Sep 24, 09 — If no name is mentioned, it is due to Bourgain...? ①

Cauchy problem of the cubic NLS on $\mathbb{T}$

Consider

$$\begin{cases} i u_t - \Delta u \pm |u|^{p-2} u = 0 \\ u|_{t=0} = u_0 \end{cases}$$

- Conservation laws

$$N(u) = \int |u|^2 dx$$

$$H(u) = \frac{1}{2}\int |\nabla u|^2 \pm \frac{1}{p}\int |u|^p$$

① Basic properties of $X^{s,b}$ spaces

$$\|u\|_{X^{s,b}} = \| \langle n\rangle^s \langle \tau - n^2\rangle^b \, \hat{u}(n,\tau)\|_{L^2_{n,\tau}}$$
$$= \| S(-t) u \|_{H^b_t H^s_x} \cdot \quad S(t) = e^{-it\Delta}.$$

<u>Note</u>: The linear soln $\begin{cases} i u_t - \Delta u = 0 \\ u|_{t=0} = u_0 \end{cases}$

is a measure supported on $\{\tau = n^2\}$

$\Rightarrow$ The $X^{s,b}$ norm "measures" how close u is to the linear soln (for $b > 0$.)

② ·

- By Sobolev embedding,

$$u \in X^{s,b},\ b > 1/2 \Rightarrow u \in C_t(H^s_x)$$

- <u>Linear estimates</u>:

(a) $\eta(t)$ = smooth cutoff on $[-1, 1]$.

$$\Rightarrow \left(\eta(t) S(t) u_0\right)^\wedge(n) = \hat{\eta}(\tau - n^2)\, \hat{u}_0(n)$$

$$\Rightarrow \quad \| \eta(t) S(t) u_0 \|_{X^{s,b}} \lesssim_b \| u_0 \|_{H^s}.$$

(b) <u>Duhamel term</u>: $\displaystyle\int_0^t S(t-t') F(t')\, dt'$.

$$= \int_0^t \sum e^{inx} \int_{\mathbb{R}} e^{it'\tau} e^{itn^2} \hat{F}(n, \tau + n^2)\, d\tau\, dt'.$$

$$\left(\begin{aligned} &\because \left(S(t-t') F(x, t')\right)^\wedge \\ &= \int_{\mathbb{R}} e^{-it'\tau} e^{i(t-t')n^2} \hat{F}(n, t')\, dt' \\ &= e^{itn^2} \hat{F}(n, \tau + n^2) \end{aligned}\right.$$

$$= \sum e^{inx} \int_{\mathbb{R}} \int_0^t e^{it'\tau} e^{itn^2} \hat{F}(n, \tau + n^2)\, dt'\, d\tau$$

$$= \sum e^{inx} \int_{\mathbb{R}} \underbrace{\int_0^t e^{it'(\tau - n^2)}\, e^{itn^2}\, \hat{F}(n, \tau)\, dt'}_{\frac{1}{i(\tau - n^2)}\left(e^{it\tau} - e^{itn^2}\right)}\, d\tau$$

$\Rightarrow$ Duhamel term

$$= c \sum e^{inx} \int_{\mathbb{R}} \frac{e^{it\tau} - e^{itn^2}}{\tau - n^2} \, d\tau$$

$$= c \sum_{k \geq 1} \frac{i^k}{k!} t^k \eta(t) \left[\sum_n \int \eta(\tau - n^2)(\tau - n^2)^{k-1} \hat{F}(n, \tau) \, d\tau \right] \times e^{i(nx + tn^2)}$$

$$+ \eta(t) \sum_n e^{inx} \int \frac{(1-\eta)(\tau - n^2) e^{it\tau}}{\tau - n^2} \hat{F}(n,\tau) \, d\tau$$

$$+ \eta(t) \sum_n e^{i(nx + tn^2)} \int \frac{(1-\eta)(\tau - n^2)}{\tau - n^2} \hat{F}(n,\tau) \, d\tau$$

$\Rightarrow$ <u>nonhomog. lin. estimate.</u>

$$\left\| \eta(t) \int_0^t S(t - t') F(t') \, dt' \right\|_{X^{s,b}} \lesssim \| F \|_{X^{s,b-1}}, \quad b > 1/2$$

(For $b \leq 1/2$, add $\| F \|_{Y^{s, b - 3/2}}$ on RHS,
where $\| F \|_{Y^{s,b}} = \| \langle n \rangle^s \langle \tau - n^2 \rangle^b \hat{F}(n,\tau) \|_{L^2_n L^1_\tau}$

• <u>time localization</u>: Let $T \le 1$.

Let $\eta_T(t) = \eta\left(t/T\right)$, localized on $[-T, T]$.

• You lose $\log T$ when $b = 1/2$

<u>e.g.</u> • For $b' > -1/2$, we have

$$\left\| \eta_T \int_0^t S(t-t') F(t')\,dt' \right\|_{X^{s,b}} \lesssim T^{1-b+b'} \|F\|_{X^{s,b'}}$$

i.e. for $b - b' < 1$, we gain T^{0+}.

• $\|\eta_T u\|_{X^{s,b}} \lesssim T^{\frac{1}{2}-b-} \|u\|_{X^{s,1/2}}$, $b < 1/2$.

<u>Pf</u>) By interpolation,

$$\|\eta_T u\|_{X^{s,b}} \lesssim \|\eta_T u\|_{X^{s,0}}^{\alpha} \, \|\eta_T u\|_{X^{s,1/2}}^{1-\alpha}$$

for $\alpha = 1 - 2b \in (0, 1)$

Note that $\|\widehat{\eta_T}\|_{L^q} \sim T^{\frac{q-1}{q}}$.

$$\Rightarrow \|\widehat{\eta}_T * \widehat{u}(n,\cdot)\|_{L^2_\tau} \le \|\widehat{\eta}_T\|_{L^{2-}_\tau} \|\widehat{u}(n,\cdot)\|_{L^{1+}_\tau}$$

$$\lesssim T^{\frac{1}{2}-} \underbrace{\|\langle \tau - n^2 \rangle^{-1/2}\|_{L^{2+}_\tau}}_{<\infty} \|\langle \tau - n^2 \rangle^{\frac{1}{2}} \widehat{u}(n,\cdot)\|_{L^2_\tau}$$

- <u>Transference principle</u> : $b > 1/2$

$u \in X^{s,b}$ if and only if

$\exists\ f_{\tau,0} \in L^2(\mathbb{R}, H^s)$ s.t.

$$u = \int \langle \tau \rangle^{-b} e^{it\tau} f_\tau(x,t)\, d\tau$$

where f_τ is a linear soln to

$$\begin{cases} i\partial_t f_\tau - \partial_x^2 f_\tau = 0 \\ f_\tau |_{t=0} = f_{\tau,0} \end{cases}$$

<u>Lemma 1</u>: T, k linear operator s.t.

$$\| T(S(t)f_1, \cdots, S(t)f_k) \|_{L^q_t L^r_x} \lesssim \prod_{j=1}^{k} \| f_j \|_{H^{s_j}}$$

Then,

$$\| T(u_1, \cdots, u_k) \|_{L^q_t L^r_x} \lesssim \prod_{j=1}^{k} \| u_j \|_{X^{s_j, b}}$$

for $b > 1/2$.

