Codensity monads

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These slides: available on my web page

Preview

This talk is about a canonical categorical construction: the codensity monad of a functor.

When applied to some familiar functors, it produces concepts including these:

- prime ideals and fields of fractions
- the radical of an integer
- ultrafilters and ultraproducts
- compact Hausdorff spaces and sheaves on them
- double dualization of vector spaces
- linearly compact vector spaces
- probability measures
- ...

The overall idea

Any functor with a left adjoint gives rise to a monad.

But even many functors *without* a left adjoint give rise to a monad. That's the 'codensity monad' of the functor.

If the functor (G) does have a left adjoint (F) then the codensity monad is the familiar thing ($G \circ F$). But there are many other interesting and significant examples.

Plan

1. The definition, from many angles

2. Examples

1. The definition, from many angles

(Isbell, Ulmer; Applegate & Tierney, A. Kock)

The shape of the definition

Let $G\colon \mathscr{B} \longrightarrow \mathscr{A}$ be a functor.

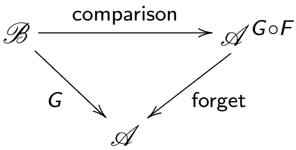
The codensity monad of G, if defined, is a monad T^G on $\mathscr{A}$.

(I'll define it later.)

It's not defined for *all* functors G, but it is defined for *many*, e.g. if $\mathscr{B}$ is small and $\mathscr{A}$ has small limits.

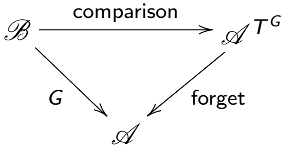
Motivation for the definition

Let $G \colon \mathscr{B} \longrightarrow \mathscr{A}$ be a functor that *does* have a left adjoint, F. We have categories and functors



and this is initial among all maps in $\mathbf{CAT}/\mathscr{A}$ from G to a monadic functor.

Theorem (Dubuc) Let $G \colon \mathscr{B} \longrightarrow \mathscr{A}$ be a functor whose codensity monad T^G is defined. Then

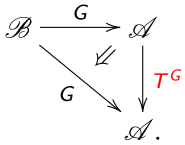


is initial among all maps in $\mathbf{CAT}/\mathscr{A}$ from G to a monadic functor.

Corollary Let G be a functor with a left adjoint, F. Then $T^G = G \circ F$.

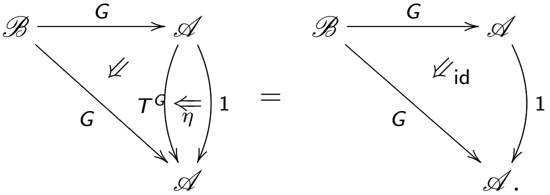
The definition, via Kan extensions

Definition The codensity monad T^G of a functor G is the right Kan extension of G along itself (if it exists).



The monad structure on T^G comes from the universal property of Kan extensions.

E.g. the unit map $\eta\colon 1_{\mathscr{A}} \longrightarrow T^G$ is unique such that



The definition, via ends

By the end formula for Kan extensions, the codensity monad T^G of $G\colon \mathscr{B} \longrightarrow \mathscr{A}$ is given by

$$T^G(A) = \int_B [\mathscr{A}(A, GB), GB] \ \in \mathscr{A}$$

$(A \in \mathscr{A})$.

Here $[-,-]$ denotes a power: if $S \in \mathbf{Set}$ and $X \in \mathscr{A}$ then $[S, X] := \prod_{s \in S} X$.

The definition, via limits

Recall: $G \colon \mathscr{B} \longrightarrow \mathscr{A}$ and T^G is a monad on $\mathscr{A}$, with

$$T^G(A) = \int_B [\mathscr{A}(A, GB), GB].$$

Equivalently,

$$T^G(A) = \lim\big((A \downarrow G) \xrightarrow{\text{proj}} \mathscr{B} \xrightarrow{G} \mathscr{A}\big) = \lim_{\substack{B \in \mathscr{B}, \\ f \colon A \longrightarrow GB}} GB.$$

The definition, for **Set**-valued functors

If $\mathscr{A} = \mathbf{Set}$ then $G\colon \mathscr{B} \longrightarrow \mathbf{Set}$ and T^G is a monad on **Set**, with

$$T^G(A) = \int_B [[A, GB], GB] = \lim_{\substack{B \in \mathscr{B}, \\ f\colon A \longrightarrow GB}} GB.$$

