

Singular stochastic dispersive PDEs. 17/02/21 (Wed) ①

• Dispersive PDEs

nonlinear Schrödinger eqn : u , $\mathbb{C}$ -valued

$$(NLS) \quad i\partial_t u - \Delta u + \underbrace{|u|^{k-1} u}_{\text{nonlinear term}} = 0, \quad k \in 2\mathbb{N}+1$$

nonlinear wave eqn : u , $\mathbb{R}$ -valued

$$(NLW) \quad \partial_t^2 u - \Delta u + \underbrace{u^k}_{\text{nonlinear term}} = 0$$

+ sign : defocusing case
($k \in 2\mathbb{N}+1$)
- sign : focusing case

Main goal: Understand how given initial data is propagated by nonlinear dynamics

1st question: Well-posedness (existence, uniqueness,
stability under perturbation)

local well-posedness . i.e for short times.

$\xrightarrow{\text{LWP}}$ Q: global well-posedness or finite time blowup?

Stochastic dispersive PDEs

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SNLS

$$i \partial_t u - \Delta u + |u|^{k-1} u = \phi \xi$$

← stoch forcing

SNLW

$$\partial_t^2 u - \Delta u + u^k = \xi$$

↖ bdd op on L^2
(smoothing operator in x)

SdNLW (stochastic damped NLW):

$$\partial_t^2 u + \underline{\partial_t u} - \Delta u + u^k = \xi$$

In terms of LWP, SNLW and SdNLW are the "same".

$\xi =$ space-time white noise $\xi(t, x)$
(Gaussian)

$$\mathbb{E}[\xi(t_1, x_1) \xi(t_2, x_2)] = \delta(t_1 - t_2) \delta(x_1 - x_2)$$

⇐ random behavior at diff pts are indep

⇒ very rough: reg $-\frac{d}{2} - \varepsilon$ in x
 $-\frac{1}{2} - \varepsilon$ in t .

⇒ makes the relevant analysis challenging. ③

① analytically challenging (need analysis, PDEs, stoch analysis)

SNLS: 1-d cubic, $\phi = \text{Id}$ (i.e. space-time white noise)

LWP: OPEN, critical

SNLW: 3-d cubic is open.

② SdNLW formally preserves the Gibbs measure
measure on functions/distributions

Stochastic nonlinear heat eqn
(reaction-diffusion eqn)

$$\partial_t u - \Delta u + u^k = \xi.$$

formally preserves Φ_d^{k+1} -measure: $dp = Z^{-1} e^{-\frac{1}{k+1} \int u^{k+1} dx}$
 $\times e^{-\frac{1}{2} \int |u|^2 dx} du$
Gaussian free field.

⇐ Euclidean QFT (quantum field theory)
constructive QFT

④

'70 - '80 : construction of Φ_d^{k+1} - measures

$k+1$ = deg of nonlinearity
 d = spatial dimension

$d=1, 2$; any $k \in 2\mathbb{N}+1$

$-\frac{1}{k+1} \int u^{k+1} dx$
↑ good sign

$d=2$ requires
renormalization

$d=3$: $k=3$ only Φ_3^4 - measure

• stochastic quantization : Introduce a stochastic PDE which preserves Φ_d^{k+1} - measure. (Parisi - Wu)

⇒ SNLH : $d=1$: easy
(LWP) $d=2$: Da Prato - Debussche '03

$d=3$ ($h=3$) SQE = stoch quantization eqn

$$\partial_t u + (1-\Delta)u + \underbrace{u^3 - \infty \cdot u}_{\text{renormalization}} = \xi$$

- Haïrér 2014 : LWP of dynamical Φ_3^4 -model
(parabolic Φ_3^4 -model)
- regularity structures

- Gubinelli - Imkeller - Perkowski 2015

- paracontrolled distributions

- Kupiainen 2016

- RG method (renormalization group method)

Wave analogue: $\partial_t^2 u + \partial_t u + (1-\Delta)u + u^3 - \infty \cdot u = \xi$

↑
hyperbolic Φ_3^4 -model $(u, \partial_t u) \sim \underbrace{\Phi_3^4\text{-measure}}_u \otimes \underbrace{\text{white noise}}_{\partial_t u}$ formally invariant

So far, we discussed results on $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$

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$\Leftarrow$ spatial roughness is the issue.

On $\mathbb{R}^d$: noise does not decay as $|x| \rightarrow \infty$.
soln, not integrable in the WSP-sense.
heat: Mourrat - Weber '17 A.P.
Hofmannova - Gubinelli
SNLW: 2-d cubic. Tolomeo A.P.

SNLS: $i\partial_t u - \Delta u + |u|^{k-1}u = \Phi \xi$

Ito: $i du = (\Delta u - |u|^{k-1}u)dt + \Phi dW$

$W(t, x) = L^2$ -cylindrical Wiener process.

$$= \sum_{n \in \mathbb{Z}^d} \beta_n(t) e^{in \cdot x}$$

$\{\beta_n\}_{n \in \mathbb{Z}^d} =$ independent standard $\mathbb{C}$ -valued B.M.

$\text{Var } \beta_n(t) = t$

$\beta_n = \text{Re } \beta_n + i \text{Im } \beta_n$

$\nwarrow \nearrow$
indep $\mathbb{R}$ -valued B.M.

SNLW, SNLH.

impose $\beta_{-n} = \overline{\beta_n}$

Mild formulation: (Duhamel formulation)

$$u(t) = S(t)u_0 - \int_0^t S(t-t') |u|^{k-1} u(t') dt' + \int_0^t S(t-t') \Phi dW(t')$$

$S(t) = e^{-it\Delta} =$ linear Schrödinger propagator

stoch. convolution.

• Stoch convolution (for SNLS)

⑧

$$\Psi(t) = \int_0^t S(t-t') \phi dW(t')$$

$$= \sum_{n \in \mathbb{Z}^d} e_n \int_0^t \underbrace{e^{i(t-t')|m|^2} \phi_n d\beta_n(t')}_{\text{Wiener integral}}$$

if ϕ is 'diagonal'

$$\phi(e_n) = \phi_n e_n$$

i.e. Fourier multiplier

$$W = \sum_n \beta_n e_n$$

$$e_n(x) = e^{in \cdot x}$$

$$S(t) = e^{-it\Delta}$$

$$\widehat{S(t)f(n)} = e^{it|m|^2} \widehat{f(m)}$$

• Wiener integral: ($\mathbb{R}$ -valued case)

Kuo: intro to stoch integ

Given $f \in L^2([a, b])$,

f , deterministic

$$I(f) = \int_a^b f(t) dB(t)$$

$B = \text{B.M.}$

$\Leftarrow$ • mean 0 Gaussian r.v. ($\mathbb{E}[I(f)] = 0$)

$$\begin{aligned} \bullet \text{Var}(I(f)) &= \|f\|_{L^2([a, b])}^2 \\ &\stackrel{||}{=} \mathbb{E}[|I(f)|^2] \end{aligned}$$

$\Leftarrow$ check for step functions
 and approximate f in $L^2(\Omega)$

• $B(0) = 0$ (9)

• indep increment
over disjoint intervals

• $B(t_2) - B(t_1)$
 $\sim N(0, \underline{t_2 - t_1})$

→ If f is C^1 , we can define it as
 a Paley-Wiener-Zygmund integral

$$I(f) = \int_a^b f dB = - \int_a^b f'(t) B(t) dt \quad \text{pathwise}$$

• I is an isometry from $L^2([a, b])$ onto its image in $L^2(\Omega)$

Kolmogorov continuity criterion: $\{X_t\}$ with values in a metric space

Suppose $E [d(X_s, X_t)^p] \leq C_0 |s - t|^{1+\alpha}$

Then, for some $p, \alpha > 0$

$$P \left(\sup_{s \neq t} \frac{d(X_s, X_t)}{|s - t|^{\alpha/p - \varepsilon}} \geq \lambda \right) \leq \frac{C_1}{\lambda^p} \quad \forall 0 < \varepsilon < \frac{\alpha}{p}$$

i.e. X_t is a.s. $(\frac{\alpha}{p} - \varepsilon)$ -Hölder continuous.

Ex: B.M.

$$\mathbb{E} [|B(t_2) - B(t_1)|^2] = t_2 - t_1$$

$$\Rightarrow \mathbb{E} [|B(t_2) - B(t_1)|^p] \sim (t_2 - t_1)^{p/2} \quad // 1+\alpha$$

$$\frac{\alpha}{p} - = \frac{\frac{p}{2} - 1}{p} = \frac{1}{2} - \frac{1}{p} \rightarrow \frac{1}{2} - \quad \text{as } p \rightarrow \infty$$

i.e. BM is a.s. $(\frac{1}{2} - \varepsilon)$ -Hölder.

Prop: $\Phi \in \text{HS}(L^2; H^s)$

$$\|\Phi\|_{\text{HS}(X; Y)} = \left(\sum_{n \in \mathbb{Z}^d} \|\Phi(e_n)\|_Y^2 \right)^{1/2}$$

Then, $\Psi \in C^{\frac{\alpha}{2}-}_t W^{s-d, r}_x(\mathbb{T}^d)$ *slight loss of regularity*
 $\implies$ a.s. $r \leq \infty, \alpha > 0$

$\{e_n\}$ O.N.B. of X

Also,

$r=2$:

$\Psi \in C_t H^s_x$, a.s.

Banach setting:
 γ -radonifying op.
Appendix: Forlano-Oh-Wang
on SNLS.

$$\frac{\alpha}{2} - = \frac{\alpha}{2} - \varepsilon \quad \text{for any small } \varepsilon > 0$$

(1)

Sobolev spaces (Bessel potential space L_s^p)

$$\|f\|_{W^{s,p}} = \|\langle \nabla \rangle^s f\|_{L^p}$$

$$\langle \nabla \rangle = \sqrt{1 - \Delta}$$

$$= \|\mathcal{F}^{-1}(\langle m \rangle^s \hat{f}(m))\|_{L^p}$$

$$\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$$

↑ Japanese bracket.

$p=2$: $W^{s,2} = H^s$

$$\|f\|_{H^s} = \left(\sum_{n \in \mathbb{Z}^d} \langle m \rangle^{2s} |\hat{f}(m)|^2 \right)^{1/2}$$

Hölder-Besov space : $\mathcal{C}^s = B_{\infty,\infty}^s$

Besov space : $B_{p,q}^s$

$s \in \mathbb{R}$
($1 \leq p \leq \infty$)

Lec 2. 24/02/21 (Wed)

①

Stochastic convolution (for SNLS)

$$\mathbb{I}(t) = \int_0^t S(t-t') \phi \, dW(t')$$

$$= \sum_{n \in \mathbb{Z}^d} e_n \int_0^t \underbrace{e^{i(t-t')|n|^2} \phi_n}_{\text{Wiener integral}} d\beta_n(t')$$

$$S(t) = e^{-it\Delta}$$

$$\widehat{S(t)f}(n) = e^{it|n|^2} \widehat{f}(n)$$

$$\phi(e_n) = \widehat{\phi}_n e_n$$

Prop: $s \in \mathbb{R}$, $\phi \in HS(L^2(\mathbb{T}^d); H^s(\mathbb{T}^d))$

Then, $\mathbb{I} \in C_t^{\frac{\alpha}{2}-} W_x^{s-\alpha, r}$ a.s.
 $\forall r \leq \infty, \alpha \geq 0$

$r=2$: $\mathbb{I} \in C_t H_x^s$ a.s.

$$\begin{aligned} \|\phi\|_{HS(L^2; H^s)} &= \left(\sum_{n \in \mathbb{Z}^d} \|\phi(e_n)\|_{H^s}^2 \right)^{1/2} \\ &= \left(\sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2s} |\widehat{\phi}_n|^2 \right)^{1/2} \end{aligned}$$

On $\mathbb{R}^d$: $\mathbb{I} \in L_t^q W_x^{s, r}$ $q \neq \infty, r \leq \frac{2d}{d-2}$
 $0 \leq t \leq \tau$

de Bouard-Debussche '03
 Oh-Pocovnicu-Wang
 Kyoto '20.

pf: $t \leq T$

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$$\mathbb{E} [\Psi(x, t) \overline{\Psi(y, \tau)}] = \sum_{n \in \mathbb{Z}^d} |\hat{\Phi}_n|^2 e_{n(x-y)} \underbrace{\int_0^t e^{i(t-t')|n|^2} e^{-i(\tau-t')|n|^2} dt'}_{= \int_0^t e^{i(t-\tau)|n|^2} dt'}$$

$\Leftarrow$ Apply $\langle \nabla_x \rangle^s$ and $\langle \nabla_y \rangle^s$

$$\langle \nabla \rangle = \sqrt{1 - \Delta}$$

$$\widehat{\langle \nabla \rangle^s f(m)} = \langle m \rangle^s \hat{f}(m)$$

$$= \int_0^t e^{i(t-\tau)|n|^2} dt'$$

$$= \underline{t e^{i(t-\tau)|n|^2}}$$

$$\Rightarrow \mathbb{E} [\langle \nabla_x \rangle^s \Psi(x, t) \cdot \overline{\langle \nabla_y \rangle^s \Psi(y, \tau)}] = \sum_{n \in \mathbb{Z}^d} |\hat{\Phi}_n|^2 \underbrace{\langle m \rangle^{2s}}_{\text{HS norm}} e_{n(x-y)} \cdot \underline{t e^{i(t-\tau)|n|^2}}$$

Set $x=y, t=\tau$

$$\begin{aligned} \bullet \mathbb{E} [|\langle \nabla \rangle^s \Psi(x, t)|^p] &\leq p^{p/2} \left(\mathbb{E} [|\langle \nabla \rangle^s \Psi(x, t)|^2] \right)^{p/2} \\ &= p^{p/2} t^{p/2} \|\Phi\|_{H^s(L^2; H^s)}^p \end{aligned}$$

$= 1$
when $x=y$
 $t=T$.

$\underline{p \geq r}$
 $\underline{r < \infty}$

$$\frac{\|\|\Psi(t)\|_{W_x^{s,r}}\|_{L^p(\Omega)}}{\|\langle \nabla \rangle^s \Psi(x, t)\|_{L_x^r}} \stackrel{\text{Mink ineq}}{\leq}$$

$$\|\|\langle \nabla \rangle^s \Psi(x, t)\|_{L^p(\Omega)}\|_{L_x^r} \stackrel{\text{Mink ineq}}{\leq} p^{1/2} t^{1/2} \|\Phi\|_{H^s(L^2; H^s)} \cdot \underline{\neq p \geq r}$$

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$$\Leftarrow \cdot \Psi(t) \in W_x^{s,r}(\mathbb{T}^d), \text{ a.s. } (r < \infty)$$

• Here, we used the fact $|\mathbb{T}^d| \sim 1$.

• When $r = \infty$,

$$\| \Psi(t) \|_{W_x^{s,\infty}} \| \cdot \|_{L^p(\Omega)} \stackrel{\text{Sobolev}}{\lesssim} \| \Psi(t) \|_{W_x^{s+\varepsilon,r}} \| \cdot \|_{L^p(\Omega)} \quad \underline{\varepsilon r > d}$$

$$\stackrel{\text{for } p \geq r}{\lesssim} p^{1/2} t^{1/2} \| \Phi \|_{HS(L^2; H^{s+\varepsilon})}.$$

In other words, $\Psi(t) \in W_x^{s-\varepsilon,\infty}$, if $\Phi \in HS(L^2; H^s)$

$$\begin{aligned} & P(\| \Psi(t) \|_{W_x^{s,r}} > \lambda) \\ & \leq C e^{-c \lambda^2 / t \| \Phi \|_{HS(L^2; H^s)}} \end{aligned}, \quad r < \infty$$

• Sobolev embedding thm

$$\| f \|_{L^\infty} \lesssim \| f \|_{W^{s,r}} \quad sr > d$$

• Sobolev inequality

$$\| f \|_{L^q} \lesssim \| f \|_{\dot{W}^{s,p}}$$

→ Grafakos
Benyi-Oh
 f , mean 0
on $\mathbb{T}^d$.

$$\frac{s}{d} = \frac{1}{p} - \frac{1}{q} \quad 1 < p < q < \infty$$

(4)

Given $h \in \mathbb{R}$, let

$$\delta_h \Psi(x, t) = \Psi(x, t+h) - \Psi(x, t)$$

$$\begin{aligned} \Rightarrow \mathbb{E} [\delta_h \Psi(x, t) \overline{\delta_h \Psi(y, t)}] \\ = \mathbb{E} [\Psi(x, t+h) \overline{\Psi(y, t+h)}] \\ - \mathbb{E} [\Psi(x, t) \overline{\Psi(y, t+h)}] \\ - \mathbb{E} [\Psi(x, t+h) \overline{\Psi(y, t)}] \\ + \mathbb{E} [\Psi(x, t) \overline{\Psi(y, t)}] \end{aligned}$$

$$\begin{aligned} t &> 0 \\ h &> 0 \end{aligned}$$

$$= \sum_{n \in \mathbb{Z}^d} |\hat{\Phi}_n|^2 e_n(x-y) \underbrace{\left\{ (t+h) - t e^{-i h |n|^2} - t e^{+i h |n|^2} + t \right\}}$$

$$=: F_n(t, h) \rightarrow 0 \text{ as } h \rightarrow 0$$

$$\text{and } |F_n(t, h)| \leq |t| + |h|.$$

$$|F_n(t, h)| \leq |h| + t \underbrace{|1 - e^{-ih|m|^2}|}_{\leq 1} + t \underbrace{|1 - e^{+ih|m|^2}|}_{\leq 1} \quad (5)$$

$$\lesssim |h| + t \min(1, |h|m^2)$$

$$\lesssim |h| + t |h|^\alpha |m|^{2\alpha}, \quad 0 \leq \alpha \leq 1$$

$$\begin{aligned} \Rightarrow \| \langle \nabla \rangle^s \delta_h \Psi(x, t) \|_{L^p(\Omega)} &\leq p^{1/2} \| \langle \nabla \rangle^s \delta_h \Psi(x, t) \|_{L^2(\Omega)} \\ &\lesssim p^{1/2} (1+T)^{1/2} |h|^{\alpha/2} \left(\sum_n \langle n \rangle^{2(s+\alpha)} |\hat{\Phi}_n|^2 \right)^{1/2}, \quad \forall t \in [0, T] \end{aligned}$$