<u>pf)</u> <u>$k=1$</u>, $T = Id$.

$$u(x,t) = \int_{\mathbb{R}^{n+1}} \hat{u}(\xi, \tau) e^{i(t\tau + x\cdot\xi)}\, d\xi\, d\tau$$

Let $\tau = \tau' + |\xi|^2$

and

$$f_{\tau'}(x) = \int_{\mathbb{R}^n} \hat{u}(\tau' + |\xi|^2, \xi)\, e^{ix\cdot\xi}\, d\xi$$

$$\Rightarrow S(t) f_{\tau'}(x) = \int \hat{u}(\tau' + |\xi|^2, \xi)\, e^{it|\xi|^2} e^{ix\cdot\xi}\, d\xi$$

$$\Rightarrow u(t) = \int_{\mathbb{R}} e^{it\tau'} S(t) f_{\tau'}\, d\tau'$$

$$\Rightarrow \|u\|_{L^q_t L^r_x} \lesssim \int \|f_{\tau'}\|_{H^s_x}\, d\tau'$$

$$\lesssim_b \left(\int \langle \tau' \rangle^{2b} \|f_{\tau'}\|^2_{H^s_x}\, d\tau' \right)^{1/2}$$

$$\sim \|u\|_{X^{s,b}}. \quad \boxtimes$$

---

② Strichartz estimate:

On $\mathbb{R}^d$, we have

$$\|S(t) u_0\|_{L^q_t L^r_x} \lesssim \|u_0\|_{L^2_x}$$

- dimension counting. "2 space derivatives = 1 time deriv".

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$$

· **Idea of Proof**

· dispersive estimate: $\| S(t) u_0 \|_{L^\infty_x} \lesssim |t|^{-d/2} \| u_0 \|_{L^1_x}$

· unitary operator: $\| S(t) u_0 \|_{L^2_x} = \| u_0 \|_{L^2_x}$

$\Rightarrow$ $L^p - L^q$ estimate: $\| S(t) u_0 \|_{L^p_x} \lesssim |t|^{-(\frac{d}{2} - \frac{d}{p})} \| u_0 \|_{L^q_x}$

$\Rightarrow$ Duality, TT^* argument,

Hardy–Littlewood–Sobolev inequality

$1 < p, q, r < \infty, \quad \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$

$\Rightarrow \quad \| |x|^{-n/p} * g \|_{L^r(\mathbb{R}^n)} \lesssim \| g \|_{L^q(\mathbb{R}^n)}$

Note: $|x|^{-n/p} \notin L^p(\mathbb{R}^n)$ but almost...

i.e. H-L-S is the endpoint Young's ineq.

· **On $\mathbb{T}$**:

· L^4 Strichartz (Zygmund '74)

Lemma 2:

$$\Big\| \sum_{n \in \mathbb{Z}} e^{i(nx + tn^2)} \hat{u}_0(n) \Big\|_{L^4_{x,t}(\mathbb{T}^2)} \lesssim \| u_0 \|_{L^2(\mathbb{T})}$$

Pf) Let $f(x,t) = \sum_n e^{i(nx+tn^2)} \hat{u}_0(n)$. ⑧

Then,

$$\| f\bar{f} \|_{L^2(\mathbb{T}^2)} = \Big\| \sum_{n_1,n_2} \hat{u}_0(n_1)\overline{\hat{u}_0(n_2)}\, e^{it(n_1^2-n_2^2)+i(n_1-n_2)x} \Big\|_{L^2(\mathbb{T}^2)}$$

- Rewrite this sum as a Fourier series in (x,t)

$$\sum_{(\tau,n)\in\mathbb{Z}^2} a(n,\tau)\, e^{i(nx+\tau t)}$$

where $$a(n,\tau) = \sum_{(n_1,n_2)\in P(n,\tau)} \hat{u}_0(n_1)\overline{\hat{u}_0(n_2)}$$

and

$$P(n,\tau) = \left\{ (n_1,n_2) \,\middle|\, n_1^2 - n_2^2 = \tau,\ n_1 - n_2 = n \right\}$$

- Given $(n,\tau) \neq (0,0)$ of the form $(n,\tau) = (n_1-n_2, n_1^2-n_2^2)$, there is at most one soln (n_1,n_2)

$$\left(\because\ \tau - n^2 = n_1^2 - n_2^2 - n^2 = -2n_2(n_2-n_1) = -2n n_2 \right.$$
$$\left. \Rightarrow n_2 = \frac{\tau - n^2}{-2n} \text{ is determined.} \Rightarrow \text{so is } n_1 \right.$$

- Moreover, $a(0,0) = \sum_n |\hat{u}_0(n)|^2$

$$\left(\because\ n = 0 \Rightarrow n_1 = n_2 \Rightarrow \tau = n_1^2 - n_2^2 = 0 \right.$$

By Plancherel,

$$\|f\|_{L^4(\mathbb{T}^2)} = \Big(\sum_{\tau, n \in \mathbb{Z}} |a(n,\tau)|^2\Big)^{1/4}$$

$$\sim \underbrace{\Big(\sum_{(\tau,n)\neq(0,0)} |a(n,\tau)|^2\Big)^{1/4}} + \underbrace{|a(0,0)|^{1/2}}_{=\|u_0\|_{L^2}}$$

$$\lesssim \Big(\sum_{n_1, n_2} |\hat{u}_0(n_1)\overline{\hat{u}_0(n_2)}|^2\Big)^{1/4} \qquad (n_1, n_2) \Longleftrightarrow (\tau, n)$$

$$\lesssim \Big(\sum |\hat{u}_0(n)|^2\Big)^{1/2} = \|u_0\|_{L^2} \quad \boxtimes$$

