The Convex Magnitude Conjecture

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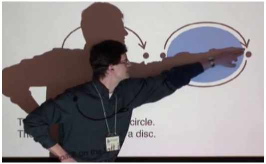
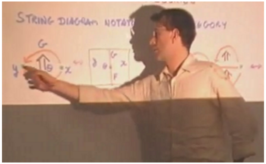


Mark Meckes

Simon Willerton



These slides: available on my web page

Executive summary

Magnitude is a real-valued invariant of metric spaces.

It seems not to have been previously investigated.

Conjecturally, it captures a great deal of geometric information.

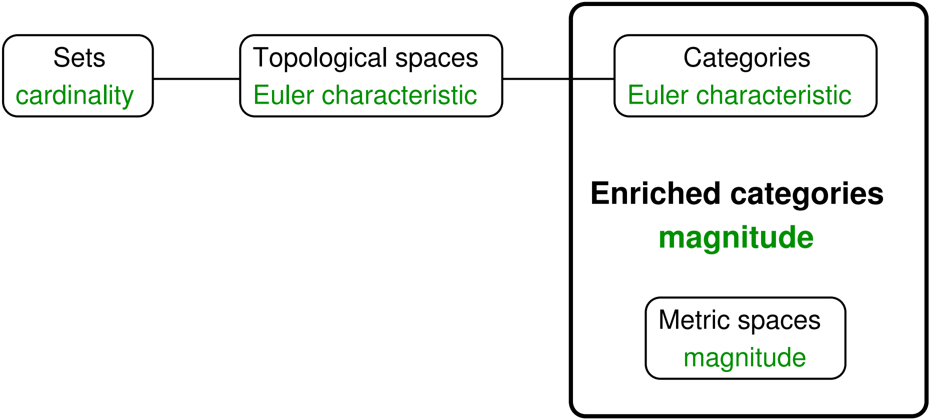
It arose from a general study of ‘size’ in mathematics.

Plan

1. Where does magnitude come from?
2. The magnitude of a finite space
3. The magnitude of a compact space
4. The convex magnitude conjecture

1. Where does magnitude come from?

Concepts of counting and size



A category has:	A metric space has:
objects $a, b, \ldots$	points $a, b, \ldots$
sets $\mathrm{Hom}(a, b)$	numbers $d(a, b)$
composition operations	triangle inequalities
$\mathrm{Hom}(a, b) \times \mathrm{Hom}(b, c) \to \mathrm{Hom}(a, c)$	$d(a, b) + d(b, c) \geq d(a, c)$

2. The magnitude of a finite space

The definition

Let $A = \{a_1, \ldots, a_n\}$ be a finite metric space.

Write Z or Z_A for the $n \times n$ matrix given by $Z_{ij} = e^{-d(a_i, a_j)}$.

A weighting on A is a column vector $w \in \mathbb{R}^n$ such that

$$Zw = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Suppose there is at least one weighting w on A. The magnitude of A is

$$|A| = \sum_j w_j \in \mathbb{R}.$$

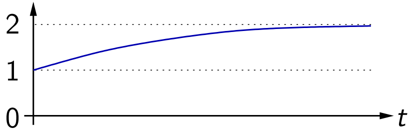
This is independent of the weighting chosen.

E.g.: Usually Z is invertible. Then A has magnitude

$$|A| = \sum_{i,j} (Z^{-1})_{ij}.$$

Examples

- $|\emptyset| = 0$ and $|\bullet| = 1$.
- $\left|\bullet \overset{\leftarrow\ t\ \rightarrow}{} \bullet\right| = 1 + \tanh(t/2)$:



- If $d(a_i, a_j) = \infty$ for all $i \neq j$ then $|\{a_1, \ldots, a_n\}| = n$.

Digression: magnitude as maximum diversity

Take a probability distribution p on a finite metric space $A = \{a_1, \ldots, a_n\}$.

Its entropy (of order 1) is

$$H_A(p) = -\sum_{i=1}^{n} p_i \log(Z_A p)_i$$

Ecological interpretation:

- points of A represent species
- distances represent differences between species
- probabilities represent frequencies
- exponential of entropy measures biological diversity.

Given a list of species, which distribution maximizes diversity?

Theorem: Under hypotheses,

- the maximizing distribution p is the weighting w, normalized
- the maximum diversity is the magnitude: $\max_p e^{H_A(p)} = |A|$.

Magnitude functions

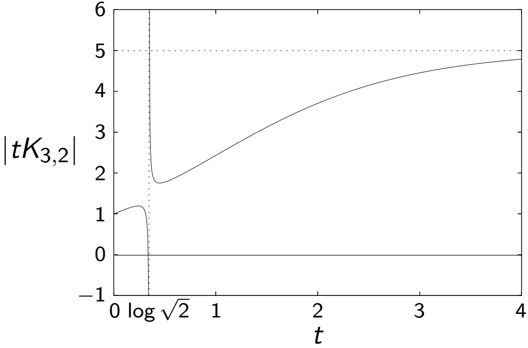
Magnitude changes unpredictably as a space is rescaled.