• $\frac{p \geq r}{r < \infty}$:

$$\| \delta_h \Psi(t) \|_{W_x^{s,r}} \|_{L^p(\Omega)}$$

Mink Ineq

$$\lesssim C_T p^{1/2} \| \Phi \|_{HS(L^2; H^{s+\alpha})} |h|^{\alpha/2}, \quad \forall t \in [0, T]$$

$\Rightarrow$ Kolmogorov conti criterion

$$\Rightarrow \Psi \in C_t^{\frac{\alpha}{2} - \frac{1}{p}} W_x^{s-\alpha, r}, \quad \text{a.s. if } \Phi \in HS(L^2; H^s)$$

(here, we replaced $s+\alpha$ by s)

$$p \rightarrow \infty \Rightarrow \Psi \in C_t^{\frac{\alpha}{2}-} W_x^{s-\alpha, r}, \text{ a.s. if } \phi \in HS(L^2; H^s) \quad (6)$$

$r \leq \infty, \alpha > 0$

• When $r=2$, it suffices to study the continuity property of

$$\tilde{\Psi}(t) = S(t-t) \Psi(t) = \int_0^t S(t-\tau) \phi dW(\tau)$$

$$\Rightarrow E[\tilde{\Psi}(x, t) \overline{\tilde{\Psi}(y, \tau)}] = \sum_n |\hat{\phi}_n|^2 e_n(x-y) \cdot \underline{\underline{t}} \quad t \leq \tau$$

By repeating analogous computation,

$$\| \|\delta_h \tilde{\Psi}(t)\|_{H_x^s} \|_{L^p(\Omega)} \leq C_T p^{1/2} \|\phi\|_{HS(L^2; H^s)} |h|^{1/2}$$

$\Rightarrow$ Kolmogorov conti criterion,

$$\tilde{\Psi} \in C_t^{1/2-} H_x^s, \text{ a.s.}$$

$$\rightarrow \Psi \in C_t H_x^s, \text{ a.s.}$$

□

$$\begin{cases} \Psi(t+h) - \Psi(t) \\ = S(t+h) [\tilde{\Psi}(t+h) - \tilde{\Psi}(t)] \\ + [S(t+h) - S(t)] \tilde{\Psi}(t) \end{cases}$$

ex: $\phi = \text{Id} \Rightarrow \phi \in \text{HS}(L^2; H^s)$, $s < -\frac{d}{2}$

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$\Rightarrow \Psi_{\text{Sch}} \in C_t H_x^s, \text{ a.s. }, s < -\frac{d}{2}$

spatial regularity of
(space-time) white noise

Wave & heat? $\phi = \text{Id}$.

$\Psi_{\text{wave}}(t) = \int_0^t \frac{\sin(t-t') \langle \nabla \rangle}{\langle \nabla \rangle} dW(t'), \quad (\partial_t^2 + 1 - \Delta) \Psi = \xi$

$\langle \nabla \rangle$ one gain of derivative

$\Psi_{\text{heat}}(t) = \int_0^t P(t-t') dW(t'), \quad (\partial_t + 1 - \Delta) \Psi = \xi$

↑
In the first step, we get $\int_0^t e^{+2t' \langle m \rangle^2} dt'$

↑ $e^{t \langle m \rangle^2} e^{-t \langle m \rangle^2}$ ↓ $\frac{1}{\langle m \rangle^2}$

↑
 $\mathbb{E}[\Psi(x,t) \Psi(y,t)]$

$P(t) = e^{t(\Delta-1)}$

$\widehat{P(t)f(m)} = e^{-t(1+|m|^2)} \widehat{f(m)}$

Taking a square root, it gives one gain of deriv.

$$\Rightarrow \Psi_{\text{wave}}, \Psi_{\text{heat}} \in C_t W_x^{s, \infty}(\mathbb{T}^d), \quad \text{a.s.} \quad \underline{s < -\frac{d}{2} + 1} \quad \textcircled{8}$$

($\Phi = \text{Id}$)

Chap 1: One-dimensional case

$$\text{SNLS} : \begin{cases} i\partial_t u - \Delta u + |u|^{k-1} u = \Phi \Xi \\ u|_{t=0} = u_0 \end{cases}, \quad x \in \mathbb{T}^d.$$

$$\bullet \quad \underline{\underline{s > \frac{d}{2}}}: \quad \begin{aligned} \Phi &\in \text{HS}(L^2; H^s) \\ u_0 &\in H^s(\mathbb{T}^d) \end{aligned}$$

$$\begin{aligned} \text{Duhamel: } u(t) &= S(t) u_0 + i \int_0^t S(t-t') |u|^{k-1} u(t') dt' + \Psi(t) \\ &=: \Gamma_{u_0, \Phi}(u) \end{aligned}$$

When $s > \frac{d}{2}$, H^s is an algebra.

$$\|fg\|_{H^s} \lesssim \|f\|_{H^s} \|g\|_{H^s}$$

$$\left(\begin{array}{l} \cdot \langle m_1 + m_2 \rangle^s \lesssim \langle m_1 \rangle^s + \langle m_2 \rangle^s, \quad s \geq 0 \\ \cdot \|\hat{f} * \hat{g}\|_{l_n^2} \leq \|\hat{f}\|_{l_n^2} \underbrace{\|\hat{g}\|_{l_n^1}}_{\substack{\lesssim \|g\|_{H^s} \\ \text{C-S.}}} \end{array} \right.$$

$$\|P_{u_0, \phi}(u)\|_{C_T H_x^s}$$

$$C_T H_x^s = C([0, T]; H^s)$$

$S(t)$, unitary on H^s

$$\leq \underbrace{\|u_0\|_{H^s}}_{\leq T} + \left\| \int_0^t \underbrace{\|S(t-t')(u|^{h-1}u(t')\|_{H_x^s}}_{\lesssim \|u(t')\|_{H_x^s}^k} dt' \right\|_{L_T^\infty}$$

$$h \in 2\mathbb{N} + 1.$$

$$+ \underbrace{\|\Phi\|_{C_T H_x^s}}$$

$$\leq C_1 T \|u\|_{C_T H_x^s}^k$$

$$R = R_{\omega} = 2(\|u_0\|_{H^s} + \|\Phi\|_{C([0,1]; H^s)})$$

random ↑ NOT T.

$$\| \Gamma_{u_0, \Phi}(u) \|_{C_T H^s_x} \leq \frac{1}{2} R + C_1 T R^k$$

$$\leq R.$$

$$T \leq 1.$$

$$\forall u \in B_R \subset C_T H^s_x$$

by choosing $T = T_{\omega} = T(R_{\omega}) \ll 1$

$$T \sim R^{-\frac{1}{k-1}}$$

$$\Rightarrow \Gamma_{u_0, \Phi} : B_R \hookrightarrow B_R.$$

$$\| \Gamma_{u_0, \Phi}(u) - \Gamma_{u_0, \Phi}(v) \|_{C_T H^s_x} \leq C_2 T \left(\|u\|_{C_T H^s_x}^{k-1} + \|v\|_{C_T H^s_x}^{k-1} \right)$$

$$|u|^{k-1}u - |v|^{k-1}v \leq C_2 T R^{k-1} \|u-v\|_{C_T H^s_x}$$

⌚ write in a telescoping sum. $\leq \frac{1}{k}$

⌚ Young's ineq $ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1.$

(11)

$\Rightarrow \Gamma_{u_0, \phi}$ is a contraction on $B_R \subset C_T H_x^s$.

By Banach fixed pt thm, $\exists! u \in B_{R_\omega}$ s.t. $u = \Gamma_{u_0, \phi}(u)$
 $[0, T_\omega]$

- time of local existence is random
- conti dependence on $u_0 \in H^s$ and $\phi \in HS(L^2, H^s)$

Suppose $u = \Gamma_{u_0, \phi_1}(u)$, $v = \Gamma_{v_0, \phi_2}(v)$

$$\begin{aligned} \Rightarrow \|u - v\|_{C_T H_x^s} &\lesssim \|u_0 - v_0\|_{H^s} \\ &\quad + \|\Psi_1 - \Psi_2\|_{C_T H_x^s} \\ &\quad \quad \quad \underbrace{\int_0^t S(t-t')(\phi_1 - \phi_2) dW(t')} \\ &\leq C_\omega \|\phi_1 - \phi_2\|_{HS(L^2, H^s)} \end{aligned}$$

②

· Here, we hid the nonlinear part on LHS.

$$\|\Psi_1 - \Psi_2\|_{C_T H_x^s} \leq C_{\infty} \|\Phi_1 - \Phi_2\|_{HS(L^2; H^s)}$$

↑ random const

$$\Leftrightarrow \mathbb{E} [\|\Psi_1 - \Psi_2\|_{C_T H_x^s}^2] \leq C_T \|\Phi_1 - \Phi_2\|_{HS(L^2; H^s)}^2$$

and Chebyshev's ineq.

Lec 3 : 03 / 03 / 21 (Wed)

①

Last time : LWP of 1-d SNLS in $H^s(\mathbb{T}^d)$, $s > \frac{d}{2}$
 $\Phi \in HS(L^2; H^s)$

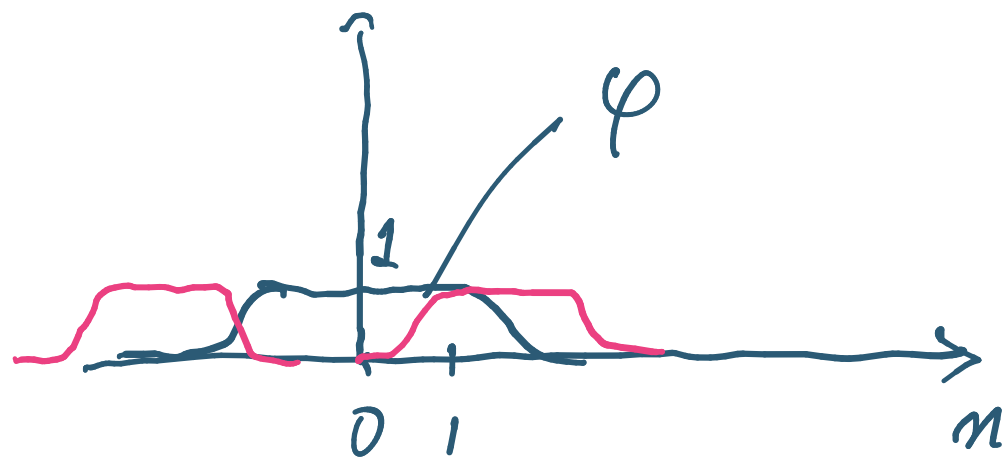
• Littlewood-Paley decomposition.

$$f = \sum_{\substack{N \geq 1 \\ \text{dyadic}}} P_N f$$

$$N \in 2^{\mathbb{Z}}, j \geq 0.$$

$$= \sum_{j=0}^{\infty} P_j f$$

$P_N =$ LP projector = "projection" onto freq $\{ |n| \sim N = 2^j \}$.



$\varphi(\xi)$ radial, supported on $\{ |\xi| \leq 2 \}$
 $\equiv 1$ on $\{ |\xi| \leq \frac{6}{5} \}$

$$\varphi_j(\xi) = \varphi\left(\frac{\xi}{2^j}\right) - \varphi\left(\frac{\xi}{2^{j-1}}\right), j \geq 1$$

$$\widehat{P_j f}(\xi) = \varphi_j(\xi) \widehat{f}(\xi).$$

(2)

Need to normalize φ_j

$$\psi_j(\vec{z}) = \frac{\varphi_j(\vec{z})}{\sum_{k=0}^{\infty} \varphi_k(\vec{z})} \leftarrow \text{finite sum}$$

$$\Rightarrow P_j f = F^{-1}(\psi_j f)$$

LP theorem:

$$1 < p < \infty$$

$$\|f\|_{L^p} \sim \left\| \underbrace{\left(\sum_{\substack{N \geq 1 \\ \text{dyadic}}} |P_N f|^2 \right)^{1/2}}_{\text{square function.}} \right\|_{L^p}$$

$$\left(\begin{array}{l} \text{if } p \geq 2 \\ \leq \end{array} \right\| \underbrace{\|P_N f\|_{L^p}}_{\text{simpler object}} \left\|_{\ell_{N, \text{dyadic}}^2}$$

Sobolev spaces

$$\|f\|_{H^s} \sim \left(\sum_{\substack{N \geq 1 \\ \text{dyadic}}} \underline{N^{2s}} \|P_N f\|_{L_x^2}^2 \right)^{1/2}$$

Besov spaces

$$B_{p,q}^s \quad \text{or} \quad B_q^{s,p}$$

$$\|f\|_{B_{p,q}^s} = \left\| \left\| \underbrace{N^s}_{2^{js}} \|P_N f\|_{L_x^p} \right\|_{\substack{N \geq 1 \\ \text{dyadic}}} \right\|_{\ell_j^s(\mathbb{Z}_{\geq 0})}$$

$$\cdot \quad H^s = B_{2,2}^s$$

$$\cdot \quad \underline{\underline{B_{p,1}^s}} \subset \underline{\underline{W_{L_s^p}^{s,p}}} \subset \underline{\underline{B_{p,\infty}^s}}$$

$$\cup$$

$$B_{p,\infty}^{s+\varepsilon}$$

• Hölder - Besov space $C^s = B_{\infty, \infty}^s$, $s \in \mathbb{R}$

(4)

• natural extension of the classical Hölder space. C^s , $0 < s < 1$

$$\|f\|_{C^s} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^s}$$

$\dot{\Lambda}^s = (\text{homog})$ Lipschitz space

$$\Lambda^s = \dot{\Lambda}^s \cap L^\infty$$

FACT: $\dot{\Lambda}^s = \dot{B}_{\infty, \infty}^s$, $0 < s < 1$

$$\|f\|_{\dot{B}_{p, q}^s} = \left\| 2^j \|P_j f\|_{L^p_x} \right\|_{\dot{L}_j^q(\mathbb{Z})}$$

• $s > 0$: C^s is an algebra

$$\|fg\|_{C^s} \lesssim \|f\|_{C^s} \|g\|_{C^s}$$

(5)

• 1-d SNLW: $(\partial_t^2 + 1 - \Delta) u + u^k = \underline{\xi}$ on Π

Duhamel

$$u(t) = \underbrace{\partial_t S(t) u_0 + S(t) u_1}_{\substack{\text{"} \\ \omega(t \langle \nabla \rangle) u_0 + \frac{\sin(t \langle \nabla \rangle)}{\langle \nabla \rangle} u_1}} - \int_0^t S(t-t') u^k(t') dt' + \underbrace{\int_0^t S(t-t') dW(t')}_{\Psi}$$

 $\Psi = \text{stoch convolution}$

$$S(t) = \frac{\sin t \langle \nabla \rangle}{\langle \nabla \rangle}$$

$$= \int_0^t S(t-t') \xi(dt')$$

Denote RHS by $\Gamma(u) = \Gamma_{(u_0, u_1), \xi}(u)$

• Recall $\Psi = \Psi_{\text{wave}} \in C_t W_x^{\frac{1}{2}, \infty}(\Pi)$

• $\| \partial_t S(t) u_0 + S(t) u_1 \|_{H^s} \lesssim \| (u_0, u_1) \|_{\mathcal{H}^s}$

$$\mathcal{H}^s = H^s \times H^{s-1}$$

$$\Rightarrow \| \Gamma(u) \|_{C_T H^s} \lesssim \| (u_0, u_1) \|_{\mathcal{H}^s} + \underbrace{\int_0^t \| u^k(t') \|_{H^{s-1}} dt'}_{\leq \| u^k(t) \|_{L^2} = \| u(t) \|_{L^{2k}}^{k-1}} + \| \Phi \|_{C_T H^s}$$

$s < 1/2$
↓
 $\leq C_\omega < \infty$
a.s.

$s \geq \frac{1}{2} - \frac{1}{2k}$
↓
 $< 1/2$

Sob
 $\lesssim \| u(t) \|_{H^s}^{k-1}$

$\Rightarrow$ By choosing $\frac{1}{2} - \frac{1}{2k} \leq s < 1/2$, we have

$$\| \Gamma(u) \|_{C_T H^s} \lesssim \| (u_0, u_1) \|_{\mathcal{H}^s} + T \| u \|_{C_T H^s}^{k-1} + \| \Phi \|_{C_T H^s}$$

Similarly,

$$\| \Gamma(u) - \Gamma(v) \|_{C_T H^s} \lesssim T \left(\| u \|_{C_T H^s}^{k-1} + \| v \|_{C_T H^s}^{k-1} \right) \| u - v \|_{C_T H^s}$$

(6)

$$R = R_\omega \sim \| (u_0, u_1) \|_{H^S} + \| \mathbb{F} \|_{C(\underline{\tau}, \underline{\tau}]; H^S)} \quad (7)$$

$\Rightarrow \Gamma$ is a contraction on $B_R \subset C_T H^S$, $T = T_\omega = T(R_\omega) \ll 1$.

