General Topology

Course overview

Non-examinable

Tom Leinster, 2014–15

What is topology?

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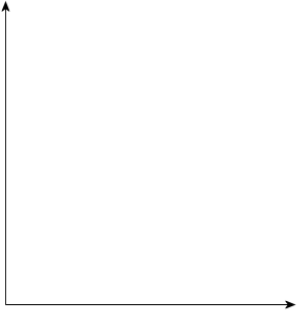
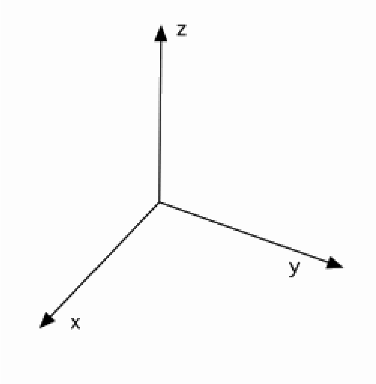
Here are some examples of spaces—that is, worlds in which we might move around continuously.

Euclidean space

The spaces we're most familiar with are $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, ...





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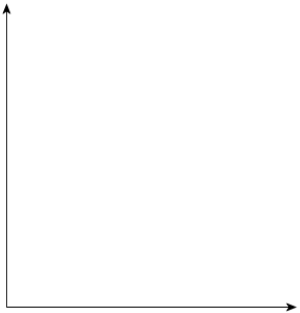

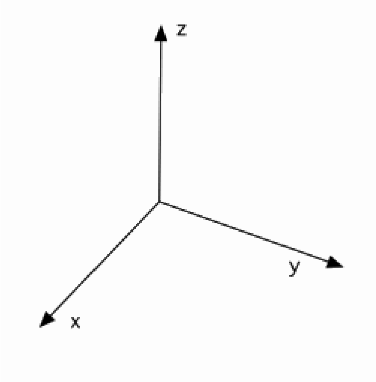

But there are many less familiar spaces...

Some simple surfaces

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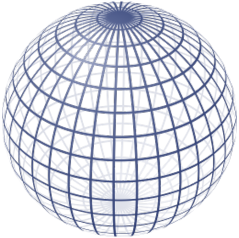

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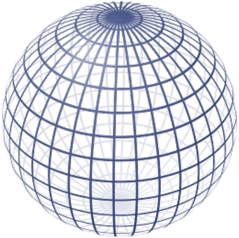

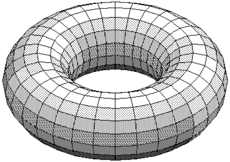

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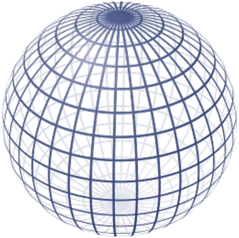

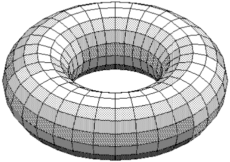

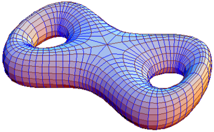

Deforming one space into another

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Some non-orientable spaces

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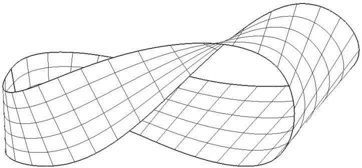

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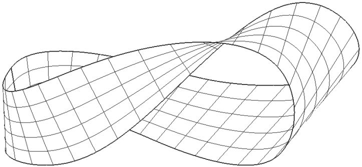
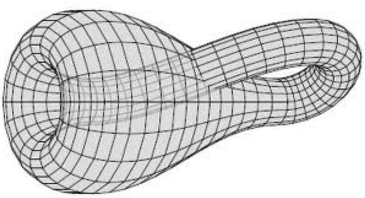


Klein bottle:



Some highly convoluted spaces

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Swiss cheese:

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Spaghetti Junction, Birmingham:

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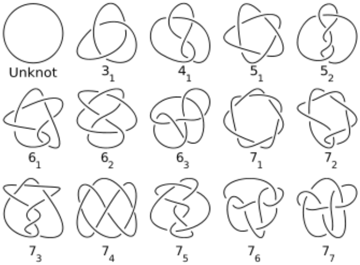

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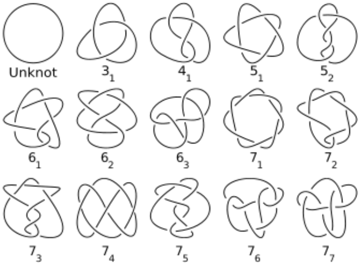