· L^6 Strichartz:

Recall that

$\#$ of divisors of an integer $n \sim 2^{\frac{\log n}{\log\log n}}$

$= o(n^\delta)$ for any $\delta > 0$ as $n \to \infty$

· Lemma 3:

$$\Big\|\eta(t) \sum_{|n|\leq N} e^{i(nx+tn^2)} \hat{u}_0(n)\Big\|_{L^6(\mathbb{T}\times\mathbb{R})}$$

$$\lesssim N^{0+} \|u_0\|_{L^2_x}$$

By Transference principle,

$$\|\eta\, u\|_{L^6_{x,t}(\mathbb{T}\times\mathbb{R})} \lesssim \|u\|_{X^{\varepsilon,\frac{1}{2}+}}, \quad \forall\, \varepsilon>0.$$

<u>Pf)</u>

Dirichlet projection onto $|n|\le N$.

$$\left(\eta(t)\, S(t)\, \mathbb{P}_N f\right)^{\wedge}(n,\tau) = \hat{f}(n)\, \hat{\eta}(\tau-n^2)$$

for $|n|\le N$.

· By Plancherel,

$$\|\eta(t)\, S(t)\, \mathbb{P}_N f\|^6_{L^6_{x,t}} = \left\|\left(\eta(t)\, S(t)\, \mathbb{P}_N f\right)^3\right\|^2_{L^2_{x,t}}$$

really $\square\,\bar{\square}\,\square$

$$(*) \quad = \Big\| \sum_{n=n_1-n_2+n_3} \int_{\tau=\tau_1-\tau_2+\tau_3} \prod_{j=1}^{3} \hat{\eta}(\tau_j-n_j^2)\, \hat{g}(n_j)\, d\tau_1\, d\tau_2 \Big\|^2_{L^2_{n,\tau}}$$

$\hat{g} = \hat{f}\, \chi_{|n|\le N}$

<u>Note</u>:

$$\int_{\tau=\tau_1-\tau_2+\tau_3} \prod_{j=1}^{3} \hat{\eta}(\tau_j-n_j^2)\, d\tau_1\, d\tau_2$$

$$= \int \hat{\eta}(\tau_1-n_1^2)\, \Psi(\tau-\tau_1+n_2^2-n_3^2)\, d\tau_1$$

$$= \hat{\Psi}\left(\tau-(n_1^2-n_2^2+n_3^2)\right)$$

for some $\Psi,\ \hat{\Psi}\in \mathcal{S}(\mathbb{R})$

- Now, assume $n_1, n_3 \neq n$.

By Hölder,

$$\circledast \leq \Big\| \sum_{*} \hat{\Psi}(\tau - (n_1^2 - n_2^2 + n_3^2)) \Big\|_{L^\infty_{n,\tau}}$$

$$\times \Big\| \sum_{*} \hat{\Psi}(\tau - (n_1^2 - n_2^2 + n_3^2)) \prod_{j=1}^{3} |\hat{g}(n_j)|^2 \Big\|_{L^1_{n,\tau}}$$

$$* = \{ n = n_1 - n_2 + n_3 ,\ |n_j| \leq N \}$$

- $$(2^{nd} \text{ factor}) = C_{\hat{\Psi}} \sum_{n_1, n_2, n_3} \prod_{j=1}^{3} |g(n_j)|^2 \lesssim \| \mathbb{P}_N f \|^6_{L^2_x}$$

$$\int \hat{\Psi}(\tau - (n_1^2 - n_2^2 + n_3^2)) d\tau < \infty$$

- $$\Big\| \sum_{*} \hat{\Psi}(\tau - (n_1^2 - n_2^2 + n_3^2)) \Big\|_{L^\infty_{n,\tau}} \sim \sup_{n,\tau} \sum_{n_1, n_2} \hat{\Psi}(\tau - (n_1^2 - n_2^2 + n_3^2))$$

$$\sim \sup_{n,\tau} \# \{ (n_1, n_2) \in \mathbb{Z}^2 \mid 2(n_2 - n_1)(n_2 - n_3) = C_{n,\tau} + O(1) \}$$

$$\left(\begin{array}{l} \because \ \tau - (n_1^2 - n_2^2 + n_3^2) = O(1) \\ \Rightarrow 2(n_2 - n_1)(n_2 - n_3) = \underbrace{-\tau + n^2}_{\text{fixed}} + O(1) \end{array} \right.$$

Note: $|n| \leq 3N$
$|\tau| \lesssim N^2$.

Given $n \in \mathbb{Z}$, $\tau \in \mathbb{R}$,

$\exists$ at most $O(1)$ many integer k s.t.

$$|k - C_{n,\tau}| \lesssim 1$$

$\Rightarrow$ $|k| = |-\tau + n^2| + O(1) \lesssim N^2$.

$\Rightarrow$ By divisor counting, $\exists$ at most N^{0+} many

$$\underbrace{n_2 - n_1}_{\substack{\shortparallel \\ -n+n_3}} \mid k \text{ and } \underbrace{n_2 - n_3}_{\substack{\shortparallel \\ -n+n_1}} \mid k.$$

$\Rightarrow$ at most $O(N^{0+})$ many $n_1, n_3 \Rightarrow n_2$.

Now, assume $n_1 = n$. $\Rightarrow$ $n_2 = n_3$

$$\circledast = \Big\| \sum_{n_2} \hat{\psi}(\tau - n^2)\, \hat{g}(n) \overline{\hat{g}(-n_2)}\, \hat{g}(-n_2) \Big\|^2_{L^2_{n,\tau}}$$

$$= \|\hat{\psi}\|^2_{L^2_\tau} \Big(\sum_{n_2} |\hat{g}(n_2)|^2 \Big)^2 \|\hat{g}(n)\|^2_{L^2_n}$$

$$\sim \|\mathbb{P}_N f\|^6_{L^2_x} \quad \boxtimes$$

<u>Remark</u>: Lemma 3 is <u>FALSE</u> if we replace N^{0+} by constant.

