A SURVEY OF THE THEORY OF BICATEGORIES

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0. Preliminaries
1. What structure do bicategories form?
2. Coherence: general thoughts
3. Coherence: the Yoneda method
4. Coherence: the strictification method

0. PRELIMINARIES

Terminology

Classical

bicategory

homomorphism

pseudonatural transformation

monoidal category

strong monoidal functor

biequivalence

2-category

Uniform

weak 2-category

weak functor

weak transformation

(weak) monoidal category

weak monoidal functor

(weak 2-) equivalence

strict 2-category

Unbiased vs. biased

An **unbiased bicategory** is like a classical bicat, but comes equipped with an n-fold composition

$$(a_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} a_n) \mapsto (a_0 \xrightarrow{(f_n \circ \cdots \circ f_1)} a_n)$$

for each $n \geq 0$ (not just $n \in \{0,2\}$).

The coherence isos are

$$((f_n^{k_n} \circ \cdots \circ f_n^1) \circ \cdots \circ (f_1^{k_1} \circ \cdots \circ f_1^1)) \xrightarrow{\sim} (f_n^{k_n} \circ \cdots\cdots \circ f_1^1),$$

$$f \xrightarrow{\sim} (f)$$

and satisfy obvious axioms.

Lemma: There is an equivalence of categories

(unbiased bicats + unbiased weak functors)

$\simeq$ (classical bicats + classical weak functors).

So we can use unbiased/classical interchangeably.

Unbiased can make the theory easier.

1. WHAT STRUCTURE DO 2-CATEGORIES FORM?

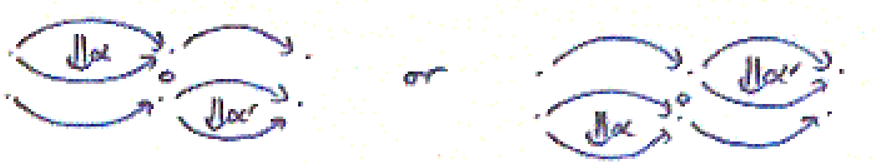
Weak 2-categories

- (Weak 2-categories + weak functors) is a category.
- (Weak 2-categories + weak functors + weak transformations) <u>don't</u> form a weak 2-category
- (Weak 2-categories + weak functors + weak transformations + modifications) form a weak 3-category $2\mathrm{Cat}_w$.

(Really they form two different 3-categories, $2\mathrm{Cat}_w^L$ and $2\mathrm{Cat}_w^R$, depending on whether we define the composite of

$$\begin{tikzcd} A \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & A' \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\alpha'"] & A'' \end{tikzcd}$$

to be



But $2\mathrm{Cat}_w^L$ & $2\mathrm{Cat}_w^R$ are equivalent, so we call them both $2\mathrm{Cat}_w$.)

Strict 2-categories

- (Strict 2-categories + strict functors + strict transformations + modifications) form a strict 3-category $2\mathrm{Cat}_s$.

Monoidal categories

- (Weak monoidal cats + weak monoidal functors + monoidal transformations) form a strict 2-category MonCat_w.
- Similarly, MonCat_s for strict everything.

Warning:

A monoidal category is just a one-object 2-category.

A monoidal functor is just a functor between the corresponding 2-categories.

But a monoidal transformation is not the same as a transformation between the corresponding 2-functors

3. COHERENCE: GENERAL THOUGHTS

- We'll take "all diagrams commute" as read. The serious question is: when can a weak thing be replaced by a strict thing?
- Strictness is not a categorical property!
- How not to prove coherence: quotient out by coherence isos.
- We want a holistic version of the coherence theorem: not just "every weak 2-category is equivalent to a strict one", but something about functors and transformations too.

4. COHERENCE: THE YONEDA METHOD

Theorem: Every weak 2-category $\mathbb{A}$ is equivalent to some strict 2-category $\mathbb{B}$.

Proof: Let $\mathbb{B}$ be the full image of

$$\text{Yoneda}: \mathbb{A} \hookrightarrow 2\mathbf{Cat}_w(\mathbb{A}^{op}, \mathbf{Cat}).$$ $\square$

Warning: Not every weak 2-functor is equivalent to a strict one! Can write down a weak 2-functor

$$\mathbb{B} \xrightarrow{F} \mathbb{T}$$

with $\mathbb{B}, \mathbb{T}$ strict and F not equivalent (in $2\mathbf{Cat}_w$) to any strict 2-functor. (E.g. can take $\mathbb{B}, \mathbb{T}$ to be monoidal groupoids: see Lack, TAC, 2007.)

More coherence from Yoneda

Theorem: Let $\mathbb{S}$ be a strict 2-category. Then every weak functor $X : \mathbb{S} \longrightarrow \mathrm{Cat}$ is equivalent to some strict functor Z.

Proof: Let $Z(s) = \big(2\mathrm{Cat}_w(\mathbb{S}, \mathrm{Cat})\big)\big(\mathbb{S}(s,-), X\big)$. $\square$

for all $s \in \mathbb{S}$.