For a space A and $t > 0$, let tA be A scaled up by a factor of t.

The magnitude function of A is the function $t \mapsto |tA|$ on $(0, \infty)$.
(It may have a finite number of singularities.)

E.g.: The magnitude function of $\bullet \overset{\leftarrow 1 \rightarrow}{} \bullet$ is $t \mapsto 1 + \tanh(t/2)$.

E.g.: Magnitude functions can be wild for small t, but behave well for large t:



Positive definite spaces

A (possibly infinite) metric space A is positive definite if
for each finite $B \subseteq A$, the matrix Z_B is positive definite.

Examples of positive definite spaces:

- Euclidean space $\mathbb{R}^N$
- the sphere S^N with the geodesic metric.

For finite positive definite spaces, magnitude behaves intuitively, e.g.:

- the magnitude is always defined
- $|A| \geq 1$ for nonempty A
- if $B \subseteq A$ then $|B| \leq |A|$.

3. The magnitude of a compact space

Extending the definition beyond finite spaces

How could we define the magnitude of an *infinite* space?

- Idea: approximate it by finite spaces.
- Alternative idea: replace sums by integrals.
  (Then weightings become measures or distributions.)

Meckes has shown: for compact, positive definite spaces, both ideas work.

Moreover, they give the same answer.

The definition

Theorem (Meckes)

Let A be a positive definite compact metric space (e.g. $A \subseteq \mathbb{R}^N$).

Let $B_1 \subseteq B_2 \subseteq \cdots \subseteq A$ be finite subspaces with $\overline{\bigcup B_i} = A$.

Then $\lim_{i \to \infty} |B_i|$ exists and depends only on A (not on the sequence (B_i)).

The magnitude $|A|$ of A is defined as $\lim_{i \to \infty} |B_i|$.

Digression: Meckes has also shown:

$$|A| = \sup\left\{ \frac{\mu(A)^2}{\int_A \int_A e^{-d(x,y)}\, d\mu(x)\, d\mu(y)} \,\middle|\, \text{signed Borel measures } \mu \text{ on } A \right\}.$$

Examples

- $|[0,t]| = 1 + \frac{1}{2}t$.
- For $A = [0,p] \times [0,q]$ *with the ℓ^1 (taxicab) metric*,

$$\begin{aligned} |A| &= (1 + \tfrac{1}{2}p)(1 + \tfrac{1}{2}q) \\ &= 1 + \tfrac{1}{4}\,\mathrm{perimeter}(A) + \tfrac{1}{4}\,\mathrm{area}(A). \end{aligned}$$

So the magnitude function of A is

$$t \mapsto \boxed{|tA| = 1 + \tfrac{1}{4}\,\mathrm{perimeter}(A)\cdot t + \tfrac{1}{4}\,\mathrm{area}(A)\cdot t^2}$$

(a polynomial of degree 2).

Magnitude dimension

Let A be a compact positive definite metric space (e.g. $A \subseteq \mathbb{R}^N$).

The magnitude dimension of A is

$$\dim_{\text{mag}} A = \inf\left\{ r \geq 0 \ : \ \frac{|tA|}{t^r} \text{ is bounded for } t \gg 0 \right\}.$$

Examples:

- Finite sets have magnitude dimension 0.
- Line segments have magnitude dimension 1.
- The Cantor set has magnitude dimension $\log_3 2$.
- The N-sphere with geodesic metric has magnitude dimension N.

Theorem (with Meckes): For compact $A \subseteq \mathbb{R}^N$,

$$\dim_{\text{Hausdorff}}(A) \leq \dim_{\text{mag}}(A) \leq N.$$

(But magnitude dimension and Hausdorff dimension sometimes disagree.)

4. The Convex Magnitude Conjecture

The conjecture (in 2 dimensions)

Recall that for rectangles A in $\mathbb{R}^2$ with the ℓ^1 metric,

$$|A| = 1 + \frac{1}{4}\,\text{perimeter}(A) + \frac{1}{4}\,\text{area}(A).$$

Conjecture (with Willerton)

For compact convex $A \subseteq \mathbb{R}^2$ (with the Euclidean metric),

$$|A| = \chi(A) + \frac{1}{4}\,\text{perimeter}(A) + \frac{1}{2\pi}\,\text{area}(A).$$

Equivalently: for compact convex $A \subseteq \mathbb{R}^2$ and $t > 0$,

$$|tA| = \chi(A) + \frac{1}{4}\,\text{perimeter}(A) \cdot t + \frac{1}{2\pi}\,\text{area}(A) \cdot t^2.$$

In particular: the magnitude function of a convex planar set is a polynomial, from which we can recover its Euler characteristic, perimeter and area.

The conjecture (in arbitrary dimension)

Let ω_i denote the volume of the i-dimensional unit ball.

Let V_i denote i-dimensional intrinsic volume.