Amk: By a similar argument, we can show LWP for

$$\partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta - 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ \partial_t u \end{pmatrix} + \begin{pmatrix} 0 \\ -u^k \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbb{F} \end{pmatrix}$$

$\Leftarrow$ important for global-in-time study.

1-d SNLH: $(\partial_t + 1 - \Delta) u + u^k = \mathbb{F}$

Schauder estimate: $1 \leq p \leq q \leq \infty$

$$\| D^\alpha P(t) f \|_{L_x^q} \lesssim \underbrace{t^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q}) - \frac{\alpha}{2}}}_{\text{pink underline}} \| f \|_{L_x^p}$$

$$P(t) = e^{t(\Delta - 1)}$$

on $\mathbb{R}^d$ or $\mathbb{T}^d$, $t > 0$, $\alpha \geq 0$.

Pf: On $\mathbb{R}^d$.

• Beinhouri - Chemin - Danchin (8)
• Grafakos

$$e^{t\Delta} f(x) = \int \underline{K_t(x-y)} f(y) dy$$

$$\hat{K}_t(\xi) = e^{-t|\xi|^2}$$

$$= \hat{K}(\underline{t^{1/2} \xi}), \text{ where } K = K_1.$$

inv F.T.

$$\Rightarrow K_t(x) = \frac{1}{\underline{t^{d/2}}} K\left(\underline{\frac{x}{t^{1/2}}}\right)$$

$$\Rightarrow \|K_t\|_{L_x^r} = t^{-d/2} \|K(\frac{x}{t^{1/2}})\|_{L_x^r} = t^{-d/2} \underline{t^{\frac{d}{2r}}} C_K.$$

ch of var

$$\sim t^{-\frac{d}{2}(\frac{1}{r} - \frac{1}{q})}$$

Apply
Young

$$\frac{1}{q} + 1 = \frac{1}{r} + \frac{1}{p}$$

$$\Rightarrow \text{By Young's ineq, } \|e^{t\Delta} f\|_{L^q(\mathbb{R}^d)} \lesssim t^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}$$

$\alpha > 0$:

$$D^\alpha(e^{t\Delta} f) = D^\alpha(K_t * f) = (\underline{D^\alpha K_t}) * f$$

$$\widehat{D^\alpha K_t(\xi)} = |\xi|^\alpha e^{-t|\xi|^2}$$

$$D = \sqrt{-\Delta} = |\nabla|$$

⑨

$$= t^{-\frac{\alpha}{2}} \underbrace{(t^{1/2} |\xi|)^\alpha e^{-t|\xi|^2}}$$

$$=: \widehat{G_t(\xi)} = \widehat{G}(t^{1/2} \xi), \quad G = G, \in \mathcal{S}(\mathbb{R}^d)$$

repeating the same comp
 $\Rightarrow$

$$\|D^\alpha K_t\|_{L^r} = t^{-\frac{\alpha}{2}} \|G_t(x)\|_{L^r} \sim t^{-\frac{\alpha}{2}} t^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q})}$$

$\Rightarrow$ Apply Young's ineq.

$$\frac{1}{q} + 1 = \frac{1}{r} + \frac{1}{p}$$

On $\mathbb{T}^d$:

$$e^{t\Delta} f = R_t * f, \quad \widehat{R_t(m)} = e^{-t|m|^2}$$

- can not use the scaling argument.

$\Rightarrow$ use the Poisson summation formula: $|f(x)| + |\widehat{f}(x)| \lesssim \langle x \rangle^{-d-\varepsilon}$
 f on $\mathbb{R}^d$

$$\sum_{n \in \mathbb{Z}^d} \widehat{f}(n) e^{in \cdot x} = \sum_{m \in \mathbb{Z}^d} f(x+m)$$

pf:

$$F(x) = \sum_{n \in \mathbb{Z}^d} f(x+n)$$

//

periodic func on $\mathbb{T}^d$

(10)

$$\sum_{n \in \mathbb{Z}^d} \hat{F}(n) e^{in \cdot x}$$

$$\hat{F}(n) = \int_{\mathbb{T}^d} F(x) e^{-in \cdot x} dx = \int_{[-1/2, 1/2]^d} \sum_m f(x+m) e^{-in \cdot x} dx$$

$$= \sum_{m \in \mathbb{Z}^d} \underbrace{\int_{m+\mathbb{T}^d} f(y) e^{-in \cdot y} dy}_{= \int_{\mathbb{R}^d}} = \hat{f}(n)$$

□

• Back to the proof of the Schauder esti. on $\mathbb{T}^d$.

$$\frac{1}{r} = 1 - \left(\frac{1}{p} - \frac{1}{q}\right)$$

$$\|R_t\|_{L^r(\mathbb{T}^d)}$$

$$\hat{R}_t(m) = \hat{K}_t(m) = e^{-t|m|^2}$$

$$= \left\| \sum \hat{K}_t(m) e^{in \cdot x} \right\|_{L^r(\mathbb{T}^d)}$$

$$\stackrel{\text{Poisson}}{=} \left\| \sum_{n \in \mathbb{Z}^d} K_t(x+n) \right\|_{L^r(\mathbb{T}^d)}$$

$$\begin{matrix} \uparrow \\ \langle m \rangle^{-\beta} & \langle m \rangle^{\beta} \end{matrix}$$

$$\beta r' > d$$

$$\lesssim \left\| \underbrace{\left(\sum \langle m \rangle^{\beta r'} \right)^{1/r'}}_{< \infty} \left\| \langle m \rangle^{\beta} K_t(x+n) \right\|_{\ell_n^r} \right\|_{L_x^r(\mathbb{T}^d)}$$

$$\lesssim \left\| \langle x \rangle^{\beta} K_t(x) \right\|_{L_x^r(\mathbb{R}^d)}$$

$$\sim \left\| K_t(x) \right\|_{L_x^r(\mathbb{R}^d)} + \underbrace{\left\| |x|^{\beta} K_t(x) \right\|_{L_x^r(\mathbb{R}^d)}}_{= \frac{1}{t^{d/2}} K\left(\frac{x}{t^{1/2}}\right)}$$

$\Rightarrow$ Repeat the prev argument.

$$\leq \left| \frac{x}{t^{1/2}} \right|^{\beta}, \quad 0 < t \leq 1.$$

$\Rightarrow$ Schauder est for $e^{t\Delta}$ on $\mathbb{T}^d$, $0 < t \leq 1$.

• As for $P(t) = e^{t(\Delta-1)} = \bar{e}^t e^{t\Delta}$.

$\uparrow$
can absorb $t^{B/2}$, $\forall t$.

$\Rightarrow$ Schauder est for $P(t)$ on $\mathbb{T}^d$, $t > 0$

□

Cor: $\|P(t)f\|_{C^{s_1}} \lesssim t^{-\frac{s_1-s_2}{2}} \|f\|_{C^{s_2}}, \quad s_1 \geq s_2$

• Back to SNLH on $\mathbb{T}$:

$$u(t) = P(t)u_0 - \int_0^t P(t-t') u^k(t') dt' + \underbrace{\int_0^t P(t-t') dW(t')}_{=: \Psi}$$

• $\Psi \in C_t C_x^{\frac{1}{2}-}$

$$\stackrel{\frac{1}{2}-2\varepsilon}{\parallel} W^{s,\infty} \supset C^{s+\varepsilon}$$

$\Gamma(u)$ = RHS of the Duhamel formulation.

$$\Rightarrow \|\Gamma(u)(t)\|_{C_x^s} \lesssim \|u_0\|_{C_x^s} + \int_0^t \|u(t')\|_{C_x^s}^k dt' + \|\Psi(t)\|_{C_x^s}$$

$\nwarrow s > 0$
 $\swarrow s < 1/2$

$$\Rightarrow \|\Gamma(u)\|_{C_T C_x^s} \lesssim \|u_0\|_{C_x^s} + T \|u\|_{C_T C_x^s}^k + \|\Psi\|_{C_T C_x^s}$$

$T \leq 1$

& difference estimate

$$\Rightarrow \text{LWP in } C^s(T^d), \quad 0 < s < 1/2.$$

Rougher data?

$$u_0 \in \mathcal{C}^s, \quad s < 0$$

(14)

$$\|P(t) u_0\|_{\mathcal{C}^\sigma} \lesssim \underline{t^{-\frac{\sigma-s}{2}}} \|u_0\|_{\mathcal{C}^s}, \quad \sigma \geq s.$$

↓ blows up as $t \rightarrow 0+$

• $\sigma > s$:

$$\|u\|_{Y^\sigma(\tau)} = \sup_{0 \leq t \leq \tau} t^\theta \|u(t)\|_{\mathcal{C}^\sigma}$$

$$\Rightarrow t^\theta \|P(u)(t)\|_{\mathcal{C}^\sigma} \lesssim \underbrace{t^\theta t^{-\frac{\sigma-s}{2}} \|u_0\|_{\mathcal{C}^s}}_{\text{put in sup } t} + t^\theta \int_0^t \underbrace{(t')^{-\theta k}}_{\text{need } \theta k < 1} \left(\underbrace{(t')^\theta \|u(t')\|_{\mathcal{C}^\sigma}}_{\text{put in sup } t} \right)^k dt' + t^\theta \|\Phi(t)\|_{\mathcal{C}^\sigma}$$

$\Rightarrow$ Take $\sup_{0 \leq t \leq \tau}$ and get estimate for the $Y^\sigma(\tau)$ -norm

$$\theta - \frac{\sigma-s}{2} \geq 0$$

$$\begin{aligned}
 & S < 0 \\
 & \| \Gamma(u) \|_{C_T C_x^S} \lesssim \| u_0 \|_{C^S} \\
 & \quad + \underbrace{\left\| \int_0^t (t')^{-\theta k} \left((t')^\theta \| u(t') \|_{C^\sigma} \right)^k dt' \right\|_{L_T^\infty}}_{T^{1-\theta k} \| u \|_{Y^\sigma(T)}^k} \\
 & \quad + \| \Psi \|_{C_T C_x^S}
 \end{aligned}$$

C^S is not an algebra.

(15)

$$\theta - \frac{\sigma - S}{2} \geq 0$$

$$\begin{aligned}
 \Rightarrow \underline{S} & \geq \sigma - 2\theta \\
 & > -2\theta \\
 & > -\frac{2}{k}
 \end{aligned}$$

$$\sigma > 0$$

$$\theta < \frac{1}{k}$$

① Run a contraction argument on a ball in $Y^\sigma(T)$ and show $u \in C_T C_x^S$

$$\Rightarrow u \in C([0, T_\omega]; C^S) \cap C(\underline{0}, T_\omega; C^\sigma).$$

1-d cubic SNLS

$$i \partial_t u - \partial_x^2 u + |u|^2 u = \phi \xi$$

ii) on $\mathbb{R}$. $\phi \in HS(L^2; L^2)$

de Bouard - Debussche '03
↓ Oh-Poc-Wang '20

$$\Psi = \int_0^t S(t-t') \phi dW(t') \in C_T L_x^2 \cap \underbrace{L_T^q W_x^{s,r}}_{\substack{\uparrow \\ \text{on } [0, T]}} \text{ if } \phi \in HS(L^2; \underline{H^s})$$

$\forall q < \infty, \quad \underline{r \leq \frac{2d}{d-2}}$

($d = 1, 2$, $r < \infty$.

$$S(t) = e^{-it\Delta}$$

$$\widehat{S(t)f}(\xi) = e^{it|\xi|^2} \widehat{f}(\xi)$$

$$\begin{aligned} \| S(t)f \|_{L^2(\mathbb{R}^d)} &\stackrel{\text{Plancherel}}{=} \| e^{it|\xi|^2} \widehat{f} \|_{L_{\xi}^2(\mathbb{R}^d)} \\ &= \| f \|_{L_x^2} \end{aligned}$$

• unitary on $H^s(\mathbb{R}^d)$.

• (q, r) is called Schrödinger admissible if

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$$

(scaling condition)

$$2 \leq q, r \leq \infty$$

$$(q, r, d) \neq (2, \infty, 2)$$

• Strichartz estimates on $\mathbb{R}^d$ (NOT on $\mathbb{T}^d$)

① (homog est): $\|S(t)f\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|f\|_{L_x^2}$ $v \geq 2$

② (dual homog est): func of t and x .

$$\left\| \int_{\mathbb{R}} S(-t') \underline{F(t')} dt' \right\|_{L^2(\mathbb{R}^d)} \lesssim \|F\|_{L_t^{q'} L_x^{r'}}$$

③ (nonhomog est / retarded est)

$$\| \int_0^t S(t-t') F(t') dt' \|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|F\|_{L_t^{q'} L_x^{r'}}$$

$(q, r), (\tilde{q}, \tilde{r})$, admissible

(3)

- On time averaged sense, there is a smoothing
(in terms of integrability, NOT in terms of differentiability).

Idea of proof: • dispersive estimate $\|S(t)f\|_{L_x^\infty} \lesssim \frac{1}{|t|^{d/2}} \|f\|_{L_x^1}$

$\Leftarrow$ • soln formula $S(t)f = \frac{1}{(4\pi it)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{-|x-y|^2}{4ti}} f(y) dy$
 or
 • Method of stationary phase (see my lecture note)
 $S(t)f(x) = \int_{\mathbb{R}^d} e^{ix \cdot \xi} \underline{e^{it|\xi|^2}} f(\xi) d\xi$

- unitary in $L^2(\mathbb{R}^d)$

$\Rightarrow$ interpolate $\|S(t)f\|_{L_x^p} \lesssim |t|^{-(\frac{d}{2} - \frac{d}{p})} \|f\|_{L^{p'}} , \quad p \geq 2.$

(4)

$$T = S(t), \quad bdd : L^2 \rightarrow B = L_t^q L_x^r$$

$$\Leftrightarrow \text{Dual op, } bdd : B' = L_t^{q'} L_x^{r'} \rightarrow L_x^2$$

$$T^* F = \int_{\mathbb{R}} S(t-t') F(t') dt'.$$

$$\begin{aligned} \left(\langle S(t)f, F \rangle_{L_{t,x}^2} \right. &= \iint S(t)f \overline{F(t,x)} dt dx \\ &= \int f(x) \int S(t) F(t,x) dt dx \\ &= \langle f, T^* F \rangle_{L_x^2} \end{aligned}$$

$$\Leftrightarrow TT^* bdd : B' \rightarrow B.$$

$$TT^* F = \int_{\mathbb{R}} \underline{S(t-t')} F(t') dt'.$$

↑
put in $L_t^q L_x^r$, $r \geq 2$

$\Rightarrow$ integrate in t'

$$\| S(t-t') F(t') \|_{L_x^r} \lesssim \underbrace{|t-t'|^{-\left(\frac{d}{2}-\frac{d}{r}\right)}}_{\text{convolution}} \| F(t') \|_{L_x^{r'}}.$$

(5)

Use Hardy-Littlewood-Sobolev ineq
 $1 < p, q, r < \infty, \quad \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$

$$\| |x|^{-d/p} * g \|_{L^r(\mathbb{R}^d)} \lesssim \|g\|_{L^q(\mathbb{R}^d)}$$

$$|x|^{-d/p} \notin L^p(\mathbb{R}^d)$$

but "almost".

H-L-S ineq is an endpoint ver of
Young's ineq

$\Rightarrow TT^* \text{ bdd} : B \rightarrow B' \quad (\text{nonendpoint})$

$\Rightarrow$ ① & ②.

As for ③, $\int_0^+ = \int_{\mathbb{R}} \mathbb{1}_{[0, t]}^{(+)} \quad \text{and prove by hand}$

or use Christ-Kiselev lemma (see Tao's book)

endpt case : $q=2, r=\frac{2d}{d-2}$ by Keel - Tao AJM '99.

⑥

• Back to 1-d cubic SNLS on $\mathbb{R}$

Duhamel formulation: $u(t) = S(t)u_0 - \int_0^t S(t-t')(|u|^2 u(t'))dt' + \Psi.$

$(q, r) = (\infty, 2), (8, 4)$
admissible

$\frac{2}{8} + \frac{1}{4} = \frac{1}{2}.$

• $\Gamma u :=$ RHS of Duhamel

$X(T) = C_T L_x^2 \cap L_T^8 L_x^4$

$\frac{7}{8} = \frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$

$C_w < \infty$
v.i. a.s.

$\Rightarrow \| \Gamma u \|_{X(T)} \lesssim \| u_0 \|_{L_x^2} + \| |u|^2 u \|_{L_T^{8/7} L_x^{4/3}} + \| \Psi \|_{X(T)}$

$\leq T^{1/2} \| u \|^3_{L_T^8 L_x^4} \leq T^{1/2} \| u \|^3_{X(T)}$

$$T \leq 1 \quad \cdot \quad \| \Gamma u \|_{X(T)} \leq C_1 \left(\| u_0 \|_{L^2_x} + \| \Phi \|_{X(1)} \right) + T^{1/2} \| u \|_{X(T)}^3 \quad (7)$$

Also, difference estimate.

$\Rightarrow$ LWP in $L^2(\mathbb{R})$ if $\Phi \in HS(L^2; L^2)$

On $\mathbb{T}$?

• Strichartz estimates on $\mathbb{T}^d$

• only local in time.

• NOT as good as those on $\mathbb{R}^d$

• proof, much harder.