$= \mathrm{Hom}\,(\mathbb{S}(s,-), X$

Theorem: Let $\mathbb{A}$ be a weak 2-category, $a \in \mathbb{A}$, and $Z : \mathbb{A} \longrightarrow \mathrm{Cat}$ a strict functor. Then every weak transformation
$$\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "\mathbb{A}(a{,}-)"] \arrow[r, bend right, "Z"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathrm{Cat} \end{tikzcd}$$
is isomorphic to a strict transformation σ.

Proof: Let
$$\begin{aligned} \sigma_b : \mathbb{A}(a,b) &\longrightarrow Zb \\ \text{be} \qquad p &\longmapsto (Zp)(\alpha_a(1_a)). \end{aligned}$$
$\square$

where $b \in \mathbb{A}$.

5. COHERENCE: THE STRICTIFICATION METHOD

Strictification of monoidal categories

There is a strict 2-adjunction

$$\begin{tikzcd} \mathbf{MonCat}_s \arrow[r, hook, shift left, "U"] \arrow[r, phantom, "\top"] & \mathbf{MonCat}_w \arrow[l, shift left, "st"] \end{tikzcd}.$$

If $\mathbb{A}$ is a weak monoidal cat, then $st\,\mathbb{A}$ has:

- objects: sequences $\langle A_1, \ldots, A_n \rangle$ $(A_i \in \mathrm{ob}\,\mathbb{A})$
- maps $\langle A_1, \ldots, A_n \rangle \longrightarrow \langle B_1, \ldots, B_m \rangle$:

  maps $(A_1 \otimes \cdots \otimes A_n) \longrightarrow (B_1 \otimes \cdots \otimes B_m)$ in $\mathbb{A}$

  (easiest if unbiased)
- $\otimes$ is concatenation.

(Joyal & Street, "Braided tensor categories".)

Stratification of monoidal categories (cont.)

The adjunction $\mathrm{MonCat}_s \underset{st}{\overset{U}{\rightleftarrows}} \mathrm{MonCat}_w$ ($\perp$) is a weak coreflection, i.e. for all weak $\mathbb{A}$,

$$\eta_{\mathbb{A}} : \mathbb{A} \longrightarrow st\,\mathbb{A}$$

is an equivalence. Hence:

Theorem: (i) for every weak mon cat $\mathbb{A}$, $\mathbb{A} \simeq st\,\mathbb{A}$

(ii) for every weak monoidal functor $\mathbb{A} \xrightarrow{F} \mathbb{B}$, have

$$\begin{tikzcd} st\,\mathbb{A} \arrow[r, "st\,F"] & st\,\mathbb{B} \\ \mathbb{A} \arrow[u, "\eta_{\mathbb{A}}", "\simeq"'] \arrow[r, "F"'] \arrow[ur, phantom, "\cong"] & \mathbb{B} \arrow[u, "\simeq", "\eta_{\mathbb{B}}"'] \end{tikzcd} \quad \} \text{ strict}$$

$\square$

In this sense, <u>can</u> replace any weak monoidal functor by a strict one.

Alternative way to say "weak coreflection": st is locally an equivalence. So MonCat_w is a full sub-2-category of MonCat_s !

Strictification of 2-categories

For a weak 2-category $\mathbb{A}$, define a strict 2-category $\mathrm{st}\,\mathbb{A}$ as follows:

- objects are those of $\mathbb{A}$
- 1-cells $a \to a'$ are paths $a = a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} a_n = a'$ in $\mathbb{A}$

etc (as for monoidal categories).

For a weak 2-functor $\mathbb{A} \xrightarrow{F} \mathbb{B}$, can define a strict 2-functor $\mathrm{st}\mathbb{A} \xrightarrow{\mathrm{st}F} \mathrm{st}\mathbb{B}$ in an obvious way.

But: given a weak transformation $\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbb{B} \end{tikzcd}$,

the resulting transformation $\begin{tikzcd} \mathrm{st}\mathbb{A} \arrow[r, bend left, "\mathrm{st}F"] \arrow[r, bend right, "\mathrm{st}G"'] \arrow[r, phantom, "\Downarrow"] & \mathrm{st}\mathbb{B} \end{tikzcd}$ is <u>weak</u>!

Strictification of 2-categories (cont.)

Let $2Cat_G$ (= Gray) be the weak 3-category

(strict 2-categories + strict functors

+ weak transformations + modifications).

Then there is a 3-adjunction

$$2Cat_G \underset{st}{\overset{U}{\underset{\top}{\rightleftarrows}}} 2Cat_w .$$

This is a weak coreflection, i.e.

$$\eta_{\mathbb{A}} : \mathbb{A} \longrightarrow st\, \mathbb{A}$$

is an equivalence for all $\mathbb{A}$. So we have a coherence theorem for weak 2-categories and functors (as for mon cats), but not for transformations.

Moreover, st is locally an equivalence.

(Cf. Gordon, Power & Street, "Coherence for tricats".)

Warning: $2Cat_G \not\simeq 2Cat_w$ and $MonCat_s \not\simeq MonCat_w$

(See Lack, TAC, 2007.)

CONCLUSIONS

- "Strictification" (the left adjoint to the inclusion strict $\hookrightarrow$ weak) provides a canonical, holistic coherence theorem for 2-categories.

- You can strictify a weak 2-category.

  You can strictify a weak 2-functor (provided you're willing to change the domain & codomain up to equivalence).

  But you can't strictify a weak transformation!