General Topology Course overview

Non-examinable

Tom Leinster, 2014-15

What is topology?

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Topology is the study of continuous change.

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Here are some examples of spaces—that is, worlds in which we might move around continuously.

Euclidean space

The spaces we're most familiar with are \mathbb{R} , \mathbb{R}^2 , \mathbb{R}^3 ,



Euclidean space

The spaces we're most familiar with are \mathbb{R} , \mathbb{R}^2 , \mathbb{R}^3 ,



But there are many less familiar spaces...

















Some non-orientable spaces

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Möbius band:



Some non-orientable spaces





Klein bottle:



Some highly convoluted spaces

Some highly convoluted spaces

Swiss cheese:



Some highly convoluted spaces

Swiss cheese:



Spaghetti Junction, Birmingham:



Some topologists specialize in the study of knots. Some simple knots:



Some topologists specialize in the study of knots. Some simple knots:



Some more complex ones:



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Knot theory is more applicable than it might sound:

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FIGURE 1. In these examples the recombinase complex B meets the substrate in the two crossover sites (highlighted in black).

Fractal spaces arising from dynamical systems Take a function $f : \mathbb{C} \to \mathbb{C}$ of the form

$$f(z) = \frac{a_n z^n + \cdots + a_1 z + a_0}{b_m z^m + \cdots + b_1 z + b_0}$$

 $(a_i, b_i \in \mathbb{C}).$

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Example: If $f(z) = (2z/(1+z^2))^2$ then J(f) looks like this:



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Something else...



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Are f and g close together? The area between them is small, but the maximum difference between them is large.

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We can't really move *continuously* within the space of all messages or DNA sequences.

But there is still a sense of 'closeness' between messages or sequences.

Some biological spaces

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We can usefully consider spaces of species...



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We can usefully consider spaces of species...



... or the space of possible strains of a virus ...



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- Example: Take b = 10. Then 19658 is far from 19659, but much closer to 5489658.

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It's hard to visualize! For instance, here's an attempt to draw $Spec(\mathbb{Z}[x])$:



Continuity without distance

Continuity without distance

We have no notion of the 'distance' between colours, but we know what it means for colours to change continuously:

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Continuity without distance
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Consider planes in \mathbb{R}^3 (not necessarily through the origin).



Continuity without distance

Consider planes in \mathbb{R}^3 (not necessarily through the origin).



It's not so easy to define the 'distance' between two such planes. But we know what it means for a plane to move continuously in space.

Where do we begin?

There are many different types of space, belonging to different branches of mathematics.

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- Topological spaces are floppy and rubbery: you can stretch them, pull them, and deform them without it making any difference.



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But you're not allowed to tear them.

We have to think carefully about what tearing means:



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We have to think carefully about what tearing means:

[-1, 1]		
	[-1, 1]	



We have to think carefully about what tearing means:





We have to think carefully about what tearing means:





We have to think carefully about what tearing means:

For instance, we have to distinguish between these three spaces:



And that means thinking carefully about open and closed sets ...



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And that means thinking carefully about open and closed sets ...

... which is exactly where we'll begin.