Codensity and the ultrafilter monad

Tom Leinster

Edinburgh

These slides: available on my web page

n-Category Café posts: 1, 2, 3, 4

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The moral of this talk

Whenever you meet a functor,
ask
"What is its codensity monad?"

Plan

1. What codensity monads are

With codensity monads as part of our toolkit:

The notion of ...	automatically gives rise to	the notion of ...
2. finiteness of a set		ultrafilter
3. finite-dimensionality of a vector space		double dualization
4. finiteness of a family		ultraproduct

1. What codensity monads are

(Isbell, Ulmer; Appelgate & Tierney, A. Kock)

Loosely

The codensity monad of a functor $G \colon \mathscr{B} \longrightarrow \mathscr{A}$
is what the composite of G with its left adjoint would be
if G had a left adjoint

Grammar: given a functor $G \colon \mathscr{B} \longrightarrow \mathscr{A}$, the codensity monad T^G of G is a certain monad on $\mathscr{A}$.

It is defined as long as $\mathscr{A}$ has enough limits.

The definition will be given later.

Characterization of the codensity monad

Motivation: Let $G\colon \mathscr{B} \longrightarrow \mathscr{A}$ be a functor that *does* have a left adjoint, F. We have categories and functors

$$\begin{tikzcd}
\mathscr{B} \arrow[rr, "\text{comparison}"] \arrow[dr, "G"'] & & \mathscr{A}^{G\circ F} \arrow[dl, "\text{forget}"] \\
& \mathscr{A} &
\end{tikzcd}$$

and this is initial among all maps in $\mathbf{CAT}/\mathscr{A}$ from G to a monadic functor.

Theorem (Dubuc) Let $G\colon \mathscr{B} \longrightarrow \mathscr{A}$ be a functor whose codensity monad T^G is defined. Then

$$\begin{tikzcd}
\mathscr{B} \arrow[rr, "\text{comparison}"] \arrow[dr, "G"'] & & \mathscr{A}^{T^G} \arrow[dl, "\text{forget}"] \\
& \mathscr{A} &
\end{tikzcd}$$

is initial among all maps in $\mathbf{CAT}/\mathscr{A}$ from G to a monadic functor.

Corollary Let G be a functor with a left adjoint, F. Then $T^G = G \circ F$.

Three definitions of the codensity monad

Let $G\colon \mathscr{B} \longrightarrow \mathscr{A}$ be a functor. Three equivalent definitions:

- The codensity monad of G is the right Kan extension of G along itself:

$$\begin{tikzcd} \mathscr{B} \arrow[r, "G"] \arrow[dr, "G"'] & \mathscr{A} \arrow[d, "T^G"] \arrow[dl, phantom, "\swarrow"] \\ & \mathscr{A} \end{tikzcd}$$

(and is defined iff the Kan extension exists).

- $T^G(A) = \int_B \left[\mathscr{A}(A, G(B)), G(B)\right] = \varprojlim_{B\in\mathscr{B},\ f\colon A\longrightarrow G(B)} G(B).$

- Recall: if $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ with $\mathscr{A}$ small and $\mathscr{B}$ cocomplete, get adjunction

$$\begin{tikzcd} \mathscr{B} \arrow[r, shift left=2, "\mathrm{Hom}(F,-)"] \arrow[r, phantom, "\top"] & {[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]} \arrow[l, shift left=2, "-\otimes F"] \end{tikzcd}, \qquad \text{e.g.} \qquad \begin{tikzcd} \mathbf{Top} \arrow[r, shift left=2, "\text{singular}"] \arrow[r, phantom, "\top"] & {[\Delta^{\mathrm{op}}, \mathbf{Set}]} \arrow[l, shift left=2, "|-|"] \end{tikzcd}.$$

Three definitions of the codensity monad

Let $G \colon \mathscr{B} \longrightarrow \mathscr{A}$ be a functor. Three equivalent definitions:

- The codensity monad of G is the right Kan extension of G along itself:

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(and is defined iff the Kan extension exists).