• Zygmund '74, Bourgain '93, '13

Bourgain-Demeter '15, Killip-Viřan '16.

analytic number theory
(HL circle method)

• L^4 - Strichartz on $\mathbb{T}$ (Zygmund '74)

(8)

$$\underbrace{\left\| \sum_{n \in \mathbb{Z}} e^{inx} e^{itn^2} \hat{u}_0(n) \right\|_{L^4_{x,t}(\mathbb{T}^2)}}_{\text{"} F(t,x) \text{"}} \lesssim \|u_0\|_{L^2(\mathbb{T})}$$

$\Pi_t \times \Pi_x$

Pf:

$$\|F \bar{F}\|_{L^2(\mathbb{T}^2)} = \left\| \sum_{n_1} \sum_{n_2} \hat{u}_0(n_1) \overline{\hat{u}_0(n_2)} \underbrace{e^{it(n_1^2 - n_2^2)}}_{\text{red}} \underbrace{e^{i(n_1 - n_2)x}}_{\text{red}} \right\|_{L^2(\mathbb{T}^2)}$$

• write this sum as a Fourier series in t and x

$$\sum_{(\tau, n) \in \mathbb{Z}^2} a(\tau, n) e^{i(\tau t + nx)}.$$

where $a(\tau, n) = \sum_{(n_1, n_2) \in \underline{P(\tau, n)}} \hat{u}_0(n_1) \overline{\hat{u}_0(n_2)}$

$$P(\tau, n) = \left\{ (n_1, n_2) : n_1^2 - n_2^2 = \tau, \quad n_1 - n_2 = n \right\}$$

9

- Given $(\tau, n) \neq (0, 0)$ and $(\tau, n) = (n_1^2 - n_2^2, n_1 - n_2)$,
there exists at most one soln (n_1, n_2)

$$\begin{cases} \tau - n^2 = n_1^2 - n_2^2 - n^2 = -2n_2(n_2 - n_1) = 2nn_2 \\ \Rightarrow \text{determines } n_2 \text{ (and } n_1 = n + n_2) \end{cases}$$

- Also, $a(0, 0) = \sum_n |\hat{u}_0(n)|^2$

Putting together,

$$\|F\|_{L^4(\mathbb{T}^2)} \stackrel{\text{Plancherel}}{=} \left(\sum_{\tau, n \in \mathbb{Z}} |a(\tau, n)|^2 \right)^{1/4}$$

$$\sim \left(\sum_{(\tau, n) \neq (0, 0)} \dots \right)^{1/4} + \underbrace{|a(0, 0)|^{1/2}}_{= \|u_0\|_{L^2}}$$

$$\lesssim \left(\sum_{n_1, n_2} |\hat{u}_0(n_1) \overline{\hat{u}_0(n_2)}|^2 \right)^{1/4}$$

↖ disjoint sum in n_1 and n_2

$$\sim \|u_0\|_{L^2} \quad \square$$

- Zygmund's L^4 -Strichartz is NOT enough to prove
LWP of cubic NLS in $L^2(\mathbb{T})$ (any $H^s(\mathbb{T})$, $s \leq \frac{1}{2}$)

'80 ~: Kenig - Ponce - Vega, harmonic analytic approach
on $\mathbb{R}^d$.

- Fourier restriction norm method: Bourgain '93

$X^{s,b}$ - spaces

(wave : hyperbolic Sobolev space
by Klainerman-Machedon '93

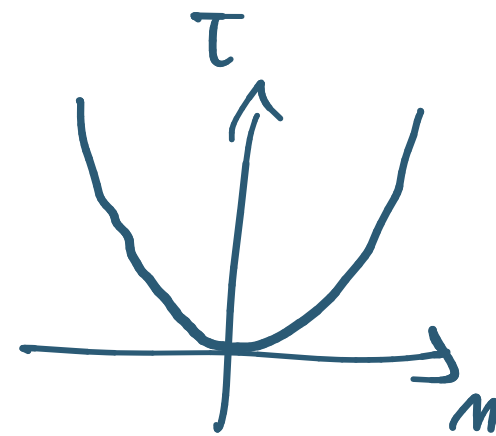
$$\|u\|_{X^{s,b}} = \left\| \underbrace{\langle m \rangle^s}_{\text{Sob weight in } x} \underbrace{\langle \tau - |m|^2 \rangle^b}_{\text{modulation}} \hat{u}(\tau, m) \right\|_{l_n^2 L_\tau^2(\mathbb{Z}^d \times \mathbb{R})}$$

- useful for "perturbative" study
"close to being linear soln"

linear Schrödinger: $i \partial_t u - \Delta u = 0$

space-time F.T.
 $\Rightarrow$

$$-(\tau - |m|^2) \hat{u}(\tau, m) = 0$$



(11)

$\hat{u}(\tau, m)$ is a measure supported on $\{\tau = |m|^2\}$

• $\langle \tau - |m|^2 \rangle^b$, $b > 0$, "penalizes" functions

whose Fourier support is away from $\{\tau = |m|^2\}$

• Basic properties:

① $b > 1/2$, $X^{s,b} \subset C_t H_x^s$

($b = 1/2$, need some other spaces

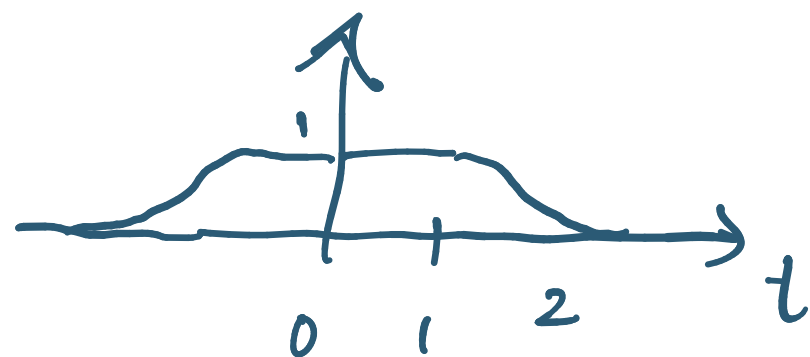
$b = 1/2$, may use $B_{2,1}^{1/2}$
 may V^2 , $V^2 = \text{predual of } V^2$

② $\|u\|_{X^{s,b}} = \|S(-t)u(t)\|_{\underset{=}{H_x^s} \underset{=}{H_t^b}} := \| \underbrace{\langle \partial_x \rangle^s}_{\langle \tau \rangle^b} \langle \partial_t \rangle^b (S(-t)u(t)) \|_{L_{t,x}^2}$

(3)

 $\eta(t) = \text{smooth cut off func on } [0,1]$

(12)



$$\|\eta(t) S(t) f\|_{X^{s,b}} \lesssim \|f\|_{H^s}.$$

Pf: $\widehat{\eta(t) S(t) f}(\tau, n) = \widehat{\eta}(\tau - |n|^2) \widehat{f}(n)$

$$(LHS) = \left\| \langle n \rangle^s \underbrace{\langle \tau - |n|^2 \rangle^b}_{=: \tau} \underbrace{\widehat{\eta}(\tau - |n|^2)}_{\tau} \widehat{f}(n) \right\|_{l_n^2 L_\tau^2}$$

$$= \|\eta\|_{H^b} \|f\|_{H^s}$$

④ Duhamel term: $b > 1/2$

⑬

$$\| \varphi(t) \int_0^t S(t-t') F(t') dt' \|_{X^{s,b}} \lesssim \| F \|_{X^{s,b-1}}$$

- Bourgain '93, Tao's book
- Andreia's 1st project (2018) $\Leftarrow$ by taking space-time Fourier transform

$$\cdot \| \varphi(\frac{t}{T}) \int_0^t S(t-t') F(t') dt' \|_{X^{s,b}} \lesssim T^\theta \| F \|_{X^{s,b-1+\theta}}$$

cut off on
 $[-T, T]$

$$b > 1/2, \quad \theta > 0, \text{ small}, \quad T \leq 1$$

$$\left(\underline{\text{but}} \right) \| \varphi(\frac{t}{T}) S(t) f \|_{X^{s,b}} \lesssim \underline{T^{\frac{1}{2}-b}} \| f \|_{H^s}, \quad \begin{matrix} b > 0 \\ b \neq 1/2 \end{matrix}$$

bad for $b > 1/2$
 $T \ll 1$.

We consider

$$u(t) = \underline{\underline{\eta(t)}} S(t) u_0 - \underline{\underline{\eta\left(\frac{t}{T}\right)}} \int_0^t S(t-\tau) |u|^2 u(\tau) d\tau \\ + \underline{\underline{\eta(t)}} \int_0^t S(t-\tau) \phi dW(\tau).$$

⑤ L^4 - Strichartz estimate by Bourgain 1993

$$\|u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{0, \frac{3}{8}}} \leftarrow \text{better than } \frac{1}{2}^+$$

$$\left(\begin{array}{l} \text{Zygmund '74: } \|S(t)f\|_{L^4_{t,x}}^{\text{local int}} \lesssim \|f\|_{L^2_x} \end{array} \right.$$

$$\Rightarrow \| \eta(t) u \|_{L^4_{t,x}} \lesssim \|u\|_{\underline{\underline{X^{0,b}}}}, \quad \underline{\underline{b > 1/2}}$$

↑
transference principle

• Tao's book: by Tzvetkov.

Prop: $b < 1/2$. $\Phi \in HS(L^2; H^s)$

$$\| \mathbb{1}_{[0,1]} \Psi \|_{X^{s,b}} \leq C_\infty < \infty, \text{ a.s.}$$


 $\Psi \sim \text{B.M. in time}$

Next time: • pf of Prop on Ψ

• LWP of 1-d cubic SNLS in $L^2(\mathbb{T})$

Lec 5 17 / 03 / 21 (Wed)

$$S(t) = e^{-it\Delta}$$

①

$$\Psi = \int_0^t S(t-t') \Phi dW(t')$$

$$\Phi \in HS(L^2; H^s)$$

$$\Phi e_n = \hat{\Phi}_n e_n$$

$$= \sum_{n \in \mathbb{Z}^d} \hat{\Phi}_n e_n \int_0^t e^{i(t-t')|n|^2} d\beta_n(t')$$

Prop: $b < 1/2$. Then, $\mathbb{1}_{[0,T]} \Psi \in X^{s,b}$, a.s.

pf:

Recall

$$\|u\|_{X^{s,b}} = \| \langle n \rangle^s \underbrace{\langle \tau - |n|^2 \rangle^b}_{\text{interaction representation}} \hat{u}(\tau, n) \|_{\ell_n^2 L_\tau^2}$$

$$= \| \langle \partial_x \rangle^s \langle \partial_t \rangle^b \underbrace{S(-t) u(t)}_{L_{x,t}} \|_{L_{x,t}^2}$$

interaction representation

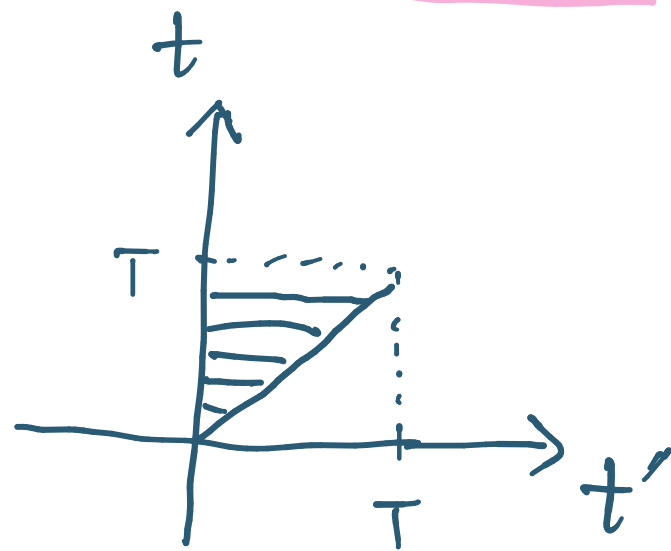
$$\mathbb{1}_{[0,T]}^{(t)} \Psi(t, n) = \mathbb{1}_{[0,T]}^{(t)} \hat{\Phi}_n \int_0^t e^{i(t-t')|n|^2} d\beta_n(t')$$

$\uparrow$ $S(-t)$

$$F(t) = \underline{S(-t)} \mathbb{1}_{[0,T]}(t) \Psi(t)$$

(2)

$$\hat{F}(\tau, n) = \hat{\Phi}_n \int_{\mathbb{R}} e^{-it\tau} \mathbb{1}_{[0,T]}(t) \int_0^t e^{-it'm|^2} d\beta_n(t') dt$$



stoch Fubini $\leftarrow$ see Da Prato-Zabczyk: "stoch eqns in ∞ dim"

$$\stackrel{\downarrow}{=} \hat{\Phi}_n \int_0^T e^{-it'm|^2} \underbrace{\int_{t'}^T e^{-it\tau} dt}_{\text{Wiener integral}} d\beta_n(t')$$

$$\lesssim_T \min(1, \frac{1}{|\tau|}) \sim \frac{1}{\langle \tau \rangle}$$

$$\mathbb{E} [\| \mathbb{1}_{[0,T]} \Psi \|_{\chi^{s,b}}^2]$$

$$\langle \tau \rangle = \sqrt{1 + |\tau|^2}$$

$$= \| \langle n \rangle^s \langle \tau \rangle^b \| \hat{F}(\tau, n) \|_{L^2(\Omega)}^2 \|_{\ell_n^2 L_\tau^2}^2$$

$$|\hat{\Phi}_n| \left(\int_0^T | \dots |^2 dt' \right)^{1/2} \lesssim_T |\hat{\Phi}_n| \frac{1}{\langle \tau \rangle}$$

$$= \sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2s} |\hat{\Phi}_n|^2 \cdot \underbrace{\int_{\mathbb{R}} \langle \tau \rangle^{2(b-1)} d\tau}_{< \infty \text{ iff } b < 1/2}$$

$$= \| \Phi \|_{HS(L^2; H^s)}^2$$

(3)

We also have, for $p \geq 2$,

$$\| \mathbb{1}_{[0,T]} \Psi \|_{X^{s,b}} \| \cdot \|_{L^p(\Omega)} \lesssim_T p^{1/2} \| \Phi \|_{HS(L^2; H^s)}$$

Chernyshov
 $\Rightarrow$

$$P \left(\| \mathbb{1}_{[0,T]} \Psi \|_{X^{s,b}} > \lambda \right) \leq C e^{-c \frac{\lambda^2}{\| \Phi \|_{HS(L^2; H^s)}}}$$

□

LWP of the 1-d cubic SNLS in $L^2(\mathbb{T})$, $\Phi \in HS(L^2; L^2)$

Duhamel formulation:

$T \leq 1$.

$$u(t) = \underline{\underline{\eta(t)}} S(t) u_0 - i \underline{\underline{\eta\left(\frac{t}{T}\right)}} \int_0^t S(t-\tau) (u)^2 u(\tau) d\tau$$

$$+ \underline{\underline{\mathbb{1}_{[0,T]}(t)}} \Psi(t).$$

$$=: \Gamma(u)$$

$$\nwarrow \underline{\underline{\mathbb{1}_{[0,1]}(t)}}$$

Fix $\frac{3}{8} \leq b < \frac{1}{2}$.

$$\|y(t) \int_0^t s(t-t') |u|^2 u(t') dt'\|_{X^{0, \frac{1}{2}+\theta}}$$

(4)

Then, $\| \Gamma u \|_{X^{0,b}} \lesssim \| u_0 \|_{L^2} + \tau^\theta \| |u|^2 u \|_{X^{0, -\frac{1}{2}+2\theta}}$

+ C_ω

$\uparrow$ $\mathbb{1}_{[0,1]} \Psi$

$\cdot \| |u|^2 u \|_{X^{0, -\frac{1}{2}+2\theta}}$

$$(X^{s,b})' = X^{-s,-b}$$

duality

$$= \sup_{\|v\|_{X^{0, \frac{1}{2}-2\theta}}} \left| \int_{\mathbb{R} \times \mathbb{T}} |u|^2 u \bar{v} dx dt \right|$$

$$\leq \|u\|_{L^4_{x,t}}^3 \|v\|_{L^4_{x,t}}$$

$$\lesssim \|u\|_{X^{0, \frac{3}{8}}}^3 \|v\|_{X^{0, \frac{3}{8}}}$$

$$\underbrace{\leq 1}$$

Recall on $\mathbb{T}$

$$\|u\|_{L^4_{x,t}} \lesssim \|u\|_{X^{0, \frac{3}{8}}}$$

$$\Rightarrow \| \Gamma u \|_{X^{0,b}} \lesssim \| u_0 \|_{L^2} + T^\theta \| u \|_{X^{0,b}}^3 + C_\omega$$

Also,

$$\| \Gamma u - \Gamma v \|_{X^{0,b}} \lesssim T^\theta \left(\| u \|_{X^{0,b}}^2 + \| v \|_{X^{0,b}}^2 \right) \| u - v \|_{X^{0,b}}$$

$\Rightarrow \Gamma$ is a contraction on $B_R \subset X^{s,b}$

$$R = R_\omega \sim \| u_0 \|_{L^2} + C_\omega$$

by choosing $T = T(R) \ll 1$.

$$\uparrow \| \mathbb{1}_{[0,1]} \Psi \|_{X^{0,b}}$$

Also, LWP in $H^s(\pi)$; $\phi \in HS(L^2; H^s)$

$$\Leftarrow \| u_1 \overline{u_2} u_3 \|_{X^{\underline{s}, -\frac{1}{2}+}} \lesssim \prod_{j=1}^3 \| u_j \|_{X^s, b}$$

$$\Leftarrow \underline{\langle n \rangle^s} \lesssim \langle n_1 \rangle^s \langle n_2 \rangle^s \langle n_3 \rangle^s, \quad s \geq 0 \text{ if } n = n_1 - n_2 + n_3.$$

(5)

• We constructed $u \in B_R \subset X^{0, \frac{1}{2}-} \not\subset C_t L_x^2$ ⑥

$\Rightarrow$ We need to show $u \in C_t L_x^2$

• lin part: $S(t)u_0 \in C_t L_x^2$

• stoch convolution $\Psi \in C_t L_x^2$ if $\Phi \in \text{HS}(L^2; L^2)$

• nonlinear part $\in X^{0, \frac{1}{2}+} \subset C_t L_x^2$.