Conjecture (with Willerton)

For compact convex $A \subseteq \mathbb{R}^N$ (with the Euclidean metric),

$$|A| = \sum_{i=0}^{N} \frac{1}{i!\omega_i} V_i(A).$$

Equivalently: for compact convex $A \subseteq \mathbb{R}^N$ and $t > 0$,

$$|tA| = \sum_{i=0}^{N} \frac{1}{i!\omega_i} V_i(A) \cdot t^i.$$

In particular: the magnitude function of a convex set in $\mathbb{R}^N$ is a polynomial, from which we can recover all of its intrinsic volumes.

A gap

There is not a single example for which the conjecture is known to be true, apart from line segments.

That is: apart from line segments, there is no compact convex set whose magnitude is known.

E.g. the magnitude of the unit disk is unknown.

(Nor is there is a single example for which it is known to be false.)

Evidence for the conjecture, I

Recall: the conjecture states that for compact convex $A \subseteq \mathbb{R}^2$,

$$|tA| = \chi(A) + \frac{1}{4}\,\mathrm{perimeter}(A) \cdot t + \frac{1}{2\pi}\,\mathrm{area}(A) \cdot t^2.$$

- The conjecture holds when A is a line segment.
- Both sides are monotone in A.
- Both sides have the same growth as $t \to \infty$.
- $|tA| \geq \chi(A)$ and $|tA| \geq \frac{1}{2\pi}\mathrm{area}(A) \cdot t^2$.
- Numerical (Willerton): computer calculations for disk, square, etc.
- Heuristic argument (Willerton) for why the top coefficient should be $\frac{1}{2\pi}$.

Evidence, II: arguments by analogy

Recall: the conjecture states that for compact convex $A \subseteq \mathbb{R}^2$,

$$|tA| = \chi(A) + \frac{1}{4}\,\mathrm{perimeter}(A) \cdot t + \frac{1}{2\pi}\,\mathrm{area}(A) \cdot t^2.$$

Theorem: For compact convex $A \subseteq \mathbb{R}^2$ with the ℓ^1 metric,

$$|tA| = \chi(A) + \frac{1}{2}\Big[\mathrm{length}(\pi_1 A) + \mathrm{length}(\pi_2 A)\Big] \cdot t + \frac{1}{4}\,\mathrm{area}(A) \cdot t^2.$$

Theorem (Willerton): For homogeneous Riemannian 2-manifolds A,

$$|tA| = O(t^{-2}) + \chi(A) + \frac{1}{2\pi}\,\mathrm{area}(A) \cdot t^2$$

as $t \to \infty$.

Evidence, III: PDE approach

Recall: the conjecture states that for compact convex $A \subseteq \mathbb{R}^2$,

$$|tA| = \chi(A) + \frac{1}{4}\,\text{perimeter}(A) \cdot t + \frac{1}{2\pi}\,\text{area}(A) \cdot t^2.$$

Juan Antonio Barcelo and Tony Carbery (following Meckes) have a PDE approach, best adapted to magnitude of convex sets in odd dimensions.

For compact convex $A \subseteq \mathbb{R}^3$, they define a quantity $[A]$ in terms of the solution to a certain PDE.

- A *nonrigorous* argument suggests that $[A] = |A|$.
- A *rigorous* argument shows that when A is the ball, $[A]$ is exactly what the convex magnitude conjecture predicts.

Similar arguments probably work for the N-ball whenever N is odd.

How could we prove the conjecture?

Recall: the conjecture states that for compact convex $A \subseteq \mathbb{R}^2$,

$$|tA| = \chi(A) + \frac{1}{4}\,\text{perimeter}(A) \cdot t + \frac{1}{2\pi}\,\text{area}(A) \cdot t^2.$$

If the conjecture holds then magnitude is a convex valuation:

$$|A \cup B| = |A| + |B| - |A \cap B| \qquad \text{for all convex } A, B, A \cup B.$$

Almost-conversely, by Hadwiger's theorem, the conjecture holds as long as:

- magnitude is a convex valuation, and
- the conjecture holds for at least one 2-dimensional set.

But currently, no one knows how to do either step.

What's the point?

- It's hard.
- If the conjecture is true, it exhibits the intrinsic volumes of convex sets as intrinsic metric invariants — independent of their embedding in $\mathbb{R}^N$. (Compare the Weyl tube formula.)
- It suggests an approach to geometric measure that works for spaces that are irregular, or not embedded in any standard space such as $\mathbb{R}^N$.
- The categorical origins suggest that magnitude is a canonical quantity of mathematics: a cousin of cardinality and Euler characteristic, and therefore worth studying.

The conjecture, in one slide

For $B = \{b_1, \ldots, b_n\} \subseteq \mathbb{R}^N$, define $Z_{ij} = e^{-d(b_i, b_j)}$ and $|B| = \sum_{i,j} (Z^{-1})_{ij}$.

For compact $A \subseteq \mathbb{R}^N$, choose finite sets $B_1 \subseteq B_2 \subseteq \cdots \subseteq A$ with $\overline{\bigcup B_i} = A$, then define $|A| = \lim_{i \to \infty} |B_i|$.

Conjecture (2-dimensional case): For compact convex $A \subseteq \mathbb{R}^2$,

$$|A| = \chi(A) + \frac{1}{4}\,\text{perimeter}(A) + \frac{1}{2\pi}\,\text{area}(A).$$

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