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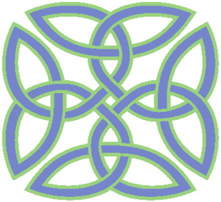

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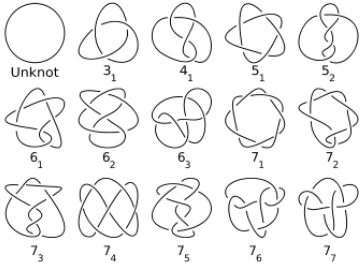

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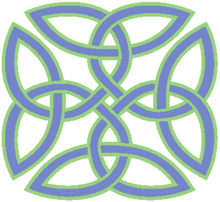

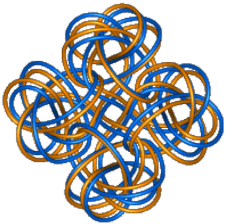

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FIGURE 1. In these examples the recombinase complex B meets the substrate in the two crossover sites (highlighted in black).

Fractal spaces arising from dynamical systems

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Take a function $f : \mathbb{C} \to \mathbb{C}$ of the form

$$f(z) = \frac{a_n z^n + \cdots + a_1 z + a_0}{b_m z^m + \cdots + b_1 z + b_0}$$

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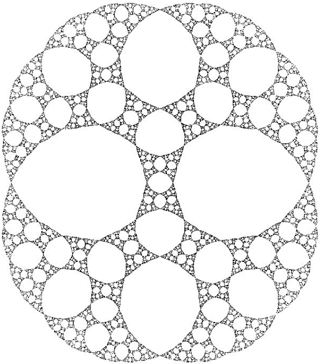
Example: If $f(z) = (2z/(1+z^2))^2$ then $J(f)$ looks like this:



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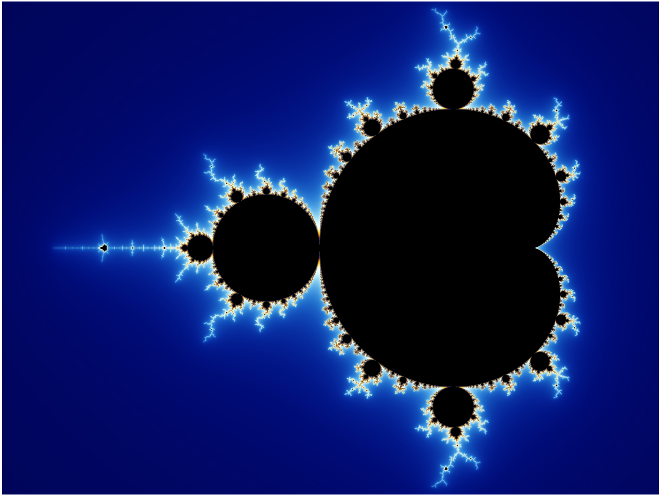

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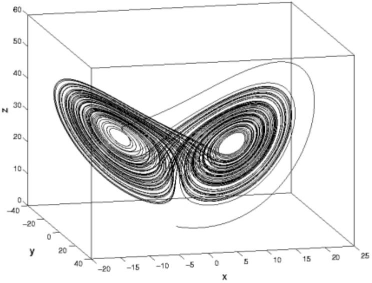

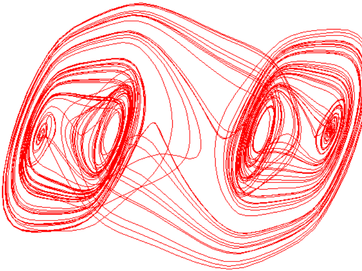

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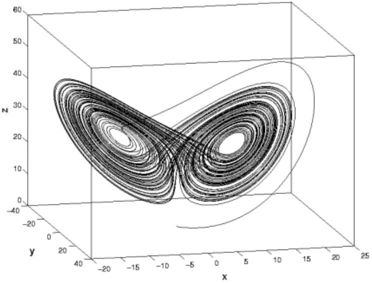

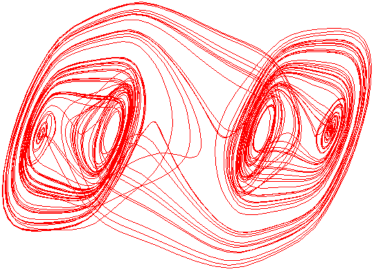

Something else. . .

