# New Perspectives on Euler Characteristic

Tom Leinster

University of Glasgow / EPSRC

# Contents



Credits: Schanuel, Rota, Baez–Dolan, . . .

Euler characteristic
is
cardinality

The Euler characteristic of an object
is
the most basic dimensionless quantity
associated with it.

# Some cardinality-like invariants

Cardinality

Euler characteristic

Measure

Probability

Truth value?

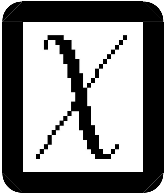
Dimension?

Write them all as $|\ \ |$.
They all satisfy the inclusion–exclusion formulas:

$$
\begin{aligned}
|A \cup B| &= |A| + |B| - |A \cap B|, \\
|\emptyset| &= 0.
\end{aligned}
$$

# Polyconvex sets

Fix $n \geq 0$. A subset of $\mathbb{R}^n$ is polyconvex if it is a finite union of compact convex subsets.
A 'measure' is a function



$$| \quad | : \{\text{polyconvex subsets of } \mathbb{R}^n\} \to \mathbb{R}$$

satisfying

- inclusion–exclusion: $|A \cup B| = |A| + |B| - |A \cap B|$, $|\emptyset| = 0$
- invariance under rigid motions
- continuity.

## Theorem (Hadwiger)

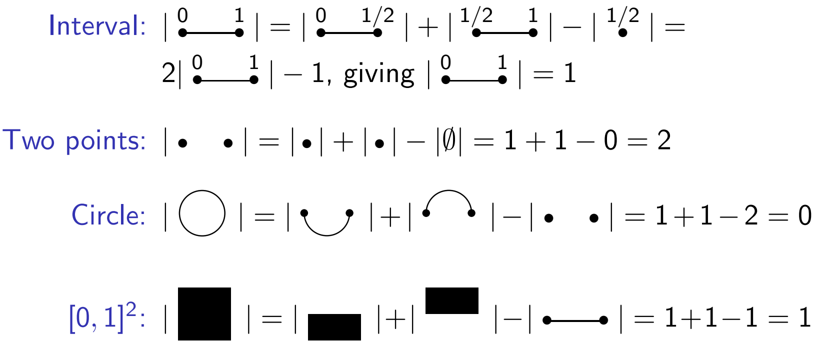
*The only measure that is dimensionless — i.e.*

$$|\lambda A| = |A| \text{ for all } \lambda > 0 \text{ and polyconvex } A$$

*— is Euler characteristic (and its scalar multiples).*

# Some easy spaces

Using only inclusion–exclusion and $|\bullet| = 1$, can calculate the Euler characteristics of many spaces:



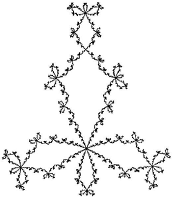

etc etc.

# Some harder spaces

Every complex rational function $f$ has a Julia set $J(f) \subseteq \mathbb{C} \cup \{\infty\}$, its 'zone of instability'.

## Example

$$J(z^3 + \frac{12}{25}z + \frac{116i}{125}) =$$



This (probably) has Euler characteristic $1/2$, by similar calculations to those above.

## Conjecture

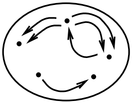
*If $f$ is a rational function then $|J(f)|$ is a well-defined rational number, $\geq 0$.*

(Image: Milnor)

# Contents

# The Euler characteristic of a category

Let $\mathcal{C}$ be a finite category with objects $c_1, \ldots, c_n$.
Write $Z$ for the $n \times n$ matrix $(|\mathrm{Hom}(c_i, c_j)|)_{i,j}$.



A weighting on $\mathcal{C}$ is an $n$-tuple $k^\bullet = (k^1, \ldots, k^n) \in \mathbb{Q}^n$ such that

$$Z \begin{pmatrix} k^1 \\ \vdots \\ k^n \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

A coweighting on $\mathcal{C}$ is a weighting on $\mathcal{C}^{\mathrm{op}}$.

### Lemma

*If $k^\bullet$ is a weighting and $k_\bullet$ a coweighting then $\sum_i k^i = \sum_i k_i$.*

### Definition

Suppose that $\mathcal{C}$ admits at least one weighting and at least one coweighting. Its Euler characteristic $|\mathcal{C}|$ is $\sum k^i = \sum k_i \in \mathbb{Q}$, for any weighting $k^\bullet$ and coweighting $k_\bullet$.