Indeed

$$\frac{1}{\sqrt{N}} \left\| \sum_{n=0}^{N} e^{2\pi i (nx + tn^2)} \right\|_{L^6_{x,t}} \to \infty \text{ as } N \to \infty$$

- $\sqrt{N} \sim \left\| \sum_{n=0}^{N} e^{2\pi i n x} \right\|_{L^2_x}$.
- Let $1 \le a < q < N^{1/2}$, $(a,q) = 1$, $0 \le b < q$

and

$$\left| x - \frac{b}{q} \right| = o\left(\frac{1}{N}\right)$$

$$\left| t - \frac{a}{q} \right| = o\left(\frac{1}{N^2}\right)$$

$\Rightarrow$ By theory of exponential sums:

$$\left| \sum_{n=0}^{N} e^{2\pi i (nx + tn^2)} \right| \sim N/\sqrt{q}$$

$\because$ Weyl's inequality.

$$\left| c - \frac{a}{q} \right| \le \frac{\gamma}{q^2}, \qquad \gamma \ge 1$$

f = poly of degree k with leading coeff c.

$\rightarrow \forall \varepsilon > 0$,

$$\sum_{n=M+1}^{M+N} e^{2\pi i f(n)} = O\left(N^{1+\varepsilon} \left(\frac{\gamma}{q} + \frac{1}{N} + \frac{\gamma}{N^{k-1}} + \frac{q}{N^k} \right)^{2^{1-k}} \right)$$

- Choose $k = 2$, $\gamma = 1$ $\Rightarrow$ $N^{1+\varepsilon}/\sqrt{q}$

$\therefore q < N^{1/2}$

Of course, this does not give a sharp lower bound but should give you some idea.

$\cdot$ Let $f(x,t) = \sum_{n=0}^{N} e^{2\pi i(nx + tn^2)}$ (14)

$$\Rightarrow \int_{\mathcal{M}_0(q,a,b)} |f|^6 \, dx\,dt \sim \frac{N^6}{q^3} \cdot \frac{1}{N^2} \cdot \frac{1}{N} = \frac{N^3}{q^3}$$

$$\Rightarrow \int |f|^6 \, dx\,dt \gtrsim \sum_{\substack{(a,q)=1 \\ a<q}} \frac{N^3}{q^2}$$ $\Leftarrow$ summed over b $0 \le b < q$.

$$\gtrsim (\log N) \cdot N^3 \Rightarrow \text{const} \gtrsim (\log N)^{1/6}$$

$\because$ Recall: Euler function

$$\varphi(q) = \# \{ a \le q : (a,q) = 1 \}.$$

and $\varphi(q) \geq \dfrac{q}{e^{\gamma} \log\log q}$ as $q \to \infty$

in the sense of lim inf.

Where γ = Euler's const $= -\Gamma'(1) = -\int_0^\infty e^{-x} \log x \, dx$.

$\left(\text{Also, recall } \prod_{\substack{p \le x \\ p,\ \text{prime}}} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log x}. \right)$

$$\Rightarrow \sum_{\substack{(a,q)=1 \\ a<q}} \frac{1}{q^2} \underset{\substack{\uparrow \\ \text{sum over } a<q}}{\gtrsim} \sum_{q<N^{1/2}} \frac{1}{q \log\log q} \sim \log N.$$

<u>Note:</u> $L^6_{x,t}$ Strichartz is the natural one for 1-d Schrödinger.

$$\frac{2+1}{6} = 1/2.$$

For KdV, $L^8_{x,t}$ Strichartz is the natural one but it is not known to hold on $\mathbb{T}$.

$$\frac{3+1}{8} = 1/2.$$

• <u>Sharper L^4 Strichartz on $\mathbb{T}$</u>

By Sobolev, we have

$$\| u \|_{L^4_{x,t}} \lesssim \| u \|_{X^{\frac{1}{4}, \frac{1}{4}}}, \quad \frac{1}{4} = \frac{1}{2} - \frac{1}{4}$$

"2 space derivatives = 1 time derivative"

$b = 3$ for KdV

$\Rightarrow$ <u>Lemma 4:</u> $\| u \|_{L^4_{x,t}(\mathbb{T}\times\mathbb{R})} \lesssim \| u \|_{X^{0, 3/8}}$

<u>Pf)</u> Write $u = \sum_M u_M$, M, dyadic.

where $u_M = u \big|_{\langle \tau - n^2 \rangle \sim M}$

$$\Rightarrow \sum_M M^{3/4} \| u_M \|^2_{L^2_{x,t}} \lesssim \| u \|^2_{X^{0,3/8}}$$

Tao's book following Tzvetkov.

(16)

<u>Note</u>: $\|u\|_{L^4_{x,t}}^2 = \|u\,u\|_{L^2} \lesssim \sum_M \sum_{M'} \|u_M u_{M'}\|_{L^2}$

$$\sum_M \sum_{M'} = \sum\sum_{M\le M'} + \sum\sum_{M'\le M}$$

$\Rightarrow$ Suffices to show

$$\sum\sum_{M\le M'} \|u_M u_{M'}\|_{L^2_{x,t}} \lesssim \sum_M M^{3/4} \|u_M\|_{L^2_{x,t}}^2$$

- Let $M' = 2^m M$, $m \ge 0$.

$\Rightarrow$ Suffices to show

summable over $m \ge 0$.

$$\sum_M \|u_M u_{2^m M}\|_{L^2_{x,t}} \lesssim 2^{-\varepsilon m} \sum_M M^{3/4} \|u_M\|_{L^2_{x,t}}^2$$

- By Cauchy-Schwarz,

suffices to show

$$\|u_M u_{2^m M}\|_{L^2_{x,t}} \lesssim 2^{-\varepsilon m} M^{3/8} \|u_M\|_{L^2} \times \underbrace{(2^m M)^{3/8} \|u_{2^m M}\|_{L^2}}_{\sim \|u_{2^m M}\|_{X^{0,3/8}}}$$

summable over M

$$\Rightarrow \sum_M \|u_M u_{2^m M}\|_{L^2} \underset{\text{c.s.}}{\lesssim} 2^{-\varepsilon m} \left(\sum_M M^{3/4} \|u_M\|_{L^2}^2\right)^{1/2} \times \left(\sum_M (2^m M)^{3/4} \|u_{2^m M}\|_{L^2}^2\right)^{1/2}$$

$$\le 2^{-\varepsilon m} \sum_M M^{3/4} \|u_M\|_{L^2_{x,t}}^2$$

Let $V_M = \dfrac{U_M}{\|U_M\|_{L^2_{x,t}}}$ and $V_{2^m M} = \dfrac{U_{2^m M}}{\|U_{2^m M}\|_{L^2_{x,t}}}$.