Spaces of functions

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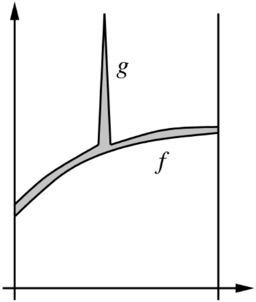

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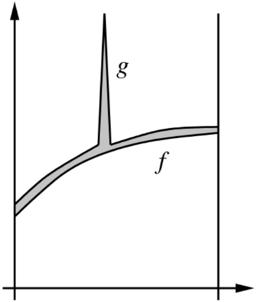

Are f and g close together? The area between them is small, but the maximum difference between them is large.

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But there is still a sense of 'closeness' between messages or sequences.

Some biological spaces

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We can usefully consider spaces of species. . .

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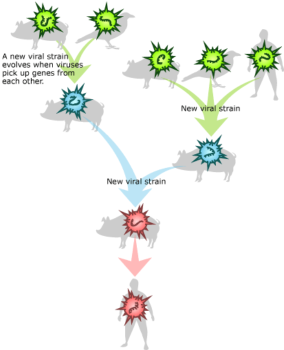
...or the space of possible strains of a virus ...



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Example: Take $b = 10$. Then $19\,658$ is far from $19\,659$, but much closer to $5\,489\,658$.

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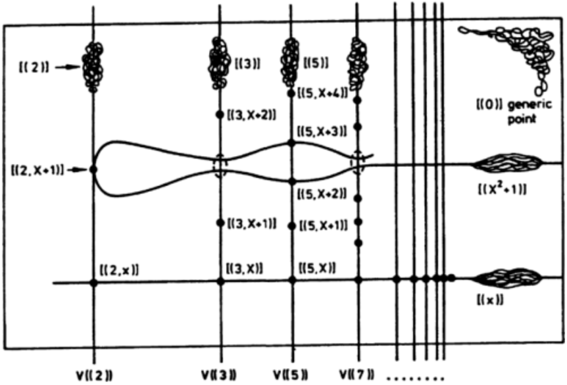
It's hard to visualize! For instance, here's an attempt to draw Spec($\mathbb{Z}[x]$):



Continuity without distance

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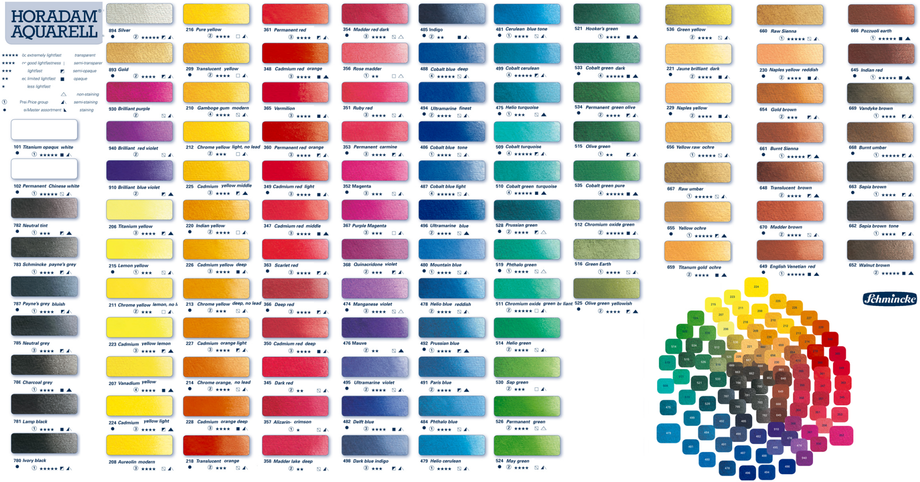
We have no notion of the 'distance' between colours, but we know what it means for colours to change continuously:



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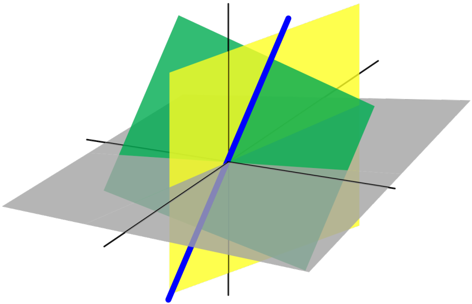

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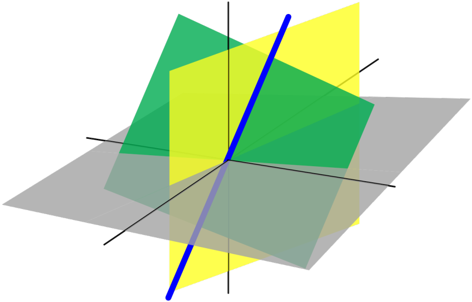

It's not so easy to define the 'distance' between two such planes.

