{E}$' and 'function' as 'map in $\mathcal{E}$'.*

WP topos with NNO and choice implies the ten axioms

Let $\mathcal{E}$ be a well-pointed topos with natural numbers object and choice. We prove Axioms 2–10.

2 (terminal set) This is immediate.

3 (empty set) Being a topos, $\mathcal{E}$ has an initial object 0. Claim: 0 has no element. Proof: suppose for a contradiction that 0 has an element, that is, there exists a map $1 \longrightarrow 0$. We already know that there exists a map $0 \longrightarrow 1$. Since 0 is initial, the only endomorphism of 0 is the identity; since 1 is terminal, the only endomorphism of 1 is the identity. Hence these two maps are mutually inverse, giving $0 \cong 1$. For all $X \in \mathcal{E}$, we have $X \cong 1 \times X \cong 0 \times X$; but $\mathcal{E}$ is cartesian closed, so $0 \times X \cong 0$. Hence every object of $\mathcal{E}$ is initial, from which it follows that $\mathcal{E}$ is equivalent to the terminal category. This contradicts well-pointedness, proving the claim.

4 (function determined by effect on elements) This states exactly that 1 is a generator, which is part of well-pointedness.

5 (products) This is immediate.

6 (function sets) The definition of function set in 'Rethinking set theory' contains the condition that $\varepsilon(\bar{q}(t), x) = q(t, x)$ for all $t \in I$ and $x \in X$. By well-pointedness, this is equivalent to the condition that $\varepsilon \circ (\bar{q} \times 1_X) = q$, as in the usual definition of exponential. Since $\mathcal{E}$ is cartesian closed, it therefore has all function sets.

7 (inverse images) Similarly, the definition of inverse image in 'Rethinking set theory' contains the conditions that $f(j(a)) = y$ for all $a \in A$ and $f(q(t)) = y$ for all $t \in I$. By well-pointedness, these are, respectively, equivalent to the conditions that the diagrams

$$\begin{array}{ccc} A & \longrightarrow & \mathbf{1} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle y} \\ X & \xrightarrow[f]{} & Y \end{array} \qquad \begin{array}{ccc} I & \longrightarrow & \mathbf{1} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

commute. So, in $\mathcal{E}$, an inverse image of $y \in Y$ under $f\colon X \longrightarrow Y$ is precisely a pullback of y along f. Since $\mathcal{E}$ has all pullbacks, it therefore satisfies Axiom 7.

8 (characteristic functions) To prove that $\mathcal{E}$ satisfies Axiom 8, it is enough to prove that every injection is monic. Let $j\colon A \longrightarrow X$ be an injection and let $f, g\colon Z \longrightarrow A$ with $j \circ f = j \circ g$. Then $j \circ f \circ z = j \circ g \circ z$ for all $z \in Z$. Since j is injective, $f \circ z = g \circ z$ for all $z \in Z$. By well-pointedness, this implies $f = g$, as required.

9 (natural numbers) The definition of natural number system before Axiom 9 contains the condition that $x(s(n)) = r(x(n))$ for all $n \in N$. By well-pointedness, this is equivalent to the condition that $x \circ s = r \circ x$. So a natural number system in $\mathcal{E}$ is exactly a natural numbers object.

10 (choice) It is enough to prove that every surjection in $\mathcal{E}$ is epic. Let $s\colon X \longrightarrow Y$ be a surjection, and let $f, g\colon Y \longrightarrow Z$ with $f \circ s = g \circ s$. Let $y \in Y$. Since s is a surjection, there exists $x \in X$ such that $s(x) = y$, and so

$$f(y) = f(s(x)) = g(s(x)) = g(y).$$

By well-pointedness, $f = g$, as required.

The ten axioms imply WP topos with NNO and choice

Now let $\mathcal{E}$ be a category satisfying Axioms 2–10. We prove that $\mathcal{E}$ is a well-pointed topos with natural numbers object and choice.

Choose a terminal object $\mathbf{1}$ in $\mathcal{E}$, a binary product $X \times Y$ for each pair X, Y of objects, and a subset classifier $\mathbf{2}$ (with distinguished element t).

Easy observations By Axiom 4 (function determined by effect on elements), a function set is exactly an exponential in $\mathcal{E}$, an inverse image of $y \in Y$ under $f\colon X \longrightarrow Y$ is exactly a pullback of $y\colon \mathbf{1} \longrightarrow Y$ along f, and a natural number system is exactly a natural numbers object.

Finite limits To prove that $\mathcal{E}$ has finite limits, it is enough to prove that $\mathcal{E}$ has a terminal object, binary products and equalizers. The first two are Axioms 2 and 5, respectively.

For equalizers, we use the argument indicated briefly after Lemma A1.6.1 of [1]. Let $f, g\colon X \longrightarrow Y$. The diagonal map $\Delta = (1_Y, 1_Y)\colon Y \longrightarrow Y \times Y$ has a left inverse; it is therefore monic, and in particular injective. By Axiom 8, there is a unique map $\delta\colon Y \times Y \longrightarrow \mathbf{2}$ such that Δ is an inverse image of $t \in \mathbf{2}$ under δ. By Axiom 7, we may form the inverse image $E \xrightarrow{m} X$ of $t \in \mathbf{2}$ under $\delta \circ (f, g)$:

$$\begin{array}{ccccc} E & & \longrightarrow & & \mathbf{1} \\ {\scriptstyle m}\downarrow & & & & \downarrow{\scriptstyle t} \\ X & \xrightarrow[(f,g)]{} & Y \times Y & \xrightarrow[\delta]{} & \mathbf{2}. \end{array}$$

One of the 'easy observations' above is that the square in the definition of inverse image is a pullback. Hence there is a unique map $n\colon E \longrightarrow Y$ making the following diagram commute:

$$\begin{array}{ccccc} E & \xrightarrow[n]{} & Y & \longrightarrow & \mathbf{1} \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle t} \\ X & \xrightarrow[(f,g)]{} & Y \times Y & \xrightarrow[\delta]{} & \mathbf{2}. \end{array}$$

The outer rectangle and the right-hand square are pullbacks, so by a standard lemma, the left-hand square is a pullback. By another standard lemma, this is equivalent to the statement that $m\colon E \longrightarrow X$ is an equalizer of f and g.

Cartesian closed We already know that $\mathcal{E}$ has finite products. By one of the easy observations, an exponential in $\mathcal{E}$ is exactly a function set, and Axiom 6 states that $\mathcal{E}$ does have function sets.

Subobject classifier Any monic is certainly an injection, so Axiom 8 implies that $\mathcal{E}$ has a subobject classifier.

We have now shown that $\mathcal{E}$ is a topos.

Well-pointed Axiom 4 states that $\mathbf{1}$ is a generator. Axiom 3 states that there exists an object $\emptyset$ of $\mathcal{E}$ with no elements, which is to say that there is no map $\mathbf{1} \longrightarrow \emptyset$. Hence $\mathcal{E}$ is not equivalent to the terminal category.

Natural numbers object One of the easy observations is that a natural numbers object is exactly a natural number system, and Axiom 9 states that one exists.

Choice It is enough to show that every epi in $\mathcal{E}$ is surjective. We already know that $\mathcal{E}$ is a well-pointed topos, and Proposition VI.10.1(iii) of [4] states that in a well-pointed topos, the epis are precisely the surjections.

References

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