$\Rightarrow$ Suffices to show

$$\Big\| \sum_{n_1+n_2=n} \int_{\tau=\tau_1+\tau_2} \widehat{V}_M(n_1,\tau_1)\widehat{V}_{2^m M}(n_2,\tau_2)\,d\tau_1 \Big\|_{L^2_{n,\tau}} \lesssim 2^{(3/8-\varepsilon)m} M^{3/4}.$$

- Note:

$$\Big\| \Big(\sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} |\widehat{V}_M(n_1,\tau_1)|^2 |\widehat{V}_{2^m M}(n_2,\tau_2)|^2 d\tau_1 \Big)^{1/2} \Big\|_{L^2_{n,\tau}}$$

$$= \sum_{n_1}\int_{\tau_1} |\widehat{V}_M(n_1,\tau_1)|^2 \sum_{n}\int_{\tau} |\widehat{V}_{2^m M}(n-n_1,\tau-\tau_1)|^2$$

$= 1$ by normalization.

- By Cauchy-Schwarz, suffices to show

$$\boxed{\sum_{n=n_1+n_2} \int_{\substack{\tau=\tau_1+\tau_2 \\ \tau_1 = n_1^2 + O(M) \\ \tau_2 = n_2^2 + O(2^m M)}} 1 \, d\tau_1} \lesssim 2^{(\frac{3}{4}-2\varepsilon)m} M^{3/2}$$

for all n,τ. $\sim \displaystyle\sum_{\substack{n=n_1+n_2 \\ \tau = n_1^2+n_2^2+O(2^m M)}} 1$

- Fix n, τ.

We have

$$\tau = \tau_1 + \tau_2 = n_1^2 + n_2^2 + O(2^m M)$$
$$n = n_1 + n_2$$

$$\rightarrow (n_1 - n_2)^2 = n_1^2 + n_2^2 - 2n_1 n_2$$
$$= 2(n_1^2 + n_2^2) - (n_1 + n_2)^2$$
$$= 2\tau - n^2 + O(2^m M)$$

$\Rightarrow$ $2^{m/2} M^{1/2}$ many choices for $n_1 - n_2$

$\Rightarrow$ n_1 and n_2 since $n = n_1 + n_2$ ← fixed.

$\Rightarrow$ choose $\varepsilon = 1/8$

$\boxtimes$ power of M gives the condition $b \geq 3/8$

③ Well-Posedness on $L^2(\mathbb{T})$

$$\begin{cases} iu_t - u_{xx} \pm |u|^2 u = 0 \\ u|_{t=0} = u_0 \end{cases}$$

η; η_T, $T < 1$

$$\Rightarrow u(t) = S(t) u_0 \pm i \int_0^t S(t-t') |u|^2 u(t') dt'$$

$\Rightarrow$ For $b \in (\frac{1}{2}, \frac{5}{8}]$,

$$\|u\|_{X^{s,b}} \lesssim \|u_0\|_{H^s} + T^\theta \| |u|^2 u \|_{X^{s,b'}}$$

$b' = b - 1 +$

NOT really needed.

For simplicity, let $b = 5/8$, $b' = b - 1 = -3/8$

Let $\|v\|_{X^{0,\frac{3}{8}}} \leq 1$.

Then,

$$\| u_1 \overline{u_2} u_3 \|_{X^{s,b'}} \sim \int (\langle \partial_x \rangle^s u_1) \overline{u_2} u_3 v \, dx dt$$

assuming u_1 has the largest spatial freq.

By $L^4_{x,t} L^4_{x,t} L^4_{x,t} L^4_{x,t}$ Hölder and L^4-Strichartz,

$$\leq \prod_{j=1}^{3} \| u_j \|_{L^4_{x,t}} \|v\|_{L^4_{x,t}} \quad (\text{for } s = 0.)$$

$$\lesssim T^{\theta} \prod_{j=1}^{3} \| u_j \|_{X^{0,3/8}}$$

from time localization

Do the same for

$$u(t) - v(t) = \int_0^t S(t-t') \left(|u|^2 u - |v|^2 v \right)(t') dt'.$$

④ Ill-posedness below $L^2(\mathbb{T})$.

(a) Failure of smoothness of solution map.

Consider

$$(*) \quad \begin{cases} i u_t - u_{xx} \pm |u|^2 u = 0 \\ u|_{t=0} = \delta \phi \end{cases}$$

for some $\phi \in H^s(\mathbb{T})$, $s < 0$.

Note: $u(x,t;\delta)$ denotes the soln of $(*)$

$\Rightarrow u(x,t;0) \equiv 0$.

Let $\Phi(t)$ denote the nonlin soln map

$$\Phi(t): u_0 \in H^s \to u(t) \, H^s.$$

- Suppose $\Phi(t)$ is well-defined and C^k smooth for some small $t > 0$. Then, by the smoothness around the zero soln, we have

$$\Phi(t)\big(\underbrace{u_0(\delta)}_{=\delta\phi}\big) = \Phi(t)\big(\underbrace{u_0(\delta=0)}_{\equiv 0}\big) + \nabla\Phi(t)(0)\cdot\delta\phi$$

$$+ \frac{1}{2}\nabla^2\Phi(t)(0)\,|\delta\phi|^2 + \cdots$$

$$\Rightarrow \Big\| \frac{d^k}{d\delta^k}\Phi(t)(\delta\phi)\Big|_{\delta=0}\Big\|_{H^s} \lesssim \underbrace{\|\nabla^k\Phi(t)(0)\|_{H^s\to H^s}}_{\lesssim \text{ const}} \|\phi\|_{H^s}^k$$

$$\partial_\delta^k u\big|_{\delta=0} \quad (\shortparallel)$$

i.e. $\left\| \left(\partial_\delta^k \Phi(t)(\delta\phi) \right)\Big|_{\delta=0} \right\|_{H^s} \lesssim \|\phi\|_{H^s}^k$

(Note: can also see this from chain rule.

$$\partial_\delta^k \Phi(t)(\delta\phi) = \nabla^k \Phi(t)(\delta\phi)\, \phi^k$$
)

- Recall: $u(t) = \delta S(t)\phi \pm i \int_0^t S(t-t') \underbrace{|u|^2 u(t')}_{\text{depends on } \delta}\, dt'$

$$\Rightarrow \frac{\partial u}{\partial \delta}\Big|_{\delta=0} = S(t)\phi =: \Phi_1$$