But we know what it means for a plane to move continuously in space.

Where do we begin?

Different types of space

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But you're not allowed to tear them.

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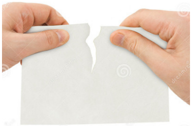
We have to think carefully about what tearing means:



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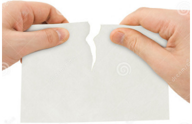

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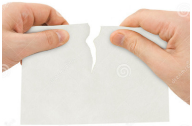
We have to think carefully about what tearing means: 

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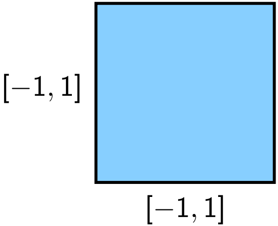

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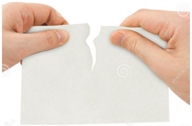

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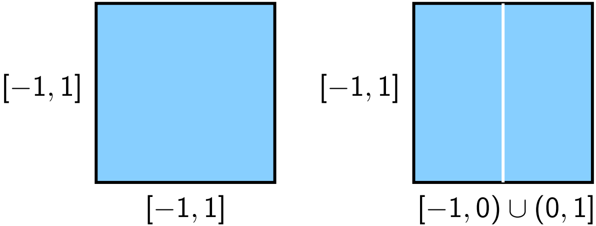

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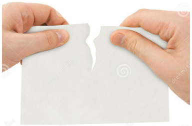

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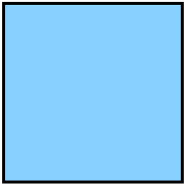

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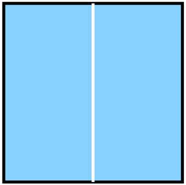

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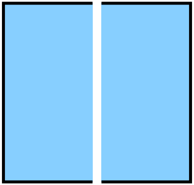

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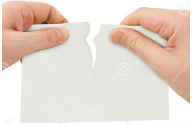

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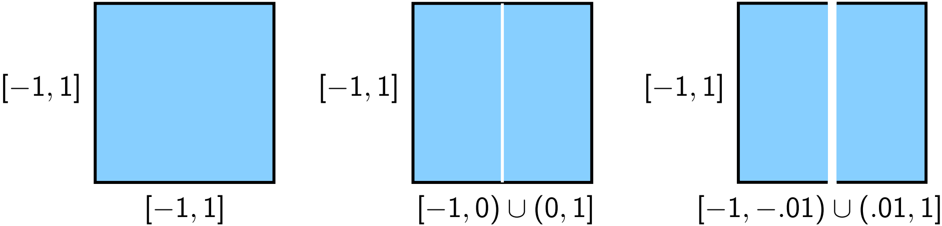
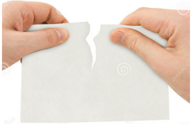


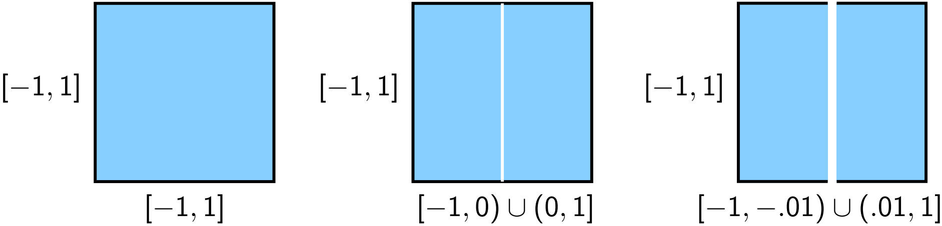
And that means thinking carefully about open and closed sets . . .

How do we make the ‘no tearing’ rule precise?

We have to think carefully about what tearing means:



For instance, we have to distinguish between these three spaces:



And that means thinking carefully about open and closed sets . . .

. . . which is exactly where we’ll begin.