- $T^G(A) = \int_B \left[\mathscr{A}(A, G(B)), G(B)\right] = \varprojlim_{B \in \mathscr{B},\ f \colon A \longrightarrow G(B)} G(B).$

- If $\mathscr{B}$ is small and $\mathscr{A}$ is complete, get adjunction

$$\begin{tikzcd} \mathscr{A} \arrow[r, shift left, "\mathrm{Hom}(-,G)"] \arrow[r, phantom, "\perp"] & {[\mathscr{B}, \mathbf{Set}]^{\mathrm{op}}}, \arrow[l, shift left, "\mathrm{Hom}(-,G)"] \end{tikzcd}$$

and T^G is the induced monad on $\mathscr{A}$.

Two short but nontrivial examples

1. Let $\mathscr{A}$ be a category and $X \in \mathscr{A}$. The codensity monad of $\mathbf{1} \xrightarrow{X} \mathscr{A}$ is the endomorphism monad $\mathbf{End}(X)$ of X, given by

$$\big(\mathbf{End}(X)\big)(A) = \big[\mathscr{A}(A, X), X\big].$$

By Dubuc, for any monad S on $\mathscr{A}$, an S-algebra structure on X amounts to a map of monads $S \longrightarrow \mathbf{End}(X)$.

2. The codensity monad of $G \colon \mathbf{Field} \hookrightarrow \mathbf{CRing}$ is given by

$$T^G(A) = \prod_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathrm{Frac}(A/\mathfrak{p})$$

$(A \in \mathbf{CRing})$. For example,

$$\begin{aligned} T^G(\mathbb{Z}) &= \mathbb{Q} \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/5\mathbb{Z}) \times \cdots, \\ T^G(\mathbb{Z}/n\mathbb{Z}) &= \mathbb{Z}/\mathrm{rad}(n)\mathbb{Z} \end{aligned}$$

where $\mathrm{rad}(n)$ is the product of the distinct prime factors of n.

2. Ultrafilters

What ultrafilters are

Lemma (Galvin and Horn) Let X be a set and $\mathscr{U} \subseteq \mathscr{P}(X)$.
The following are equivalent:

- $\mathscr{U}$ is an ultrafilter
- whenever $X = X_1 \amalg \cdots \amalg X_n$, there is a unique i such that $X_i \in \mathscr{U}$.

There is a monad U on **Set**, the ultrafilter monad, with

$$U(X) = \{\text{ultrafilters on } X\}$$

($X \in \mathbf{Set}$).

Ultrafilters as measures

Let $X \in \mathbf{Set}$ and $\mathscr{U} \in U(X)$. Think of elements of $\mathscr{U}$ as 'sets of measure 1'.

Lemma (everyone) An ultrafilter on a set X is essentially the same thing as a finitely additive probability measure on X taking values in $\{0, 1\}$.

If an ultrafilter is a kind of measure, what is integration?

Given a finite set B, define

$$\int_X - \, d\mathscr{U} : \mathbf{Set}(X, B) \longrightarrow B$$

as follows:

> for $f \in \mathbf{Set}(X, B)$, let $\int_X f \, d\mathscr{U}$ be the unique element of B whose fibre under f belongs to $\mathscr{U}$.

Justification of terminology: This 'integration' is uniquely characterized by:

- the integral of a constant function is that constant; and
- changing a function on a set of measure 0 doesn't change its integral.

Measures correspond to integration operators

Let X be a set. Given $\mathscr{U} \in U(X)$, we obtain a family of maps

$$\left(\mathbf{Set}(X, B) \xrightarrow{\int_X - \, d\mathscr{U}} B \right)_{B \in \mathbf{FinSet}}$$

natural in B. That is: given $\mathscr{U} \in U(X)$, we obtain an element

$$\int_X - \, d\mathscr{U} \in T^G(X)$$

where T^G is the codensity monad of $G \colon \mathbf{FinSet} \hookrightarrow \mathbf{Set}$. So, we have

$$\begin{array}{ccc} U(X) & \longrightarrow & T^G(X) \\ \mathscr{U} & \longmapsto & \int_X - \, d\mathscr{U}. \end{array}$$

In fact, this defines an isomorphism of monads $U \longrightarrow T^G$. Hence:

Theorem (i) (Kennison and Gildenhuys) The codensity monad of $\mathbf{FinSet} \hookrightarrow \mathbf{Set}$ is the ultrafilter monad.
(ii) (Manes) The algebras for this monad are the compact Hausdorff spaces.