$\Rightarrow u \in C([0, T]; L_x^2)$. $T = T_\omega \leq 1$.

Aside: LWP of the cubic SNLS in $L^2(\mathbb{T})$ ($\Phi \in \text{HS}(L^2; L^2)$)

• We proved

Zygmund '74: $\|S(t)u_0\|_{L_{x,t}^4(\mathbb{T}^2)} \lesssim \|u_0\|_{L^2(\mathbb{T})}$

Moyua-Vega '08: $\|S(t)u_0\|_{L_{t \in I, x}^4} \lesssim \underline{|I|^{\frac{1}{8}}} \|u_0\|_{L^2(\mathbb{T})}$
BLMS.

(Bourgain '93: $\|u\|_{L_{x,t}^4} \lesssim \|u\|_{X^{0, \frac{3}{8}}}$ $\frac{1}{2}, \frac{3}{8}, \frac{1}{8}$)

⑧

$\Rightarrow$ Christ-Kiselev lemma (JFA '01)

$$\left\| \int_0^t S(t-t') F(t') dt' \right\|_{L^4_{I,x}} \lesssim |I|^{\frac{1}{4}} \|F\|_{L^{\frac{4}{3}}_{I,x}}$$

· Now, run a contraction argument in $L^4_{T,x} \cap C_T L^2_x$

$$\left(\begin{aligned} &\Leftarrow \cdot \left\| |u|^2 u \right\|_{L^{\frac{4}{3}}_{T,x}} \leq \|u\|_{L^4_{T,x}}^3 \\ &\cdot \Psi \in L^4_{T,x} \cap C_T L^2_x \end{aligned} \right.$$

· Global well-posedness?

⑨

① Ito calculus approach (Martingale approach)

· 1-d cubic SNLS on $\mathbb{T}$: $i\partial_t u - \partial_x^2 u + |u|^2 u = \phi \Xi$
 $\phi \in \mathcal{HS}(L^2, L^2)$

deterministic case: $\phi = 0$.

$$\begin{aligned}\partial_t \int |u|^2 dx &= 2 \operatorname{Re} \int \partial_t u \cdot \bar{u} dx \\ &= -2 \operatorname{Re} \underbrace{i \int \partial_x^2 u \cdot \bar{u} dx}_{\stackrel{\text{IBP}}{=} \int | \partial_x u |^2 dx} + 2 \operatorname{Re} \underbrace{i \int |u|^2 u \cdot \bar{u} dx}_{|u|^4} \\ &= 0\end{aligned}$$

$\Rightarrow L^2$ -conservation $\Rightarrow$ GWP in $L^2(\mathbb{T})$.

For SNLS, we use Ito's lemma on

(10)

$$M(u) = \int |u|^2 dx = \sum |\hat{u}(n)|^2 = \sum (p_n^2 + q_n^2)$$

$$p_n = \operatorname{Re} \hat{u}(n), \quad q_n = \operatorname{Im} \hat{u}(n).$$

SNLS:

$$d\hat{u}_n = \left(i n^2 \hat{u}_n + i F_x(|u|^2 u)(n) \right) dt - i \hat{\Phi}_n d\beta_n$$

$$\Rightarrow dp_n = \left(-n^2 q_n - \operatorname{Im} F_x(|u|^2 u)(n) \right) dt$$

$$+ \underbrace{\operatorname{Im} (\hat{\Phi}_n d\beta_n)}$$

$$= \underbrace{\operatorname{Im} \hat{\Phi}_n}_{\text{wavy}} \underbrace{d(\operatorname{Re} \beta_n)}_{\text{pink}} + \underbrace{\operatorname{Re} \hat{\Phi}_n}_{\text{wavy}} \underbrace{d(\operatorname{Im} \beta_n)}_{\text{pink}}$$

$$dq_n = \left(n^2 p_n + \operatorname{Re} F_x(|u|^2 u)(n) \right) dt$$

$$- \underbrace{\operatorname{Re} (\hat{\Phi}_n d\beta_n)}$$

$$= - \underbrace{\operatorname{Re} \hat{\Phi}_n}_{\text{wavy}} \underbrace{d(\operatorname{Re} \beta_n)}_{\text{pink}} + \underbrace{\operatorname{Im} \hat{\Phi}_n}_{\text{wavy}} \underbrace{d(\operatorname{Im} \beta_n)}_{\text{pink}}$$

Ito's lemma: $dX = f dt + g dB$

Consider $F(X)$

$$\text{Then, } dF = \partial_x F \underline{dX} + \frac{1}{2} \partial_x^2 F \underline{(dX)^2}$$

2nd order
Taylor exp

$$= \partial_x F (f dt + g dB)$$

$$+ \frac{1}{2} \partial_x^2 F \cdot g^2 dt$$

↑
To be understood under

$$F(b) - F(a) = \int_a^b \dots$$

$$(dt)^2 = 0$$

$$dt dB = dB dt = 0$$

$$(dB)^2 = dt$$

$$\underline{dM} \stackrel{\text{Ito}}{=} 2 \underbrace{\sum (p_n \underline{dp_n} + g_n dg_n)} + \underbrace{\sum ((\underline{dp_n})^2 + (\underline{dg_n})^2)}$$

$$2 \sum (p_n \text{Im}(\hat{\Phi}_n \underline{d\beta_n}) - g_n \text{Re}(\hat{\Phi}_n \underline{d\beta_n}))$$

$$2 \|\Phi\|_{\text{HS}(L^2; L^2)}^2 dt$$

$d(\text{Re}\beta_n)d(\text{Im}\beta_n) = 0$

Burkholder - Davis - Gundy inequality : X , (local) martingale (12)

$$1 \leq p < \infty,$$

$$\mathbb{E} \left[\sup_{t \in [0, T]} |X_t|^p \right] \sim \mathbb{E} \left[\langle X \rangle_{[0, T]}^{p/2} \right]$$

quadratic variation

For Itô process $dX = fdt + gdB$

$$\langle X \rangle_{[0, t]} = \int_0^t g^2 dt'$$

ex: $2 \sum_n p_n \int_0^t \hat{\Phi}_n d(\text{Re} \beta_n)$

Take $\mathbb{E} \left[\sup_{t \in [0, T]} \int_0^t \dots \right]$

and apply B-D-G ineq.

$$\lesssim \mathbb{E} \left[\left(\int_0^T \sum_n p_n^2 |\hat{\Phi}_n|^2 dt \right)^{1/2} \right]$$

$$\leq \|u\|_{L^2}^2 \|\Phi\|_{HS(L^2, L^2)}$$

$$\leq \mathbb{E} \left[\left(\sup_{t \in [0, T]} M(u)(t) \right)^{1/2} \cdot T^{1/2} \|\Phi\|_{HS(L^2, L^2)} \right]$$

$$\leq \varepsilon \mathbb{E} \left[\sup_t M(u)(t) \right] + \frac{1}{\varepsilon} T \|\Phi\|_{HS}^2$$

Hide in LHS

$$\sum a_n b_n$$

$$\leq \sum a_n \cdot \sum b_n$$

Start with

(13)

$$\mathbb{E} \left[\sup_{t \in [0, T]} M(u)(t) \right]$$

$$\Rightarrow \mathbb{E} \left[\sup_{t \in [0, T]} M(u)(t) \right] \lesssim \|u_0\|_{L^2}^2 + C(T, \|\Phi\|_{\text{HS}(L^2; L^2)})$$

$\Rightarrow$ a.s. existence up to time T .
(but for ANY finite T).

$\Rightarrow$ GWP. in $L^2(\pi)$

• ② pathwise approach:

ex: 3-d cubic SNLW (defocusing) on $\mathbb{T}^3$

$$\partial_t^2 u + (1 - \Delta) u + u^3 = \phi \mathbb{Z}, \quad \phi \in HS(L^2; H^{s-1})$$

$$\Rightarrow \Psi \in C_t W_x^{s-\varepsilon, \infty} \quad s > 0$$

Duhamel: $u(t) = \partial_t S(t) u_0 + S(t) u_0 - \int_0^t S(t-t') u^3(t') dt'$

$$S(t) = \frac{\sin t \langle \nabla \rangle}{\langle \nabla \rangle} + \Psi.$$

"Da Prato-Debussche trick" '03:
McKean '95, Bourgain '96

1st order expansion
Write: $u = \Psi + v$

postulate "smoother".

" $v(t) \in H^1$ ".

$$v(t) = \text{lin soln} + \int_0^t \underbrace{S(t-\tau)}_{\text{one gain of deriv}} (v+\Psi)^3(\tau) d\tau. \quad \text{+ } \cancel{\Psi}$$

" $u = \Psi$ "

$$\|(Pv, \partial_t Pv)\|_{C_T \mathcal{H}'} \lesssim \|(u_0, u_1)\|_{\mathcal{H}'} + T \|v + \Psi\|_{L_T^\infty L_x^2}^3$$

$$\mathcal{H}' = H^1 \times L^2$$

v , $\partial_t v$

$\Rightarrow$ LWP. (for $(v, \partial_t v) \in C_T \mathcal{H}'$).
as long as $\Phi \in HS(L^2; H^{-1})$
(s.t. $\Psi \in C_T L_x^6$)

$$\|v\|_{L_T^\infty L_x^6}^3 + \|\Psi\|_{L_T^\infty L_x^6}^3 \leq C\omega$$

Sob $1/2$

$$\|v\|_{C_T H_x^1}^3$$

• Moral: Fix a target time $T_0 \gg 1$.
global target time

Suppose $Z \in C_{T_0} L_x^6$.

Note: The $C_{T_0} L_x^6$ -norm of Z may be very large BUT
 it is a fixed number (for fixed $T_0 \gg 1$).

The argument above yields LWP in $\mathcal{H}'(\mathbb{T}^3)$:

$$\begin{cases} \partial_t^2 v + (1 - \Delta)v + (v + Z)^3 = 0 \\ (v, \partial_t v)|_{t=0} = (u_0, u_1), \end{cases}$$

where the local existence time $T = T(\|(u_0, u_1)\|_{\mathcal{H}'}, K) > 0$

with $K = \|Z\|_{C_{T_0} L_x^6}$ ↑ small

Then, if we can control $\|(v(t), \partial_t v(t))\|_{\mathcal{H}'}$ on $[0, T_0]$, then
 we can iterate the LWP argument on $[\hat{j}T, (\hat{j}+1)T]$ to
 conclude existence on the entire interval $[0, T_0]$.

Since the choice of T_0 was arbitrary, this yields GWP.

(17)

$\Rightarrow$ Main goal: Control $\| (v(t), \partial_t v(t)) \|_{\mathcal{H}'}$
and show that it is finite on each finite time interval.

Lec 6 : 24 / 03 / 21 (Wed)

①

Back to ① Ito approach

$$i \partial_t u - \partial_x^2 u + |u|^2 u = \Phi \xi. \quad \text{on } \mathbb{T}$$

$P_{\leq N} =$ freq cutoff onto freq $\{|n| \leq N\}$
spatial (sharp cutoff, smooth cutoff)

We consider the finite dim'l approximation:

$$i \partial_t u_N - \partial_x^2 u_N + P_{\leq N} (|u_N|^2 u_N) = \Phi_N \xi$$

$$\bullet \quad \underline{u_N = P_{\leq N} u_N}, \quad \Phi_N = P_{\leq N} \circ \Phi$$

$\Rightarrow$ Write on the Fourier side

$$i d \hat{u}_N(m) = (-n^2 \hat{u}_N(m) + \overbrace{P_{\leq N} (|u_N|^2 u_N)}(m)) dt + \hat{\Phi}_N(m) d\beta_n. \quad \underline{|n| \leq N}$$

⇐ a finite dim'l system of SPDEs
for $(p_n, q_n)_{|n| \leq N}$

(2)

⇒ Apply Ito's lemma. (should check u_N is adapted.)

Since we do not know if u_N exists globally in time,
we need to use a stopping time argument.

Fix a target time $T \gg 1$.

Then, let τ be a stopping time s.t.

$$0 < \tau \leq \min(T, T_{\max}), \quad \text{a.s.}$$

where $T_{\max} = T_{\max}(\omega)$ is the maximal time of existence.

⇒ insert $\mathbb{1}_{[0, \tau]}(t)$ to the $d\hat{u}_N(t)$ equation.
and apply Ito's lemma.

(• On $\mathbb{R}^d$, also insert a cutoff in size on the nonlinearity (3)
$$O\left(\frac{\|u\|}{R}\right)$$

• By the LWP argument, we have

$$\|u_N - u\|_{X^{0, 3/8}_{\text{local in time}} \cap C_{T_{\text{local}}} L^2_x} \rightarrow 0$$

$T_{\text{local}} = T_{\text{local}}(\omega) = \text{local existence time.}$

$\Rightarrow$ verify Ito's lemma for u (with a stopping time.)

(de Bouard - Debussche '03
Oh - Okamoto '20

Back to ②: 3-d defocusing cubic SNLW on $\mathbb{T}^3$

④

$$\partial_t^2 u + (1 - \Delta) u + u^3 = \phi \Xi$$

$$\phi \in \mathcal{HS}(L^2; H^{s-1})$$

$s > 0$

Write $u = \Psi + v$.

$$\Rightarrow \partial_t^2 v + (1 - \Delta) v + \underbrace{(v + \Psi)^3}_{= v^3 + 3v^2\Psi + 3v\Psi^2 + \Psi^3} = 0.$$

$$\Psi \in L_T^\infty L_x^\infty$$

$$= v^3 + 3v^2\Psi + 3v\Psi^2 + \Psi^3.$$

• If $\Psi \equiv 0$, then the energy for $(v, \partial_t v)$

$$E(v, \partial_t v) = \frac{1}{2} \int |\nabla v|^2 dx + \frac{1}{2} \int (\partial_t v)^2 dx + \frac{1}{4} \int v^4 dx$$

is conserved.

⑤

$$\partial_t E(v, \partial_t v)$$

Fix good ω .

$$= \int \partial_t v \left(\underbrace{\partial_t^2 v + (1-\Delta)v}_{\text{use eqn}} + \cancel{v^3} \right) dx$$

$$= \cancel{-v^3} - 3v^2 \Psi - 3v\Psi^2 - \Psi^3$$

$$\leq C \|\Psi\|_{L_T^\infty L_x^\infty} \underbrace{\left(\int (\partial_t v)^2 dx \right)^{1/2} \left(\int v^4 dx \right)^{1/2}}_{\lesssim E(v, \partial_t v)}$$

$$+ C \|\Psi\|_{L_T^\infty L_x^6}^{1/2} \left(\int (\partial_t v)^2 dx \right)^{1/2}$$

$$\leq C(\Psi, T) (1 + E(v, \partial_t v)).$$

$\Rightarrow$ Apply Gronwall's inequality.

$$\Rightarrow \sup_{t \in [0, T]} \|(v(t), \partial_t v(t))\|_{\mathcal{H}^1} \leq C(\omega, T) < \infty, \text{ a.s.}$$

③ Invariant measure argument.

⑥

Consider NLW (defocusing, $k \in 2N+1$)

$$(NLW) \quad \partial_t^2 u + (1 - \Delta)u + u^k = 0$$

- NLW is a Hamiltonian equation.

$$\partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} \begin{pmatrix} \partial E / \partial u \\ \partial E / \partial (\partial_t u) \end{pmatrix}$$

$$\left(\partial p_n = \frac{\partial H}{\partial q_n}, \quad \partial q_n = -\frac{\partial H}{\partial p_n} \Leftrightarrow \partial_t \begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial H / \partial p_n \\ \partial H / \partial q_n \end{pmatrix} \right)$$

$$E = E(u, \partial_t u) = \frac{1}{2} \int |\nabla u|^2 + \frac{1}{2} \int |\partial_t u|^2 + \frac{1}{k+1} \int u^{k+1}.$$

$\Rightarrow$ " $du \, d(\partial_t u)$ " is invariant under the flow.

$\uparrow$
does not exist.

Also, $E(u, \partial_t u)$ is conserved under the NLW dynamics.

(7)

⇒ Gibbs measure

$$df(u, \partial_t u) = Z^{-1} e^{-\underline{E(u, \partial_t u)}} \underline{du d(\partial_t u)}$$

should be invariant

massive Gaussian free field

$$e^{-\frac{1}{2} \int u^{k+1} dx} \boxed{e^{-\frac{1}{2} \|u\|_{H^1}^2} du}$$

weight.

$$\otimes e^{-\frac{1}{2} \|\partial_t u\|_{L^2}^2} d(\partial_t u)$$

spatial white noise (measure)

Gaussian measures on $\mathcal{D}'(\mathbb{T}^d)$

(8)

$$d\mu_S = Z^{-1} e^{-\frac{1}{2} \|u\|_{H^S}^2} du$$

sharp freq cut-off.

$$\Leftrightarrow d\mu_{S,N} = Z_N^{-1} e^{-\frac{1}{2} \|P_{\leq N} u\|_{H^S}^2} d(P_{\leq N} u)$$

$$= Z_N^{-1} \prod_{|n| \leq N} e^{-\frac{1}{2} \frac{\langle n \rangle^{2s} |\hat{u}(n)|^2}{\|g_n\|^2}} d\hat{u}(n)$$

measure on
 $E_N = \text{span}\{e_n, |n| \leq N\}$

Lebesgue on $\mathbb{C} \cong \mathbb{R}^2$

Under $\mu_{S,N}$, we have

$$U_N(x) := \sum_{|n| \leq N} \frac{g_n(\omega)}{\langle n \rangle^s} e^{in \cdot x}$$

Take $N \rightarrow \infty$.