- $\dfrac{\partial^2 u}{\partial \delta^2}\Big|_{\delta=0} \sim \displaystyle\int_0^t S(t-t')\, \partial_\delta^2(|u|^2 u)\Big|_{\delta=0} dt' = 0$

$$\left(\because\ \partial_\delta^2 (|u|^2 u)\Big|_{\delta=0} = 2u_\delta^2 \bar{u} + 2u\, u_{\delta\delta}\, \bar{u} + 4u\, u_\delta \bar{u}_\delta + u^2 \bar{u}_{\delta\delta}\Big|_{\delta=0} = 0 \right)$$

- $\dfrac{\partial^3 u}{\partial \delta^3}\Big|_{\delta=0} \sim \displaystyle\int_0^t S(t-t')\, u_\delta^2\, \bar{u}_\delta\Big|_{\delta=0} dt'$

$$= \int_0^t S(t-t')\, |\Phi_1|^2 \Phi_1(t')\, dt' =: \Phi_3$$

- Now, choose $\phi = N^{-s} e^{iNx}$ i.e. $\hat{\phi}(N) = N^{-s}$.

$$\Rightarrow \|\phi\|_{H^s} \sim 1.$$

$$\Rightarrow \Phi_1 = N^{-s} e^{iNx + iN^2 t}$$

$$\cdot\ |\Phi_1|^2\Phi_1(t') = N^{-3s} e^{iNx + iN^2 t'}$$

$$\Rightarrow S(t-t')|\Phi_1|^2\Phi_1(t') = \underbrace{N^{-3s} e^{iNx+iN^2 t}}_{\text{indep of } t'}$$

$$\Rightarrow \Phi_3 \sim t\cdot N^{-3s} e^{iNx+iN^2 t}.$$

$$\Rightarrow \|\Phi_3\|_{H^s} \sim N^{-2s} \lesssim \|\phi\|^3_{H^s} \sim 1$$

$$\Rightarrow s \geq 0$$

(b) Failure of uniform continuity

Construct a family of (pairs of) <u>smooth</u> initial data (and solutions) such that

Given any $\underset{\text{small}}{t} > 0$ and $\delta > 0$,

$\exists\ u_0$ and v_0 s.t.

$$\|u_0 - v_0\|_{H^s} < \delta \quad \text{but} \quad \|u(t) - v(t)\|_{H^s} \gtrsim 1.$$

On $\mathbb{R}$: focusing: Kenig-Ponce-Vega '01

family of soliton solns with parameters.

defocusing: Christ-Colliander-Tao '03.

family of approximate solns.

(uses the modified scattering asymptotics by Ozawa.)

On $\mathbb{T}$: Burq-Gérard-Tzvetkov '02

Consider
$$u_{N,a} = a\, e^{i(Nx + N^2 t \pm |a|^2 t)}$$

for $a \in \mathbb{C}$ and $N \in \mathbb{N}$.

- $u_{N,a}$ solves the cubic NLS.
- Choose $a = N^{-s}\alpha$ and $a' = N^{-s}\alpha'$.

$\Rightarrow \quad \| u_{N,a}(0) - u_{N,a'}(0) \|_{H^s} = \mathcal{O}(|\alpha - \alpha'|)$

- $\| u_{N,a}(t) - u_{N,a'}(t) \|_{H^s}$
$$= \left| \alpha\, e^{\pm i N^{-2s}\alpha^2 t} - \alpha' e^{\pm i N^{-2s}(\alpha')^2 t} \right|$$
$$\sim |\alpha| + |\alpha'| \quad \text{for } N^{-2s}|\alpha^2 - (\alpha')^2| t \sim \pi$$

Choose $\alpha = 1$, $\alpha' = 1 + \delta$.

Then, t can be made small by taking N suff. large as long as $s < 0$.

Remark: As $N \to \infty$, the momentum $\to \infty$.
$$u_0 = N^{-s}\alpha\, e^{iNx}$$
$$i \int u \overline{u}_x = i \int u_0\, \partial_x \overline{u_0} = 2\pi \alpha^2 N^{1-2s} \to \infty$$

- Even when we renormalize u s.t. $\|u\|_{L^2} \sim 1$, the momentum still goes to ∞.

(c) Discontinuity

C-C-T '03 (on arXiv): $s<0$

NOT continuous: $H^s(\mathbb{T}) \to (C^\infty(\mathbb{T}))^*$

space of distributions

Thm: $\forall \rho > 0$

$\exists$ soln u s.t. $\|u(0)\|_{H^s} < \rho$

s.t. $\forall \delta > 0$ $\exists$ soln $\tilde{u}$ satisfying

$$\| u(0) - \tilde{u}(0)\|_{H^s(\mathbb{T})} < \delta$$

but $\displaystyle\sup_{0\le t\le \delta} \left| \widehat{u(t)}(0) - \widehat{\tilde{u}(t)}(0)\right| > c\rho$.

Idea: BGTz: soln supported on one mode

CCT: soln supported on two modes

- Choose $u(t,x) = \rho' e^{\pm i|\rho'|^2 t}$, $\rho' = \frac{1}{4}\rho$

and $\tilde{u}(0,x) = \rho' + \frac{1}{4}\delta N^{-s} e^{iNx}$.

- Use a nonhomog. lin. flow to approximate the flow of $\tilde{u}$.

Note: momentum $\to \infty$ as $N \to \infty$.

Aside: Galilean transf $v(x,t) = e^{i\frac{\alpha}{2}x} e^{-i\frac{\alpha^2}{4}t} u(t, x-\alpha t)$

$\Rightarrow i\int v\bar{v}_x = \frac{\alpha}{2}\int |u|^2 + i\int u u_x$

momentum is NOT preserved.

- Molinet '09 : $iu_t - u_{xx} + \gamma |u|^2 u = 0.$ (25)
- discontinuity of the map : $u_0 \in H^s(\mathbb{T}) \to u(t) \in H^s$ $s<0$.
  at any $u_0 \in L^2(\mathbb{T})$, $(u_0 \neq 0)$ and for any $t \neq 0$.
- Remark : CCT : discontinuity of the solution map
  Molinet : discontinuity of the flow map
  (for each fixed $u_0 \in L^2$.)

Main feature: He proved :

- the flow map is not continuous as a map from $L^2(\mathbb{T})$ with weak topology into the space of distributions $(C^\infty(\mathbb{T}))^*$ at any nonzero $u_0 \in L^2(\mathbb{T})$
- does not exploit the existence of exact or approximate solutions.