Equivalently,

$$T^G(A) = \{\text{natural transformations } G^A \longrightarrow G\}$$

where

$$\begin{array}{rccc} G^A\colon & \mathscr{B} & \longrightarrow & \mathbf{Set}, \\ & B & \mapsto & (GB)^A. \end{array}$$

E.g. Let $G\colon \mathbf{Grp} \longrightarrow \mathbf{Set}$ be the forgetful functor and let $A = n$ be a finite set. Then

$$\begin{aligned} T^G(n) &= \{\text{natural transformations } G^n \longrightarrow G\} \\ &= \{\text{maps } B^n \longrightarrow B \text{ defined for a generic group } B\} \\ &= \{n\text{-ary operations in the theory of groups}\} \\ &= \text{free group on } n \text{ generators}. \end{aligned}$$

2. Examples

First examples of codensity monads

- If G has a left adjoint F then $T^G = G \circ F$.
- Let $\mathscr{B} = \mathbf{1}$. A functor $\mathbf{1} \longrightarrow \mathscr{A}$ is just an object X of $\mathscr{A}$.
  Its codensity monad T^X on $\mathscr{A}$ is given by

$$T^X(A) = [\mathscr{A}(A, X), X].$$

  This is the endomorphism monad $\mathbf{End}(X) = T^X$ of X.
  It has the property that for *any* monad T on $\mathscr{A}$,

$$T\text{-algebra structures on } X \leftrightarrow \text{ monad maps } T \longrightarrow \mathbf{End}(X).$$

  (Compare group actions or representations.)

(Co)dense functors, 1

A functor $G\colon \mathscr{B} \longrightarrow \mathscr{A}$ is codense if its codensity monad is the identity.

This means that

$$A \cong \int_B [\mathscr{A}(A, GB), GB] = \lim_{f\colon A \longrightarrow GB} GB$$

naturally in $A \in \mathscr{A}$. Loosely,

> 'every object of $\mathscr{A}$ is a limit of objects in the image of G'.

Often $\mathscr{B}$ is a subcategory of $\mathscr{A}$ and G is the inclusion. Then codensity means:

> 'every object of $\mathscr{A}$ is a limit of objects of $\mathscr{B}$'.

Dually, $\mathscr{B}$ is dense in $\mathscr{A}$ if (loosely):

> 'every object of $\mathscr{A}$ is a colimit of objects of $\mathscr{B}$'.

(Co)dense functors, 2

E.g.

- Let $\mathbf{Mfd}_n$ be the category of smooth n-manifolds and smooth maps.
  Let $\mathbf{Euc}_n$ be the subcategory of open subsets of $\mathbb{R}^n$ and smooth open embeddings.
  Then $\mathbf{Euc}_n \hookrightarrow \mathbf{Mfd}_n$ is dense: 'every manifold is a colimit of Euclidean patches'.
- Similarly, $\mathbf{CRing}^{\mathrm{op}} \cong$ (affine schemes) is dense in the category of all schemes.

Warning: (Co)dense functors can behave counterintuitively! E.g. (Isbell):

- A composite of dense functors needn't be dense.
- There's an example of a full subcategory $\mathscr{B} \hookrightarrow \mathscr{A}$ such that $\mathscr{B}$ is finite and $\mathscr{A}$ is large (and not equivalent to any small category), but the inclusion is both dense and codense.

Rings and fields, 1

The inclusion $G\colon \mathbf{Field} \hookrightarrow \mathbf{CRing}$ has no left adjoint.

But it has a codensity monad! It's given by

$$T^G(A) = \prod_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathrm{Frac}(A/\mathfrak{p})$$

($A \in \mathbf{CRing}$). Key ingredients of proof:

- For a homomorphism $\phi\colon A \longrightarrow k$ into a field k, $\ker \phi$ is a prime ideal.
- Among all such homomorphisms with kernel $\mathfrak{p}$, the initial one is $A \longrightarrow A/\mathfrak{p} \hookrightarrow \mathrm{Frac}(A/\mathfrak{p})$.

E.g.

$$T^G(\mathbb{Z}) = \mathbb{Q} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \cdots$$
$$T^G(\mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/\mathrm{rad}(n)\mathbb{Z},$$

where $\mathrm{rad}(n)$ is the product of the distinct prime factors of n.