$\{g_n\}$, indep, standard
 $\mathbb{C}$ -valued
Gaussian r.v.'s.

• my course from 2017
• Also mini-courses

• Kuo: Gaussian meas in Banach sp.
• Janson
• Nuñez

$$N \geq M \geq 1$$

$$\mathbb{E}[\|u_N - u_M\|_{H^\sigma}^2] = \sum_{M < |n| \leq N} \frac{\langle n \rangle^{2\sigma} \mathbb{E}[|g_n|^2]}{\langle n \rangle^{2s}}$$

⑨

$$= \sum_{M < |n| \leq N} \langle n \rangle^{2\sigma - 2s} \rightarrow 0$$

$$\text{iff } 2\sigma - 2s < -d$$

$$\Leftrightarrow \boxed{\sigma < s - \frac{d}{2}}$$

• $\{u_N\}_{N \in \mathbb{N}}$ forms a Cauchy seq in $L^2(\Omega; H^\sigma(\mathbb{T}^d))$
(also in $L^p(\Omega; H^\sigma(\mathbb{T}^d))$, $p < \infty$.)

$\Rightarrow$ limit:

$$\omega \in \Omega \mapsto u(x) = \sum_{n \in \mathbb{Z}^d} \underbrace{g_n(\omega)}_{\langle n \rangle^s} e^{in \cdot x}$$

$$\mu_s = \lim_{N \rightarrow \infty} \mu_{s,N}$$

= induced probability measure under this map = $P \circ u^{-1}$.

Moral: μ_s is NOT a probability measure on $H^s(\mathbb{T}^d)$ (10)

and we needed to enlarge the space to H^σ , $\sigma < s - \frac{d}{2}$

ex: $W^{\sigma,p}$, $\sigma < s - \frac{d}{2}$
 $B = C^\sigma = B_{\infty,\infty}^\sigma$

Also, Fourier-Lebesgue spaces $FL^{\sigma,p}$, $\sigma < s - \frac{d}{p}$

and we say $(\mu_s, H^s, \underline{B})$ is an abstract Wiener space

$d=1$

Gibbs measure for NLW

$$d\rho = Z^{-1} \underbrace{e^{-\frac{1}{k+1} \int u^{k+1} dx}}_{\text{depends only on } u} d\mu_1 \otimes d\mu_0(u, \lambda + u)$$

depends only on u

$$u \in H^{\frac{1}{2}-}(\mathbb{T}), \text{ a.s. } \Rightarrow_{\text{Sob}} u \in L_x^r, \text{ a.s. } r < \infty.$$

$$\Rightarrow \underset{\text{a.s.}}{0 < e^{-\frac{1}{k+1} \int u^{k+1} dx} \leq 1}$$

$$\Rightarrow e^{-\frac{1}{k+1} \int u^{k+1} dx} \in L^p(\mu_1) \quad p \leq \infty$$

and a.s. positive

$$\Rightarrow \rho \sim \mu_1 \otimes \mu_0$$

↑ equivalent (i.e. $\Leftarrow$ and $\Rightarrow$)

ρ is a prob measure on $H^{\frac{1}{2}-}(\mathbb{T}) \times H^{-\frac{1}{2}-}(\mathbb{T})$

Aside: If $L(v) = \mu_0$, then

$$v = \sum_{n \in \mathbb{Z}^d} g_n e^{in \cdot x} \Leftarrow \text{spatial white noise}$$

same intensity of randomness at each freq.

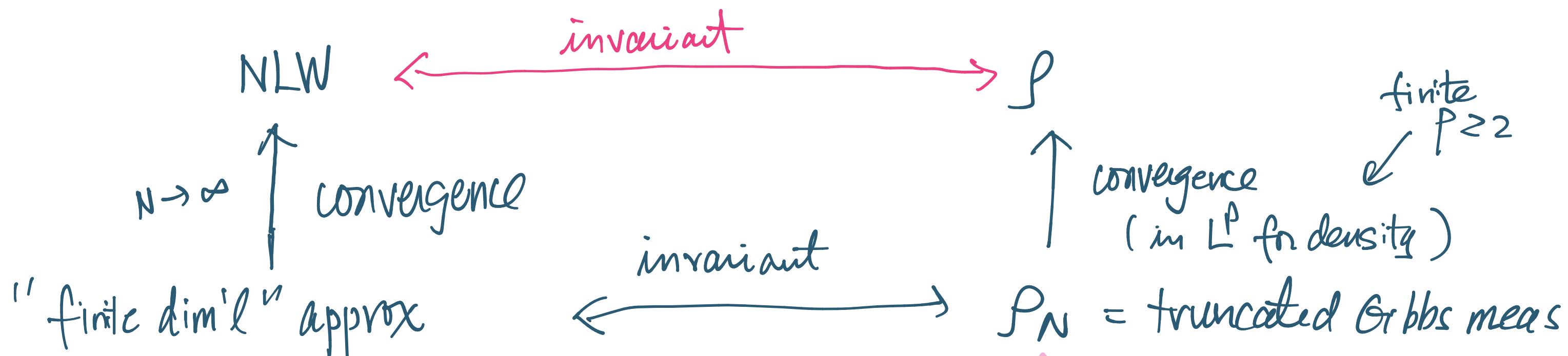
$$\text{Also, } \mathbb{E}[v(x) \overline{v(y)}] = \sum_{n \in \mathbb{Z}^d} e^{in \cdot (x-y)} = \delta_0(x-y)$$

(11)

Bourgain '94:

(12)

Bourgain's invariant measure argument.



$$\partial_t^2 u_N + (1 - \Delta) u_N + P_N((P_N u)^k) = 0$$

$$dP_N = Z_N^{-1} e^{-\frac{1}{k+1} \int (P_N u)^{k+1}} d\mu_1(x) d\mu_0(u, \partial_t u)$$

low freq ($|n| \leq N$): truncated Gibbs $\Leftarrow$ finite dim'l Hamiltonian system

high freq : Gaussian measure.
(invariant under lin dynamics.)

Main idea: Use the invariance of I_N

for the truncated dynamics. (for $(u_N, \partial_t u_N)$.)

Prop: Given $T > 0$, $\varepsilon > 0$, $\exists \Omega_{N,T,\varepsilon} \subset \Omega$

st. ① $I_N(\Omega_{N,T,\varepsilon}^c) < \varepsilon$

② For each $w \in \Omega_{N,T,\varepsilon}$, the soln $(u_N^w, \partial_t u_N^w)$ exists on $[-T, T]$ AND

$$\sup_{-T \leq t \leq T} \|(u_N(t), \partial_t u_N(t))\|_{\mathcal{H}^1} \lesssim (\log \frac{1}{\varepsilon})^{1/2}$$

indep of N .

Pf: Fix $K \gg 1$.

Set $\Omega_{N,T,\varepsilon} = \bigcap_{j=-\lceil T/\delta \rceil}^{\lceil T/\delta \rceil} \Phi_N(-j\delta) B_K,$

$\Phi_N(t): (u_0, u_1) \mapsto (u(t), \partial_t u(t))$

$\mathcal{H}^{1/2-}$ $\mathcal{H}^{1/2-}$

$\downarrow$ $\downarrow$

where B_K = ball of radius K in $\mathcal{H}^{1/2-}(\mathbb{T})$.

$\delta \sim K^{-\theta}$ local existence time for initial data of size K .

$$\textcircled{1} \quad \rho_N(\Omega_{N,T,\varepsilon}^c) \leq \sum_{j=-\lceil T/\delta \rceil}^{\lceil T/\delta \rceil} \rho_N(\Phi_N(-j\delta) B_K^c)$$

(14)

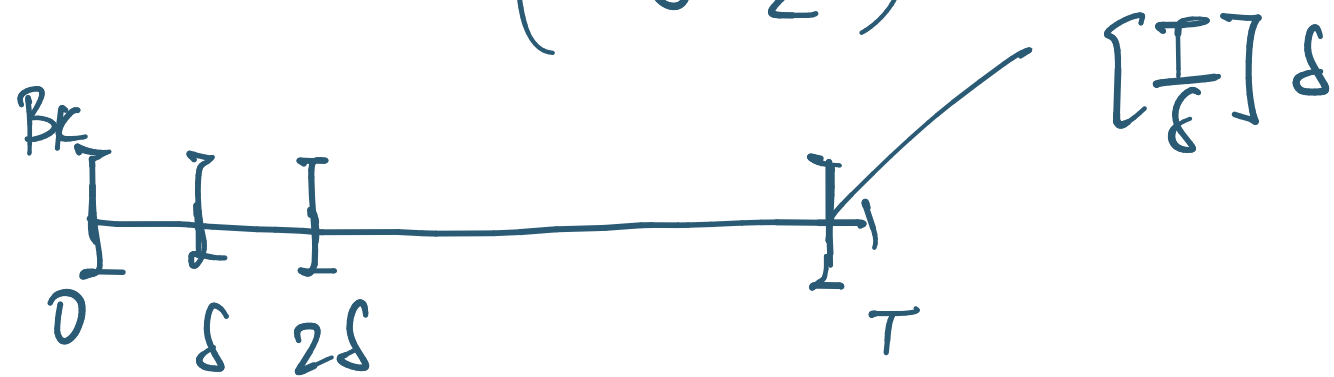
ρ_N invariant under Φ_N

$$\sim \frac{I}{\delta} \rho_N(B_K^c) \stackrel{\text{c.s.}}{\lesssim} (\mu_1 \otimes \mu_0(B_K^c))^{1/2}$$

$$\lesssim T K^\theta \underbrace{e^{-ck^2}}_{\lesssim e^{-c'k^2}} < \varepsilon$$

by choosing $K \sim \left(\log \frac{I}{\varepsilon}\right)^{1/2}$

(2)



By LWP argument, we have $\sup_{-T \leq t \leq T} \|(u_N(t), \partial_t u_N(t))\|_{H^{1/2-}} \leq CK$.

$$\sim \left(\log \frac{I}{\varepsilon}\right)^{1/2}$$

indep of N .



- The set $\Omega_{N,T,\varepsilon}$ depends on N
but the constants in the bd are indep of N .
 $\Rightarrow$ can deduce the same log bound for true soln $(u, \partial_t u)$.
(by a PDE approx argument.) $\Rightarrow$ a.s GWP.
i.e. used an invariant measure in place of a conservation law.

Lec 7 : 31 / 03 / 21 (Wed)

①

Again back to ① lto approach:

Fix $R > 0$. Set

$$\tau_R = \inf \{ t > 0 : \|u(t)\|_{L^2} \geq R \}.$$

$\Leftarrow$ Check τ_R is a stopping time.

Remark: on the construction of global-in-time solns
(i.e. a.s. GWP).

Moral: almost a.s. GWP $\Rightarrow$ a.s. GWP.

unrelated Given $T > 0, \varepsilon > 0$, $\exists \Omega_{T,\varepsilon}$ s.t. if $\omega \in \Omega_{T,\varepsilon}$,

then the corresponding soln u^ω exists on $[-T, T]$.

and $P(\Omega_{T,\varepsilon}^c) < \varepsilon$

① Fix $\varepsilon > 0$ and set

$$\Omega_j = \Omega_{2^j}, \varepsilon/2^j$$

② Set $\Omega_\varepsilon = \bigcap_{j=1}^{\infty} \Omega_j \Rightarrow P(\Omega_\varepsilon^c) \leq \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon$

Note: if $\omega \in \Omega_\varepsilon$, then u^ω exists globally in time.

③ Now, set

$$\Sigma = \bigcup_{\varepsilon > 0} \Omega_\varepsilon \Rightarrow P(\Sigma^c) = \inf_{\varepsilon > 0} P(\Omega_\varepsilon^c) = 0.$$

$\Rightarrow$ a.s. GWP.

(or use Borel-Cantelli lemma)

②

Stochastic damped NLW:

③

$$\partial_t^2 u + \partial_t u + (1 - \Delta) u + u^k = \sqrt{2} \xi, \quad k \in 2\mathbb{N} + 1.$$

Let $v = \partial_t u$. Then,

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} v \\ -(1 + \Delta) u - u^k \end{pmatrix}}_{\text{NLW}} + \underbrace{\begin{pmatrix} 0 \\ -v + \sqrt{2} \xi \end{pmatrix}}_{\text{Ornstein-Uhlenbeck process.}}$$

Gibbs meas : $d\mathcal{P} = Z^{-1} \exp\left(-\frac{1}{k+1} \int u^{k+1} dx\right) d\mu, \otimes d\mu_0(u, v)$

Note: NLW preserves $\mathcal{P}$.

white noise for v .

O.U. on v preserves $\mu_0(dv) \Rightarrow$ preserves $\mathcal{P}$.

Ornstein-Uhlenbeck process (real-valued)

(4)

$$\begin{cases} dX = -aXdt + b dB \\ X(0) = x_0 \end{cases} \quad a > 0, b \in \mathbb{R}$$

$$\Rightarrow X(t) = e^{-at} x_0 + b \int_0^t e^{-a(t-t')} dB(t')$$

Suppose $x_0 \sim N(0, \frac{b^2}{2a})$, indep from B

Then, $X(t)$ is a mean 0 Gaussian r.v. with var

$$\begin{aligned} \mathbb{E}[X^2(t)] &= e^{-2at} \mathbb{E}[x_0^2] + b^2 \int_0^t e^{-2a(t-t')} dt' \\ &= e^{-2at} \frac{b^2}{2a} + \frac{b^2}{2a} (1 - e^{-2at}) \\ &= \frac{b^2}{2a}. \end{aligned}$$

i.e. $N(0, \frac{b^2}{2a})$ is invariant under OU .

Back to SdNLW:

Look at the UV part (for v).

(5)

$$(*) \quad \partial_t v = -v + \sqrt{2} \xi$$

F.T. ind
 $\Rightarrow$

$$\partial_t \hat{v}(n) = -\hat{v}(n) + \sqrt{2} d\beta_n.$$

(Further separate into $\operatorname{Re} \hat{v}(n)$, $\operatorname{Im} \hat{v}(n)$)

$\Rightarrow g_n \sim \mathcal{N}_{\mathbb{C}}(0, 1)$ is invariant for

$\Rightarrow (*)$ preserves $\mu_0(dv)$ and hence $\mathcal{P}(du, dv)$

• How to prove a.s. GWP & invariance of $\mathcal{P}$ under SdNLW?

Consider the truncated dynamics.

⑥

$$(SdNLW_N) \quad \partial_t^2 u_N + \partial_t u_N + (1-\Delta) u_N + P_{\leq N} (P_{\leq N} u)^k = \sqrt{2} \xi.$$

Note: For high frequencies ($|m| > N$), it is

~~XX~~ $\partial_t^2 P_{>N} u_N + \partial_t P_{>N} u_N + (1-\Delta) P_{>N} u_N = \sqrt{2} P_{>N} \xi.$

$\mu_{1,N}^\perp \otimes \mu_{0,N}^\perp$ is invariant.

$\mu_{j,N}^\perp$ = marginal measure of μ_j on $E_N^\perp$
 $\parallel$
 $\text{span}\{e_n : |n| > N\}.$

and truncated Gibbs measure

$$d\rho_N = Z_N^{-1} \exp\left(-\frac{1}{k+1} \int (P_{\leq N} u)^{k+1} dx\right) d\mu_{1,N} \otimes d\mu_{0,N} (P_{\leq N}(u_N, \partial_t u_N))$$

$$\otimes \left(d\mu_{1,N}^\perp \otimes d\mu_{0,N}^\perp (P_{>N}(u_N, \partial_t u_N)) \right)$$

invariant under ~~XX~~

P_N invariant under $SdNLW_N$

(7)

$\Leftrightarrow L^N =$ generator for the Markov semigroup for $SdNLW_N$

Then, $(L^N)^*$ $P_N = 0$.
adjoint

$\downarrow$
phase space $\vec{u} = (u, v)$ $\nearrow \partial_t u$

$$\underline{P_t F(\vec{u})} = \mathbb{E}[F(\Phi_t \vec{u})]$$

$\uparrow$
soln map

$$L^N = L_1^N + L_2^N$$

$L_1^N =$ generator for ...
for NLW_N

generator = time deriv at $t=0$.

$L_2^N =$ generator for ... for OV process (for v).

$$(L^N)^* P_N = (L_1^N)^* P_N + (L_2^N)^* P_N = 0 + 0 = 0.$$

$\Rightarrow P_N$ is invariant under $SdNLW_N$.

⇒ Repeat Bourgain's invariant measures argument.

⑧

⇒ a.s. GWP of SdNLW & invariance of μ under SdNLW.