Moral of this section

$$\boxed{\text{The notion of finiteness of a set automatically gives rise to the notions of ultrafilter and compact Hausdorff space}}$$

3. Double dualization

The linear analogue of the ultrafilter theorem

Theorem (i) The codensity monad of **FDVect** $\hookrightarrow$ **Vect** is double dualization.
(ii) The algebras for this monad are the linearly compact vector spaces (certain topological vector spaces).

Table of analogues:

sets	vector spaces
finite sets	finite-dimensional vector spaces
ultrafilters	elements of the double dual
compact Hausdorff spaces	linearly compact vector spaces.

Moral of this section

The notion of finite-dimensionality of a vector space
automatically gives rise to the notions of
double dualization
and
linearly compact vector space

4. Ultraproducts

What ultraproducts are

Let $S = (S_x)_{x \in X}$ be a family of sets.

An element of the product $\prod S = \prod_{x \in X} S_x$ is a family of elements $(s_x)_{x \in X}$.

Now let $\mathscr{U}$ be an ultrafilter on X. We'll define the ultraproduct $\prod_{\mathscr{U}} S$.

Informally: An element of $\prod_{\mathscr{U}} S$ is a family of elements (s_x) defined almost everywhere and taken up to almost everywhere equality.

Formally: An element of $\prod_{\mathscr{U}} S$ is an equivalence class of families $(s_x)_{x \in Y}$ with $Y \in \mathscr{U}$, where

$$(s_x)_{x \in Y} \sim (t_x)_{x \in Z} \iff \{x \in Y \cap Z \ : \ s_x = t_x\} \in \mathscr{U}.$$

Alternatively: $\prod_{\mathscr{U}} S$ is the colimit of

$$\begin{array}{ccc} (\mathscr{U}, \subseteq)^{\mathrm{op}} & \longrightarrow & \mathbf{Set} \\ Y & \longmapsto & \prod\limits_{x \in Y} S_x. \end{array}$$

Can define ultraproducts similarly in any category with enough (co)limits.

The ultraproduct monad

Let $\mathscr{E}$ be a category with small products and filtered colimits.

Define a category $\mathbf{Fam}(\mathscr{E})$ as follows:

- an object is a family $(S_x)_{x \in X}$ of objects of $\mathscr{E}$, indexed over some set X
- a map $(S_x)_{x \in X} \longrightarrow (R_y)_{y \in Y}$ is a map of sets $f \colon X \longrightarrow Y$ together with a map $R_{f(x)} \longrightarrow S_x$ for each $x \in X$.

Given a family $S = (S_x)_{x \in X}$ of objects of $\mathscr{E}$, we get a new family

$$\left(\prod\nolimits_{\mathscr{U}} S\right)_{\mathscr{U} \in U(X)}$$

of objects of $\mathscr{E}$.

Fact (Ellerman; Kennison) This assignation is part of a monad on $\mathbf{Fam}(\mathscr{E})$, the ultraproduct monad for $\mathscr{E}$.

Ultraproducts are inevitable

Let **FinFam**($\mathscr{E}$) be the full subcategory of **Fam**($\mathscr{E}$) consisting of the families $(S_x)_{x \in X}$ in which X is finite.

Theorem (i) (with Anon.) The codensity monad of $\mathbf{FinFam}(\mathscr{E}) \hookrightarrow \mathbf{Fam}(\mathscr{E})$ is the ultraproduct monad.
(ii) (Kennison) When $\mathscr{E} = \mathbf{Set}$, the algebras for this monad are the sheaves on compact Hausdorff spaces.

Moral of this section

The notion of finiteness of a family
automatically gives rise to the notions of
ultraproduct
and
sheaf on a compact Hausdorff space.

Summary

Summary

- The codensity monad of a functor G is a substitute for $G \circ F$ (where $F \dashv G$) that makes sense even when G has no left adjoint.
- Routinely asking 'what is the codensity monad?' is worthwhile.
- For example, it establishes that the following concepts are categorically inevitable:

ultrafilter	compact Hausdorff space
double dual vector space	linearly compact vector space
ultraproduct	sheaf on a compact Hausdorff space.