Rings and fields, 2

Some puzzles:

1. What are the algebras for T^G, the codensity monad of **Field** $\hookrightarrow$ **CRing**?
2. What is $T^G(k[x])$, for a field k?
3. What is the codensity monad of the forgetful functor **Field** $\longrightarrow$ **Set** (if it has one)?
4. What are *its* algebras?

Ultrafilters

Definition Let A be a set. An ultrafilter on A is a set $\mathscr{U}$ of subsets of A such that whenever

$$A = A_1 \amalg \cdots \amalg A_n$$

($n \geq 0$), there is a unique i such that $A_i \in \mathscr{U}$.

Think of the elements of $\mathscr{U}$ as the 'large' sets.

E.g.

- Given $a \in A$, get ultrafilter $\mathscr{U}_a = \{B \subseteq A \ : \ a \in B\}$.
- On a finite set A, every ultrafilter is of the form $\mathscr{U}_a$.
  On an infinite set A, there are other ultrafilters—but proving this makes essential use of the axiom of choice.

Key properties of ultrafilters:

- they are upwards closed ($C \supseteq B \in \mathscr{U} \Rightarrow C \in \mathscr{U}$)
- they are closed under finite intersections
- if $B, B' \in \mathscr{U}$ then $B \cap B' \neq \emptyset$.

Ultrafilters from codensity monads

Theorem (Kennison and Gildenhuys) The codensity monad of the inclusion $G\colon \mathbf{FinSet} \hookrightarrow \mathbf{Set}$ is given by

$$T^G(A) = \{\text{ultrafilters on } A\}.$$

Sketch proof Recall that for $A \in \mathbf{Set}$,

$$\begin{aligned} T^G(A) &= \{\text{natural transformations } G^A \longrightarrow G\} \\ &= \{\text{natural families } (B^A \longrightarrow B)_{B \in \mathbf{FinSet}}\}. \end{aligned}$$

So an element of $T^G(A)$ is a machine that takes as input a finite set B and a function $A \longrightarrow B$, and produces as output an element of B.

Equivalently, it's a way of selecting, for each partition of A into finitely many pieces, one of those pieces.

Equivalently, it's an ultrafilter on A.

Compact Hausdorff spaces from codensity monads

We just saw that $A \mapsto \{\text{ultrafilters on } A\}$ is a monad on **Set**.

Theorem (Manes) The algebras for the ultrafilter monad are the compact Hausdorff spaces.

Background Let X be a topological space. Ultrafilters on X can be viewed as 'generalized sequences'.

An ultrafilter $\mathscr{U}$ on X converges to $x \in X$ if every neighbourhood of x belongs to $\mathscr{U}$.

Facts:

- X is compact $\iff$ every ultrafilter on X converges to *at least* one point.
- X is Hausdorff $\iff$ every ultrafilter on X converges to *at most* one point.
- X is compact Hausdorff $\iff$ every ultrafilter on X converges to *exactly* one point.

So if X is compact Hausdorff, get function $\lim\colon T^G(X) \longrightarrow X$.

Double duals

We saw that the codensity monad of **FinSet** $\hookrightarrow$ **Set** was something interesting.

What about **FDVect** $\hookrightarrow$ **Vect**?

Theorem

- The codensity monad of **FDVect** $\hookrightarrow$ **Vect** is double dualization $(\ )^{**}$.
- The algebras of $(\ )^{**}$ are the linearly compact vector spaces (certain topological vector spaces).

Details omitted! This is the linear analogue of the story of ultrafilters and compact Hausdorff spaces.

Probability measures

Let **Msbl** be the category of measurable spaces (sets equipped with a σ-algebra).

Let $\mathscr{B}$ be the subcategory consisting of the single object

$$B = \{\text{sequences in } [0,1] \text{ converging to } 0\}$$

and the affine maps $B \longrightarrow B$ (those preserving convex combinations).

Theorem (Avery) The codensity monad of $\mathscr{B} \hookrightarrow$ **Msbl** is the Giry monad,

$$T^G(A) = \{\text{probability measures on } A\}.$$

Moral

Whenever you meet a functor,
ask
"What is its codensity monad?"

References

The paper titles below are clickable links.

- Tom Leinster, Codensity and the ultrafilter monad, *Theory and Applications of Categories* 28 (2013), 332–370.
- Tom Avery, Codensity and the Giry monad, *Journal of Pure and Applied Algebra* 220 (2016), 1229–1251.

Other references, including references to work attributed to other authors in these slides, can be found in 'Codensity and the ultrafilter monad'.