(also think of For ~~***~~, Lie-Trotter product formula: $e^{(A+B)t} = \lim_{n \rightarrow \infty} (e^{\frac{A}{n}} e^{\frac{B}{n}})^n$)

• Parabolic Φ_1^{k+1} -model:

$$\partial_t u + \underbrace{(1-\Delta)u - u^k}_{\uparrow} = \sqrt{2} \xi.$$

Consider $du = -\gamma \nabla_u H(u) dt + \sqrt{2} \gamma dW.$

$$\mathcal{L}(F(u)) = \gamma \operatorname{Tr} D^2 F(u) - \gamma \langle \nabla F(u), \nabla H(u) \rangle$$

$$\begin{aligned} \mathcal{L}^*(\mu) = 0 &\Leftrightarrow \int \mathcal{L}F(u) \underline{\mu}(du) \\ &= \gamma \int \operatorname{Tr} D^2 F(u) \underline{e^{-H(u)}} du + \gamma \int \langle \nabla F(u), \underline{\nabla(e^{-H(u)})} \rangle du \\ &= 0 \quad \text{by IBP.} \end{aligned}$$

$$\partial_t u = (\Delta - 1) u - u^k + \sqrt{2} \xi.$$

⑨

$$\hat{u}_n = a_n + i b_n.$$

$$N(u) = u^k.$$

$$da_n = \left(-\underbrace{\langle n^2 \rangle a_n} - \underbrace{\operatorname{Re}(\hat{N}(u)(n))} \right) dt + \sqrt{2} d(\operatorname{Re} \beta_n)$$

When $n=0$,
 $b_0=0$.

$$db_n = \left(-\underbrace{\langle n^2 \rangle b_n} - \underbrace{\operatorname{Im}(\hat{N}(u)(n))} \right) dt + \sqrt{2} d(\operatorname{Im} \beta_n)$$

(should consider the truncated dynamics..)

$$L_1^N F(\bar{a}, \bar{b}) = \sum_n \left(-\underbrace{\langle n^2 \rangle a_n} \partial_{a_n} + \underbrace{\frac{1}{2} \partial_{a_n}^2} \right) F$$

from the noise

$$+ \sum_n \left(-\underbrace{\langle n^2 \rangle b_n} \partial_{b_n} + \frac{1}{2} \partial_{b_n}^2 \right) F$$

when $n=0$
 $\partial_{a_0}^2 F$

$$L_2^N F(\bar{a}, \bar{b}) = \sum_n \partial_{a_n} F \left(-\operatorname{Re}(\hat{N}(u))(n) \right)$$

$$+ \sum_n \partial_{b_n} F \left(-\operatorname{Im}(\hat{N}(u))(n) \right).$$

Linear case: $N(u) = 0$

$$d\mu_N = z_N^{-1} e^{-\frac{1}{2} \|P_N u\|_{H^1}^2} dP_N u.$$

(10)

$$(L_1^N)^* \mu_N = 0$$

$$\Leftrightarrow \int L_1^N F(\bar{a}, \bar{b}) d\mu_N = 0?$$

$$\stackrel{||}{=} z_N^{-1} e^{-\frac{1}{2} \sum \langle m \rangle^2 a_n^2 + \langle m \rangle^2 b_n^2} \prod da_n db_n$$

$$\int \left(-\langle m \rangle^2 a_n \partial_{a_n} F + \frac{1}{2} \partial_{a_n}^2 F \right) e^{-\square} da_n$$

$$\stackrel{\text{IBP}}{=} \int \underbrace{\langle m \rangle^2 F \partial_{a_n} (a_n e^{-\square})}_{\langle m \rangle^2 F e^{-\square} - 2 \langle m \rangle^2 F \langle m \rangle^2 a_n^2 e^{-\square}} + \frac{1}{2} F \partial_{a_n}^2 e^{-\square} da_n = 0.$$

$$\langle m \rangle^2 F e^{-\square} - 2 \langle m \rangle^2 F \langle m \rangle^2 a_n^2 e^{-\square}$$

$$a_{-n} = a_n$$

$$F \partial_{a_n} (-\langle m \rangle^2 a_n e^{-\square})$$

$$= -F \langle m \rangle^2 e^{-\square} + 2F \langle m \rangle^4 a_n^2 e^{-\square}$$

similar comp holds for the b_n -part.

(11)

$$\Rightarrow (L_1^N)^* \mu_N = 0.$$

Gibbs measure invariance (truncated).

$$L^N = L_1^N + L_2^N$$

$$(L^N)^* \mu_N = 0 \Leftrightarrow \int L^N F(\bar{a}, \bar{b}) d\mu_N(\bar{a}, \bar{b}) = 0 ?$$

$$\sim \underbrace{e^{-\frac{1}{k+1} \int (P_{\leq N} u)^{k+1}}}_{\substack{\text{"} \\ e^{-M(\bar{a}, \bar{b})}}} d\mu_N$$
$$e^{-\frac{1}{2} \sum \langle m^2 \rangle a_n^2 + \langle n^2 \rangle b_n^2} \pi da_n db_n$$

$$\partial_{a_n} M(\bar{a}, \bar{b}) = \partial_{a_n} \frac{1}{k+1} \sum_{n_1 + \dots + n_{k+1} = 0} \hat{u}_{n_1} \dots \hat{u}_{n_{k+1}}$$

$$= \underbrace{\partial_{\hat{u}_n} M}_{\downarrow} \cdot \frac{\partial \hat{u}_n}{\partial a_n} + \partial_{\hat{u}_{-n}} M \cdot \frac{\partial \hat{u}_{-n}}{\partial a_n}$$

$$\sum_{\substack{n_1 + \dots + n_k = -n}} \hat{u}_{n_1} \dots \hat{u}_{n_k}$$

$$= \hat{N}(u)(-n) + \hat{N}(u)(n) = 2 \operatorname{Re} \hat{N}(u)(n)$$

Similarly,

$$\begin{aligned} \partial_{b_n} M(\bar{a}, \bar{b}) &= i \hat{N}(u)(-n) - i \hat{N}(u)(n) \\ &= 2 \operatorname{Im} \hat{N}(u)(n). \end{aligned}$$

$$(L_1^N + L_2^N) F(\bar{a}, \bar{b})$$

(13)

$$\int \left(-\langle m^2 \rangle a_n - \text{Re} \hat{N}(u)(n) \right) \partial a_n F e^{-\square} d\bar{a} d\bar{b}$$

$\xrightarrow{L_2^N}$

$\downarrow$ includes M .

$$= \frac{1}{2} \partial a_n \square$$

$$\frac{1}{2} \partial a_n^2 F e^{-\square} \stackrel{\text{IBP}}{=} -\frac{1}{2} \partial a_n F \partial a_n \square e^{-\square}$$

$\xrightarrow{2^{\text{nd}} \text{ term in } L_1^N}$

$$\Rightarrow (L^N)^* P_N = 0.$$

Chap 2: 2-d case

(14)

Consider

$$U = \sum_{n \in \mathbb{Z}^2} \frac{g_n}{\langle n \rangle} e^{in \cdot x} \quad \text{GFF} \quad \sim d\mu, = Z^{-1} e^{-\frac{1}{2} \|u\|_{H^1}^2} du.$$

$$U_N = \sum_{|n| \leq N} \frac{g_n}{\langle n \rangle} e^{in \cdot x}$$

For fixed $x \in \mathbb{T}^2$, $U_N(x)$ is a mean 0 Gaussian r.v.

with var

$$\sigma_N^2 = \mathbb{E}[|U_N(x)|^2] = \sum_{|n| \leq N} \frac{1}{\langle n \rangle^2} \sim \log N \rightarrow \infty.$$

$\Rightarrow$ The limit u is NOT a function but a distribution.

$$\mathbb{E}[\|u\|_{H^s}^2] = \sum_{n \in \mathbb{Z}^2} \frac{1}{\langle n \rangle^{2(1-s)}} < \infty \quad \text{iff } \underline{s < 0}$$

For $s \geq 0$, $u \notin H^s(\mathbb{T}^2)$ with positive probability.

$\Rightarrow$ By Kolmogorov's 0-1 law,

$u \notin L^2(\mathbb{T}^2)$, a.s.

In particular, the power u^* is ill-defined
and thus we need to introduce a renormalization

- A word on the generator L of a Markov semigroup

Given a stochastic process X , satisfying

$$dX = f dt + g dB, \quad X(0) = x_0,$$

its Markov semigroup P_t is given by

$$P_t F(x_0) = \mathbb{E}[F(X_t)].$$

the generator L is then given by

$$LF(x_0) = \lim_{t \rightarrow 0} \frac{P_t F(x_0) - F(x_0)}{t} = \lim_{t \rightarrow 0} \frac{\mathbb{E}[F(X_t)] - F(x_0)}{t}.$$

- By Ito's lemma, we have

$$F(X_t) - F(x_0) = \int_0^t F'(X_s) f(s) ds + \int_0^t \cancel{F'(X_s) g(s) dB(s)} + \frac{1}{2} \int_0^t F''(X_s) g^2(s) ds$$

vanishes under expectation

$$\Rightarrow LF(x_0) = F'(x_0) f(0) + \frac{1}{2} F''(x_0) g^2(0)$$

ex) $X = B \Rightarrow L = \frac{1}{2} \partial_x^2.$

Lec 8 02 / 04 / 21 (Fri)

①

$$u(x) = \sum_{n \in \mathbb{Z}^2} \underbrace{g_n(\omega)}_{\langle n \rangle} e^{in \cdot x}$$

law of u

$$\downarrow$$
$$\mathcal{L}(u) = \mu, = \underbrace{\text{GFF on } \mathbb{T}^2}_{\text{massive}}$$

$\{g_n\}_{n \in \mathbb{Z}^2}$ indep std $\mathbb{C}$ -valued Gaussian r.v.'s

(In the real-valued setting,
impose $g_{-n} = \overline{g_n}$)

• Last time:

$$u \in \underline{H^s(\mathbb{T}^2)} \setminus L^2(\mathbb{T}^2), \text{ a.s. } \boxed{s < 0}$$

What about $\underline{L^p(\mathbb{T}^2)}$, $p < 2$?

$$L^p(\mathbb{T}^2) \supset L^2(\mathbb{T}^2)$$

• $p=1$:

$$\underline{u \notin L^1(\mathbb{T}^2), \text{ a.s.}}$$

Let $X_n = \frac{g_n}{\langle n \rangle} e^{in \cdot x}$ (for fixed $x \in \mathbb{T}^2$)

(2)

and $Y_n = X_n \cdot \mathbb{1}_{\{|X_n| \leq A\}}$ (for some $A > 0$)

Kolmogorov three series thm. X_n indep

Then, $\sum X_n$ converges a.s. iff the following 3 cond. hold:
for ANY $A > 0$

(i) $\sum P(|X_n| \geq A)$ converges

(ii) With $Y_n = X_n \cdot \mathbb{1}_{\{|X_n| \leq A\}}$,

$\sum \mathbb{E}[Y_n]$ converges.

(iii) $\sum \text{var}(Y_n)$ converges.

$|g_n| \leq A \langle n \rangle$



For our problem, $\sum \text{var}(Y_n) \geq \sum_{n \in \mathbb{Z}^2} \frac{1}{\langle n \rangle^2} \mathbb{E}[|g_n|^2 \cdot \mathbb{1}_{\{|g_n| \leq A\}}]$
 $= \infty$

$\Rightarrow$ By Kolmog. 3-series thm, $\sum X_n$ diverges on a set of positive prob.

$\Rightarrow$ By Kolmog 0-1 law, $\sum X_n(x)$ diverges a.s.

compute

$$\mathbb{E} \left[\int_{\mathbb{T}^2} |u(x)| dx \right] = \int_{\mathbb{T}^2} \overbrace{\mathbb{E}[|u(x)|]}^{", \infty"} dx = \infty$$

$\Rightarrow u_N$ does not converge in L^1 with positive prob.

$\Rightarrow u_N$ diverges in $L^1(\mathbb{T}^2)$, a.s.

(3)

(4)

Back to $u(x) = \sum_{n \in \mathbb{Z}^2} \frac{g_n}{\langle n \rangle} e^{in \cdot x}$.

$$\mathbb{E}[u_N(x) \overline{u_N(y)}] = \sum_{|n| \leq N} \frac{1}{\langle n \rangle^2} e_n(x-y)$$

insert $\langle \nabla_x \rangle^{-\varepsilon}$ $\langle \nabla_y \rangle^{-\varepsilon}$

and set $x=y$.

$$\Rightarrow \mathbb{E} \int |\langle \nabla \rangle^{-\varepsilon} u_N(x)|^2$$

$$\lesssim \sum_{\substack{|n| \leq N \\ n \in \mathbb{Z}^2}} \frac{1}{\langle n \rangle^{2+2\varepsilon}} < \infty \quad \text{unif in } N \in \mathbb{N}.$$

$$\Rightarrow \underbrace{\| \| u_N \|_{W^{s,r}(\mathbb{T}^2)} \|_{L^p(\Omega)}}_{\text{Mink}} \leq \| \| \langle \nabla \rangle^s u_N(x) \|_{L^p(\Omega)} \|_{L_x^r(\mathbb{T}^2)}$$

$$\lesssim p^{1/2}, \quad \text{unif in } N \in \mathbb{N}$$

$$\leq p^{1/2} \| \langle \nabla \rangle^s u_N(x) \|_{L^2(\Omega)}$$

for $\underline{s < 0}$

Note:

$$\mathbb{E}[g_n^k \overline{g_n^l}] = k! \delta_{kl}$$

In particular,

$$\mathbb{E}[g_n^2] = 0.$$

(5)

A similar computation for $u_N - u_M$ holds

$\Rightarrow \{u_N\}_{N \in \mathbb{N}}$ converges in $L^p(\Omega; W_x^{s,r}(\mathbb{T}^2))$

$s < 0$ and $r \leq \infty$

Also, we have an exponential tail estimate.

$$P(\|u\|_{W_x^{s,r}} > \lambda) \leq C e^{-c\lambda^2}, \quad \forall \lambda > 0$$

Since u is NOT a function and is only a distribution, the power u^k is ill-defined.

$\Rightarrow$ We need to introduce a renormalization.

In the following, we restrict our attention to the real-valued setting.

(6)

Hermite polynomial:

generating function: $e^{tx - \frac{1}{2}\sigma t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \underline{H_k(x; \sigma)}$

• $H_k(x; \sigma) =$ Hermite poly of deg k .

$$H_0(x; \sigma) = 1, \quad H_1(x; \sigma) = x, \quad H_2(x; \sigma) = x^2 - \sigma$$

$$H_3(x; \sigma) = x^3 - 3\sigma x, \quad H_4(x; \sigma) = x^4 - 6\sigma x^2 + 3\sigma^2, \dots$$

• Set $H_k(x) = H_k(x; 1)$

orthogonality: $\int H_k(x) H_m(x) \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = k! \delta_{km}$

FACT: $\left\{ \frac{H_k(x)}{\sqrt{k!}} \right\}_{k \in \mathbb{Z}_{\geq 0}}$ forms an O.N.B. of $L^2(\mathbb{R}, \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx)$.

Note: $\{x^k\}_{k \in \mathbb{Z}_{\geq 0}} \xrightarrow{\text{Gram-Schmidt in } L^2(\mathbb{R}; \text{Gauss})} \{H_k(x)\}_{k \in \mathbb{Z}_{\geq 0}}$

We also have the following orthogonality:

(7)

f, g , Gaussian r.v.'s with var σ_f and σ_g
(mean 0)

$$\mathbb{E} [H_k(f; \sigma_f) H_m(g; \sigma_g)] = \delta_{km} k! \{ \mathbb{E}[fg] \}^k$$

⌊ $\Leftarrow$ Take the product of the generating functions
for (f, σ_f) and (g, σ_g) .
and express in two different series
and compare the coeffs of t^k .

- In the complex-valued setting, we instead use
the (generalized) Laguerre polynomials: See Oh-Thomann
SPDE '18.

f, g -
jointly
Gaussian

(*)

Wick renormalization

Wick power, renormalized power

(8)

$$: u_N^k(x) : = H_k(u_N(x); \sigma_N) \quad \text{pointwise}$$

$$\text{where } \sigma_N = \mathbb{E}[u_N^2(x)] = \sum_{|n| \leq N} \frac{1}{\langle n \rangle^2} \sim \log N \rightarrow \infty$$

(On a mfd, σ_N depends on $x \in \mathcal{M}$.

$$\cdot \quad k=2. \quad : u_N^2 : = u_N^2 - \sigma_N$$

$$k=3 \quad : u_N^3 : = u_N^3 - 3 \sigma_N u_N$$

• Wick renormalization = orthogonal projection
onto the homog Wiener chaos of
deg k

• (H, B, μ) , abstract Wiener space

$$d\mu = Z^{-1} e^{-\frac{1}{2}\|u\|_H^2} du$$

(Think of $H = H^1(\mathbb{T}^2)$
 $B = H^{-\varepsilon}(\mathbb{T}^2)$.)

• $\{e_j\}_{j \in \mathbb{N}} \subset B^*$ complete orthonormal system of $H^* = H$.

• Consider $\prod_{j=1}^{\infty} H_{k_j}(\langle x, e_j \rangle)$, $k_j \neq 0$ only for finitely many j 's

"
 product of finitely many Hermite poly
 in different directions.

$\langle \cdot, \cdot \rangle = B - B^*$ dual pairing.

$$\begin{aligned} \underline{H_k} &= \text{homog Wiener chaoses of deg } k \\ &= \overline{\text{span} \left\{ \prod_{j=1}^{\infty} H_{k_j}(\langle x, e_j \rangle) : \sum_j k_j = k \right\}}^{\|\cdot\|_{L^2(B, \mu)}} \end{aligned}$$

Wiener - Itô decomposition:

$$L^2(B, \mu) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$$

We set $\mathcal{H}_{\leq k} = \bigoplus_{j=0}^k \mathcal{H}_j$

Lemma (Wiener chaos estimate)

Let $k \in \mathbb{N}$. Then, we have

$$\|X\|_{L^p(\Omega)} \leq (p-1)^{k/2} \|X\|_{L^2(\Omega)}$$

for any $p \geq 2$ and any $X \in \mathcal{H}_{\leq k}$

$\Leftarrow$ follows from the hypercontractivity of the OU semigroup.

Nelson '65 $e^{tL} \quad L = \Delta - x \cdot \nabla \quad (B = \mathbb{R}^d)$

elements in $\mathcal{H}_k$ = eigenfunction of L with e-val $-k$.