- Theorem 1 : $\{u_{0,n}\} \subset L^2(\mathbb{T})$
  converging weakly but not strongly to u_0 in $L^2(\mathbb{T})$
  - $\forall\, \alpha^2 \neq \int |u_0|^2$, limit pt of $\int |u_{0,n}|^2$

  and $\forall\, n_k$ s.t. $\int |u_{0,n_k}|^2 \to \alpha^2$,

  u_{n_k} converges weakly in $L^2(\mathbb{T})$ to
  $$v_\alpha(t) = \exp\left(\frac{i\gamma}{\pi} \left(\alpha^2 - \int |u_0|^2 \right) t \right) u(t)$$
  $(\neq u)$

- $u_{0,n} \rightharpoonup u_0$ in $L^2(\mathbb{T})$
but not strongly.

i.e. $\forall \varphi \in L^2(\mathbb{T})$,

$$\langle u_{0,n} - u_0, \varphi \rangle \to 0$$

but

$$\sup_{\varphi} \langle u_{0,n} - u_0, \varphi \rangle \not\to 0$$

- Weak conv & conv in norm $\Rightarrow$ strong conv.
Hence, in Thm 1,
$\exists$ at least one limit pt $\alpha^2 \neq \int |u_0|^2$.

$$\left(\because \begin{array}{l} x_n \rightharpoonup x \\ \|x_n\| \to \|x\| \end{array} \Rightarrow \begin{aligned} \|x - x_n\|^2 &= \langle x - x_n, x - x_n \rangle \\ &= \langle x, x \rangle + \langle x_n, x_n \rangle \\ &\quad - \langle x_n, x \rangle - \langle x, x_n \rangle \to 0. \end{aligned} \right.$$

- $x_n \rightharpoonup x \Rightarrow \|x\| \le \liminf \|x_n\|$.
$\sup_n \|x_n\| < \infty$ (Banach-Steinhaus)

Basically Rellich's lemma

- weak conv in $L^2(\mathbb{T})$
$\Rightarrow$ strong conv in $H^s(\mathbb{T})$, $s < 0$.

$\because$ Suppose $f_n \rightharpoonup 0$ in L^2.

Fix $\varepsilon > 0$, $\|f_n\|_{H^s} = \sup_{\|\varphi\|_{L^2}=1} \langle \langle \partial_x \rangle^s f_n, \varphi \rangle$

$$\le \underbrace{\sup_{\|\varphi\|_{L^2}=1} \sum_{|k|<K}}_{\le \sum_{|k|<K} \langle f_n, e_k \rangle} + \underbrace{\sup_{\|\varphi\|_{L^2}=1} \sum_{|k|>K} \frac{1}{(1+|k|)^{-s}} \hat{f}_n(k) \overline{\hat{\varphi}(k)}}_{\le \frac{1}{K^{-s}} \|f_n\|_{L^2} \|\varphi\|_{L^2} \le \frac{1}{K^{-s}} < \frac{1}{2}\varepsilon}$$

Choose n large s.t. $\langle f_n, e_k \rangle \le \frac{1}{4K}\varepsilon$, $\forall |k| < K$

$\le \frac{1}{2}\varepsilon$.

· __Cor 2:__ $\forall\, t \neq 0$, the flow map is

$$\text{disconti}: \underset{\substack{\text{weak topology}\\ \text{at any } u_0 \neq 0.}}{L^2(\mathbb{T}) \setminus \{0\}} \longrightarrow \left(C^\infty(\mathbb{T})\right)^*.$$

· __Remark__: Let $L^2(\mathbb{T}) \subset\subset B \subset \left(C^\infty(\mathbb{T})\right)^*$

e.g. $B = H^s$, $s < 0$.

Then, $u_0 \longmapsto u(t)$ is discontinuous from B into B.

$$\left(\begin{array}{lcl} \because\ u_{0,n} \text{ in } L^2 \rightharpoonup u_0 & \overset{\text{Cor 2}}{\Longrightarrow} & u_n(t) \not\rightharpoonup u(t) \\ \quad\Downarrow & & \quad\Downarrow \\ u_{0,n} \to u_0 \text{ in } B & & \|u_n(t) - u(t)\|_B \not\to 0. \end{array}\right.$$

· __Pf of Cor 2:__ · $u_0 \neq 0$ in $L^2(\mathbb{T})$.

· $\{\phi_n\} \subset L^2(\mathbb{T})$ s.t. $\phi_n \rightharpoonup 0$ in L^2

$$\|\phi_n\|_{L^2}^2 \to 2\pi\,\theta^2, \quad \theta \in \mathbb{R}^*.$$

e.g. $\phi_n = \theta e^{inx}$.

· Let $u_{0,n} = u_0 + \phi_n$.

Then,

· $u_{0,n} \rightharpoonup u_0$ in $L^2(\mathbb{T})$

· $\|u_{0,n}\|_{L^2}^2 \to \|u_0\|_{L^2}^2 + 2\pi\,\theta^2$.

$$\left(\begin{array}{l} \quad\| \\ \langle u_0 + \phi_n, u_0 + \phi_n \rangle \\ \quad \text{but } \langle u_0, \phi_n \rangle \to 0. \end{array}\right.$$

- By Thm 1, $u_n \rightharpoonup v(t,x) = e^{i2\gamma\theta^2 t} u(t,x)$

  i.e. $v - u = (e^{i2\gamma\theta^2 t} - 1)\, u$.

By uniqueness of solns, $u(t) \neq 0$ since $u_0 \neq 0$.

$\Rightarrow v(t) \neq u(t)$ in $(C^\infty(\mathbb{T}))^*$, $\forall\, t \in \left\{\frac{k\pi}{\gamma\theta^2},\, k \in \mathbb{Z}\right\}$

Now, given $t \neq 0$, choose suitable θ.