# The Euler characteristic of a category (continued)

If $Z = (|\mathrm{Hom}(c_i, c_j)|)_{i,j}$ is invertible over $\mathbb{Q}$, we have:

$$\begin{array}{r|c}
 & \text{SUM} \\
Z^{-1} = \begin{pmatrix} \bullet & \cdots & \bullet \\ \vdots & & \vdots \\ \bullet & \cdots & \bullet \end{pmatrix} & \begin{array}{c} k^1 \\ \vdots \\ k^n \end{array} \\ \hline
\text{SUM} \qquad k_1 \qquad \cdots \qquad k_n & |\mathcal{C}|
\end{array}$$

## Example

If $\mathcal{C} = (c_1 \rightrightarrows c_2)$ then $Z = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $Z^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$, and $|\mathcal{C}| = 1 + -2 + 0 + 1 = 0$ $(= |S^1|)$.

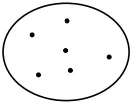
## Properties of $|\quad|$

Respects products, sums ($\amalg$) and fibrations.

If $B\mathcal{C}$ is the classifying space of $\mathcal{C}$ then $|\mathcal{C}| = |B\mathcal{C}|$.

# The Euler characteristic of a set

A set is a category in which the only arrows are identities.



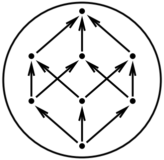
For finite sets $S$,

$$|S| = |S|$$

— the Euler characteristic of the category $S$ is the cardinality of the set $S$.

# The Euler characteristic of a poset

A poset is a category in which each hom-set has at most one element.



For a finite poset $P$,

$$|P| = \sum_{n \geq 0} (-1)^n |\{\text{chains } c_0 < c_1 < \cdots < c_n \text{ in } P\}| \in \mathbb{Z}.$$

This is closely related to Rota's Möbius inversion.

# Digression: manifolds and orbifolds

Manifolds: there is a commutative diagram

$$\begin{tikzcd}
\{\text{finitely-triangulated manifolds}\} \arrow[r, "\text{poset of simplices}"] \arrow[dr, "|\quad|"'] & \{\text{posets}\} \arrow[d, "|\quad|"] \\
& \mathbb{Z}.
\end{tikzcd}$$

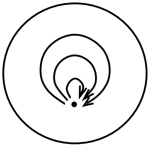
To extend to orbifolds: replace posets by categories and $\mathbb{Z}$ by $\mathbb{Q}$:

$$\begin{tikzcd}
\{\text{finitely-triangulated orbifolds}\} \arrow[r, "\text{category of simplices}"] \arrow[dr, "|\quad|"'] & \{\text{categories}\} \arrow[d, "|\quad|"] \\
& \mathbb{Q}
\end{tikzcd}$$

(joint result with Ieke Moerdijk).

# The Euler characteristic of a monoid

A monoid (= semigroup with identity) is a category in which there is only one object.



For finite monoids $M$,

$$|M| = 1/\mathrm{order}(M).$$

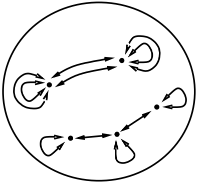
## 'Explanation' for groups $M = G$

If $G$ acts freely on a set $S$ then $|S/G| = |S|/\mathrm{order}(G)$.
But $G$ acts freely on the contractible space $EG$, suggesting $|BG| = 1/\mathrm{order}(G)$.

# The Euler characteristic of a groupoid

## (Baez–Dolan)

A groupoid is a category in which every arrow is an isomorphism.



Let $\mathcal{G}$ be a finite groupoid. Choose one object $a_i$ from each connected-component. Then

$$|\mathcal{G}| = \sum_i 1/\mathrm{order}(\mathrm{Aut}(a_i)).$$

### Example

If $\mathcal{G} = \{\text{finite sets } + \text{bijections}\}$ then

$$|\mathcal{G}| = \sum_{n \geq 0} 1/\mathrm{order}(S_n) = e = 2.718\ldots.$$

**Cardinality**

**is**

**Euler characteristic**

and we know what the cardinality / Euler characteristic of a category is.

References and details:

Tom Leinster, 'The Euler characteristic of a category'

(available on web).

These slides: on my web page.