$\boxtimes$

- Remark: $u_{0,n} = u_0 + \phi_n = u_0 + \theta e^{inx}$.

$$i\int u_{0,n}\,\overline{(u_{0,n})_x} = i\int (u_0 + \phi_n)\left(\overline{(u_0)_x} - in\theta e^{-inx}\right)$$

$$= i\int u_0 \overline{(u_0)_x} + i\int \phi_n \overline{(u_0)_x} + i\int u_0 \overline{(\phi_n)_x} + \underbrace{2\pi n\theta}_{\to\infty}$$

$\to 0$ for $u_0 \in H^1$

- $g(u_1, u_2, u_3) = u_1 \overline{u_2} u_3$

$$= \sum_{n_1, n_2, n_3 \in \mathbb{Z}} \widehat{u}_1(n_1) \overline{\widehat{u}_2(n_2)} \widehat{u}_3(n_3) e^{i(n_1 - n_2 + n_3)x}$$

$$\left.\begin{aligned} &= \sum_{n_1, n_3} \widehat{u}_1(n_1) \overline{\widehat{u}_2(n_1)} \widehat{u}_3(n_3) e^{i n_3 x} \\ &+ \sum_{n_1, n_3} \widehat{u}_1(n_1) \overline{\widehat{u}_2(n_3)} \widehat{u}_3(n_3) e^{i n_1 x} \end{aligned}\right\} \text{resonant part}$$

$$- \sum_{n} \widehat{u}_1(n) \overline{\widehat{u}_2(n)} \widehat{u}_3(n) e^{inx} =: \Lambda_2(u_1, u_2, u_3)$$

$$+ \sum_{(n_2 - n_1)(n_2 - n_3) \neq 0} \widehat{u}_1(n_1) \overline{\widehat{u}_2(n_2)} \widehat{u}_3(n_3) e^{i(n_1 - n_2 + n_3)x} =: \Lambda_1(u_1, u_2, u_3)$$

- With $g(u) := g(u, u, u) = |u|^2 u$, we have

$$g(u) = \frac{1}{\pi} \|u\|_{L^2}^2 u + \Lambda_1(u) + \Lambda_2(u)$$

- Recall the trilinear estimate: $0 < T \leq 1$

$$\|\Lambda_j(u_1, u_2, u_3)\|_{X_T^{0, -7/16}} \lesssim \prod_{i=1}^{3} \|u_i\|_{X_T^{0, 1/2}}$$

$-3/8$ $3/8$

$\Rightarrow \Lambda_1$ and Λ_2 are continuous from $(X_T^{0, 1/2})^3$ into $X_T^{0, -7/16}$.

$|t| \leq T$

<u>Lemma 3:</u> Λ_j, $j=1,2$, is conti from $(X_1^{0,\frac{1}{2}})^3$ to $X_1^{0,-7/16}$ with their respective weak topology

---

- Assume Lemma 3
- $u_{0,n} \rightharpoonup u_0$ in $L^2(\mathbb{T})$ but not strongly.
- $\|u_{0,n}\|_{L^2} \leq C \Rightarrow \exists$ limit pt $\alpha \neq \|u_0\|_{L^2}$.

Let $\|u_{0,n_k}\|_{L^2} \to \alpha$.

- Note $\{u_{n_k}\}$ is bounded in $X_1^{0,1/2}$.

$\Rightarrow \exists$ subseq converging weakly to v in $X_1^{0,1/2}$.

- $$u_n(t) = S(t)u_{0,n} + i\gamma \int_0^t S(t-t')\left[\Lambda_1(u_n(t')) + \Lambda_2(u_n(t'))\right]dt'$$

$$+ \frac{i\gamma}{\pi} \underbrace{\|u_{0,n}\|_{L^2}^2}_{\uparrow\ L^2\text{-conservation}} \int_0^t S(t-t')\, u_n(t')\, dt'.$$

$$\Rightarrow v(t) = S(t)u_0 + i\gamma \int_0^t S(t-t')\left(\Lambda_1(v) + \Lambda_2(v)(t')\right)dt'$$

$$+ \frac{i\gamma}{\pi}\alpha^2 \int_0^t S(t-t')\, v(t')\, dt'.$$

i.e. v satisfies

$$(NLS_v) \quad \begin{cases} i v_t - v_{xx} + \gamma(\Lambda_1 + \Lambda_2)(v) + \frac{\gamma}{\pi}\alpha^2 v = 0 \\ v|_{t=0} = u_0. \end{cases}$$

$\because\ u_{0,n} \rightharpoonup u_0$ in $L^2(\mathbb{T})$

$\Rightarrow \| u_{0,n} - u_0 \|_{H^{-\varepsilon}} \to 0 \quad \forall\, \varepsilon > 0$

Take test func $\phi(x,t)$

$$\iint_{\Lambda} S(t)(u_{0,n} - u_0) \cdot \phi(x,t)\, dx\, dt$$

$$\le \| S(t)(u_{0,n} - u_0) \|_{X^{-\varepsilon, 1/2}} \underbrace{\| \phi \|_{X^{\varepsilon, -1/2}}}_{<\infty}$$

$$\lesssim \| u_{0,n} - u_0 \|_{H^{-\varepsilon}} \to 0$$

etc.

- (NLS_v) is globally well-posed and

$$v \in C([-T,T]; H^s) \cap L^4((-T,T)\times\mathbb{T}) \cap X_T^{0,1/2}, \quad s \ge 0$$

unique in $L^4_{x,T}$.

$\Rightarrow$ $\underbrace{\text{Any convergent}}_{\text{weakly}}$ subseq of $\{u_{n_k}\}$ $\underbrace{\text{converges}}_{\text{weakly}}$ to v

- $\{u_n\}$ unif bdd in $L^\infty((-T,T); L^2(\mathbb{T})) \cap L^4_{x,T}$

$\Rightarrow$ $\{ t \mapsto \langle u_{n_k}(t), \phi \rangle_{L^2} \}$ is bounded in $C([-1,1])$

and equicontinuous on $[-1,1]$ or just for each t

$\Rightarrow$ Ascoli-Arzelá: $\langle u_{n_k}, \phi \rangle$ converges to $\langle v, \phi \rangle$ on $[-1,1]$

i.e. $u_{n_k} \rightharpoonup v$ in $L^2(\mathbb{T})$

for all $t \in [-1,1]$

- (NLS_v) preserves L^2-norm.

$\Rightarrow$ v satisfies

$$\begin{cases} i v_t - v_{xx} + \gamma |v|^2 v + \frac{\gamma}{\pi}(\alpha^2 - \|u_0\|_{L^2}^2) v = 0 \\ v\big|_{t=0} = u_0 \end{cases}$$

- Finally, let $\hat{v}(x,t) = e^{-\frac{i\gamma}{\pi}(\alpha^2 - \|u_0\|_{L^2}^2)t} v(x,t)$

Then, $\hat{v} \in L^4_{x,T}$ and $\hat{v}$ satisfies (NLS_3) with $\hat{v}\big|_{t=0} = u_0$.

$\Rightarrow$ By uniqueness, $\hat{v} = u$. $\boxtimes$