Prop: $\{ : u_N^k : \}_{N \in \mathbb{N}}$ forms a Cauchy seq

(11)

in $L^p(\Omega; W^{s,r}(\mathbb{T}^2))$, $s < 0$, $r \leq \infty$.

Denoting the limit by $: u^k :$, we have

$: u^k : \in W^{s,r}(\mathbb{T}^2)$, a.s. $s < 0$, $r \leq \infty$

Pf: $\mathbb{E} [: u_N^k(x) : : u_N^k(y) :]$

$$= k! \left\{ \mathbb{E} [u_N(x) u_N(y)] \right\}^k$$

$$= k! \sum_{\substack{|n_j| \leq N \\ j=1, \dots, k}} \frac{1}{\langle n_1 \rangle^2} e_{n_1}(x-y) \cdots \frac{1}{\langle n_k \rangle^2} e_{n_k}(x-y)$$

$$= k! \sum_{|n_j| \leq N} \prod_{j=1}^k \frac{1}{\langle n_j \rangle^2} \underline{e_{n_1 + \dots + n_k}(x-y)}$$

Insert $\langle \nabla x \rangle^\varepsilon$ and $\langle \nabla y \rangle^\varepsilon$ and set $x=y$.

(12)

$$\Rightarrow \mathbb{E} \left[|\langle \nabla \rangle^\varepsilon : u_N^k(x) :|^2 \right] \\ = k! \sum_{|n_j| \leq N} \prod_{j=1}^k \frac{1}{\langle m_j \rangle^2} \cdot \frac{1}{\langle n_1 + \dots + n_k \rangle^2}$$

$\lesssim_{\varepsilon, k} 1$, unif in $N \in \mathbb{N}$.

$$\Rightarrow_{r < \infty} \| \| : u_N^k : \|_{W_x^{s,r}} \|_{L^p(\Omega)} \stackrel{\text{Mink}}{\leq}_{p \geq r} \| \| \langle \nabla \rangle^s : u_N^k(x) : \|_{L^p(\Omega)} \|_{L_x^r}$$

$\lesssim p^{k/2}$. for $s < 0$

$$\leq p^{k/2} \| \dots \|_{L^2(\Omega)}$$

Wiener chaos estimate.

• A similar computation holds

for $: u_N^k : - : u_M^k :$, $N \geq M \geq 1$.

In this case, we use $\max(|m_j|) \geq M \Rightarrow \frac{1}{\langle m_j \rangle^2} \leq \frac{1}{M^\delta} \frac{1}{\langle m_j \rangle^{2-\delta}}$

□

Note:

$$P(\|:u^k:\|_{W^{s,r}} > \lambda) \leq c e^{-c \lambda^{2/k}} \text{ for any } \lambda > 0$$

 $(\Leftarrow \text{Chebyshev})$

$$\Downarrow \int e^{c \|:u^k:\|_{W^{s,r}}^{2/k}} d\mu < \infty$$

Tzvetkov

Lem 4.5

in PTRF '10

Lemma: Let $\{X_N\}_{N \in \mathbb{N}}$ and X be

spatially homogeneous stoch processes: $\mathbb{R} \rightarrow \mathcal{D}'(\mathbb{T}^d)$

i.e. for any $x_0 \in \mathbb{T}^d$,

$$\{X(t, \cdot)\}_{t \in \mathbb{R}} \text{ and } \{X(t, \cdot + x_0)\}_{t \in \mathbb{R}}$$

have the same law.

• Suppose that $X_N(t), X(t) \in \mathcal{H}^{\leq k}, \forall t \in \mathbb{R}$

(i) Fix $t \in \mathbb{R}$. If $\exists s_0 \in \mathbb{R}$ s.t.

(14)

(**)

$$\mathbb{E} [|\hat{X}(t, n)|^2] \lesssim \langle n \rangle^{-d-2s_0}, \quad \forall n \in \mathbb{Z}^d.$$

then,

$$X(t) \in W^{s, \infty}(\mathbb{T}^d), \quad \underline{s < s_0}, \text{ a.s.}$$

Moreover, if $\exists \theta > 0$ s.t.

$$\mathbb{E} [|\hat{X}_N(t, n) - \hat{X}(t, n)|^2] \lesssim N^{-\theta} \langle n \rangle^{-d-2s_0}, \quad \forall n \in \mathbb{Z}^d$$

then,

$$X_N \rightarrow X \text{ in } W^{s, \infty}(\mathbb{T}^d), \quad s < s_0, \text{ a.s.}$$

(ii') Fix $T > 0$. Suppose (i) holds on $[-T, T]$.

• If $\exists \sigma \in (0, 1)$ s.t.

$$\mathbb{E} [| \delta_h \hat{X}(t, n) |^2] \lesssim \langle n \rangle^{-d-2s_0+\sigma} |h|^\sigma$$

$$\begin{aligned} & \forall n \in \mathbb{Z}^d \\ & \forall |h| \leq 1 \\ & \forall t \in [-T, T] \end{aligned}$$

then

$$X \in C([-T, T]; W^{s, \infty}(\mathbb{T}^d))$$

for $\underline{s < s_0 - \sigma/2}$, a.s.

Furthermore, if $\exists \theta > 0$ s.t.

$$\mathbb{E} \left[|f_n \hat{X}_N(t, n) - f_n \hat{X}(t, n)|^2 \right] \lesssim N^{-\theta} \langle n \rangle^{-d-2s_0+\sigma} |h|^\sigma$$

then,

$$\forall n \in \mathbb{Z}^d, |h| \leq 1, t \in [-T, T]$$

X_N converges to X in $C([-T, T]; W^{s, \infty}(\mathbb{T}^d))$

for $s < s_0 - \frac{\sigma}{2}$, a.s.

Note: $W^{s, \infty}(\mathbb{T}^d)$ can be replaced by $C^s(\mathbb{T}^d) = B_{\infty, \infty}^s(\mathbb{T}^d)$

Mourrat-Weber-Xu '17. Oh-Okamoto-Tzvetkov

• 2-d SNLW:

(16)

$$\Psi(t, x) = \int_0^t \frac{\sin(t-t') \langle \nabla \rangle}{\langle \nabla \rangle} dW(t') \in C_t W_x^{s, \infty}(\mathbb{T}^2), \text{ a.s.}$$

$s < 0.$

$$\boxed{\Psi_N = P_{\leq N} \Psi}$$

We set

$$: \Psi_N^k(t, x) : = H_k(\Psi_N(t, x); \sigma_N(t))$$

where

$$\sigma_N(t) = \mathbb{E}[|\Psi_N(t, x)|^2] \sim \log N$$

Prop: $\{ : \Psi_N^k : \}_{N \in \mathbb{N}}$ forms a Cauchy seq

in $L^p(\Omega; C_T W_x^{-\varepsilon, \infty})$, $p < \infty$.

and also almost surely.

Gubinelli-Koch-Oh
TAMS '18

Pf: We only verify ~~xx~~ in Lemma
regularity lemma

$$\mathbb{E} \left[\left| \widehat{:\Psi_N^k:}(t, n) \right|^2 \right]$$

$$= \int \int_{\pi_x^2 \times \pi_y^2} \mathbb{E} \left[: \Psi_N^k(t, x) : : \Psi_N^k(t, y) : \right] e_n(y-x) dx dy$$

$$\stackrel{\text{Lemma}}{=} k! \underbrace{\left[\mathbb{E} \left[\Psi_N(t, x) \Psi_N(t, y) \right] \right]}^k$$

$$= \sum_{|m| \leq N} \int_0^t \left(\frac{\sin(t-t') \langle m \rangle}{\langle m \rangle} \right)^2 dt' e_m(x-y)$$

$$\lesssim \sum_{n_1 + \dots + n_k = n} \frac{k}{\prod_{j=1}^k} \frac{1}{\langle n_j \rangle^2} \lesssim \langle n \rangle^{-2+\varepsilon} \quad \parallel -2s_0.$$

$$\Rightarrow \Psi_N(t) \in W^{s, \infty}(\mathbb{T}^d), \quad s < 0, \quad \text{a.s.}$$



A word on the summation:

$$\sum_{\substack{n_1 + \dots + n_k = n \\ n_j \in \mathbb{Z}^2}} \prod_{j=1}^k \frac{1}{\langle n_j \rangle^2}$$

$\nearrow$ $k-1$ summations

• Divide the sum into regions I_ℓ

$$I_\ell = \{ |n_\ell| \geq \max_{j=1, \dots, k} |n_j| \}$$

• Under $n_1 + \dots + n_k = n$, we have $\max_{j=1, \dots, k} |n_j| \geq |n|$

$$\sum_{I_\ell} \prod_{j=1}^k \frac{1}{\langle n_j \rangle^2} \lesssim \frac{1}{\langle n \rangle^{2-\varepsilon}} \sum_{\substack{j=1 \\ j \neq \ell}}^k \prod_{j=1}^k \frac{1}{\langle n_j \rangle^{2+\frac{\varepsilon}{k-1}}} \lesssim \frac{1}{\langle n \rangle^{2-\varepsilon}}.$$

• One can also use the following lemma (Part(i)) iteratively.

Lemma: (i) $\alpha, \beta \in \mathbb{R}$ s.t. $\alpha + \beta > d$ and $\alpha, \beta < d$

Then,

$$\sum_{\substack{n=n_1+n_2 \\ n_j \in \mathbb{Z}^d}} \frac{1}{\langle n_1 \rangle^\alpha} \frac{1}{\langle n_2 \rangle^\beta} \lesssim \langle n \rangle^{d-\alpha-\beta}$$

resonant
case

(ii) $\alpha, \beta \in \mathbb{R}$ s.t. $\alpha + \beta > d$.

(19)

Then,

$$\sum_{\substack{n=n_1+n_2 \\ \underline{n_1} \sim \underline{n_2} \\ n_j \in \mathbb{Z}^d}} \frac{1}{\langle n_1 \rangle^\alpha \langle n_2 \rangle^\beta} \lesssim \langle n \rangle^{d-\alpha-\beta}.$$

This lemma follows from elementary computation.

See Moumat - Weber - Xu '17 (Also, see Ginibre - Tsutsumi - Velo
JFA 197
Lemma 4.2.

Last time, we checked the regularity of the Wick power

$$:\Psi^k: = \lim_{N \rightarrow \infty} : \Psi_N^k : , \quad \Psi_N = P_{\leq N} \Psi_{\text{wave}}$$

for the wave case.

- We only checked $(**)$ for fixed t but the other conditions in the regularity lemma.

(For the time difference part, simply use the mean value theorem to get $\langle m \rangle^\sigma |h|^\sigma$.

- $:\Psi^k: \in \mathcal{H}_k \Rightarrow$ By the Wiener chaos estimate, we get

$$P \left(\|:\Psi^k:\|_{L_T^q W_x^{-\varepsilon, \infty}} > \lambda \right) \leq C \exp \left(-c \frac{\lambda^{2/k}}{T^{1+2/qk}} \right)$$

$\forall \varepsilon > 0$, finite $q \geq 1$, $T > 0$.

For $q = \infty$, we can use the Garsia - Rodemich - Rumsey inequality. (2)

and get

$$P\left(\|:\Psi^k:\|_{L^\infty([j, j+1]; W_x^{-\varepsilon, \infty}(\mathbb{T}^2))} > \lambda\right)$$

see Friz - Victoir
GKO - Tolomeo
IMRN '21

$$\leq C \exp\left(-C \frac{\lambda^{2/k}}{j+1}\right)$$

• 2-d heat:

$$\Psi = \Psi_{\text{heat}} = \int_0^t e^{(t-t')(\Delta-1)} dW(t').$$

We only check $(**)$ in the regularity lemma. (drop N .)

$$\mathbb{E}[|\widehat{:\Psi^k:(t, n)}|^2] = \iint_{\mathbb{T}_x^2 \times \mathbb{T}_y^2} \mathbb{E}[\underbrace{:\Psi^k(t, x): : \Psi^k(t, y):}_{k! \left\{ \mathbb{E}[\Psi(t, x) \Psi(t, y)] \right\}^k}] \underbrace{e_n(y-x)}_{e_m(x-y)} dx dy$$

$$\sum_m \int_0^t e^{-2(t-t')\langle m \rangle^2} dt' \underbrace{e_m(x-y)}_{= \frac{1 - e^{-2t\langle m \rangle^2}}{\langle m \rangle^2}}$$

$$\lesssim \sum_{n_1 + \dots + n_k = n} \prod_{j=1}^k \frac{1}{\langle n_j \rangle^2} \lesssim \langle n \rangle^{-2+\varepsilon}, \quad \forall \varepsilon > 0.$$

③

as in the wave case.

Prop: $\{:\Psi_N^h:\}_{N \in \mathbb{N}}$ forms a Cauchy sequence

in $L^p(\Omega; C_t C_x^{-\varepsilon}(\mathbb{T}^2))$

$$C^{-\varepsilon} = B_{\infty, \infty}^{-\varepsilon}.$$

and in $C_t C_x^{-\varepsilon}(\mathbb{T}^2)$, a.s.

• Idea of the proof of the regularity lemma:

Translation invariance (in x)

$$\textcircled{+} \quad \Rightarrow \mathbb{E}[\hat{X}(t, n) \hat{X}(t, m)] = 0 \quad \text{for } n + m \neq 0.$$

(real-valued setting)

(4)

$$\mathbb{E} [\hat{X}(t, n) \hat{X}(t, m)]$$

$$= \iint \underbrace{\mathbb{E} [X(t, x) X(t, y)]}_{= F(t, x-y)} \underbrace{e_{-n}(x) e_{-m}(y)}_{= e_{-(n+m)}(x) e_m(x-y)} dx dy$$

$$= \hat{F}(t, m) \int_{\mathbb{T}^d} e^{-i(n+m) \cdot x} dx$$

$$= 0 \quad \text{for } n+m \neq 0.$$

Suppose ~~(X)~~ holds. (suppress t .) Then, for $p \gg 1$,

$$\| \| X \|_{W^{s, \infty}} \|_{L^p(\Omega)} \stackrel{\text{Sobolev}}{\lesssim} \| \| X \|_{W^{s+\varepsilon, r}} \|_{L^p} \quad r \gg 1.$$

$$\stackrel{\text{Mink}}{\leq} \| \| \langle \nabla \rangle^{s+\varepsilon} X(x) \|_{L^p(\Omega)} \|_{L_x^r} \quad p \geq r$$

$$\begin{aligned} \text{Wiener ch. esti.} &\leq p^{k/2} \| \| \sum_n \langle m \rangle^{s+\varepsilon} \hat{X}(m) e_n(x) \|_{L^2(\Omega)} \|_{L_x^r} \\ &\stackrel{(\oplus)}{=} \left(\sum_m \langle m \rangle^{2(s+\varepsilon)} \mathbb{E} [|\hat{X}(m)|^2] \right)^{1/2} \end{aligned}$$

$$\sum_{n \in \mathbb{Z}^d} p^{h/2} \left(\sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2(s+\varepsilon)} - d - 2s_0 \right)^{1/2} < \infty \quad (5)$$

for $s < s_0$ (by taking $0 < \varepsilon \ll 1$ s.t. $s + \varepsilon < s_0$).

□ (As for (ii), we use Kolmogorov's conti criterion.)

(2.1) LWP of SNLW on $\mathbb{T}^2$.

Basic product estimates:

(i) (fractional Leibniz rule): $1 < p_j, q_j, r < \infty$, $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{r}$, $j=1,2$

$$0 \leq s \leq 1 \quad \|\langle \nabla \rangle^s (fg)\|_{L^r(\mathbb{T}^d)} \lesssim \left(\|f\|_{L^{p_1}} \|\langle \nabla \rangle^s g\|_{L^{q_1}} + \|\langle \nabla \rangle^s f\|_{L^{p_2}} \|g\|_{L^{q_2}} \right)$$

(ii) $1 < p, q, r < \infty$

$$\frac{1}{p} + \frac{1}{q} \leq \frac{1}{r} + \frac{s}{d}$$

$s > 0$

$$\|\langle \nabla \rangle^{-s} (fg)\|_{L^r(\mathbb{T}^d)} \lesssim \|\langle \nabla \rangle^{-s} f\|_{L^p} \|\langle \nabla \rangle^s g\|_{L^q}$$

(ii) See GKO. Here, f is merely a distribution but
the product fg makes sense as long as the sum of
the regularities ≥ 0 .

(i) fractional Leibniz rule on $\mathbb{R}^d$ & transference principle.
(see GKO).

↑

Coifman-Meyer thm.

Kato-Ponce inequality '88.

$$[D^s, f]g = D^s(fg) - (D^s f)g.$$

• Christ-Weinstein JFA '91.

($\Leftarrow$ Vector-valued maximal function est.)
see Grafakos.

• Gatto JFA '02.

(6)

SNLW.
$$\begin{cases} (\partial_t^2 + 1 - \Delta) u + u^k = \xi & \text{on } \mathbb{T}^2. \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases}$$

truncated noise $\xi_N = P_{\leq N} \xi$.

(7)

Consider

$$(\partial_t^2 + 1 - \Delta) u_N + u_N^k = \xi_N$$

↑ no truncation here.

• Duhamel

$$\Rightarrow u_N = \Psi_N + \mathcal{V}_N$$

$$\Rightarrow u_N^k = \sum_{j=0}^k \binom{k}{j} \underline{\Psi_N^j} \mathcal{V}^{k-j}$$

↓
No limit for $j \geq 2$.
need to renormalize this.

Replace Ψ_N^j by $H_j(\Psi_N; \sigma_N)$
+ $\log N$.

$$\underline{H_k(x+y; \sigma)} = \sum_{\bar{j}=0}^k \binom{k}{\bar{j}} \underline{H_{\bar{j}}(x; \sigma)} y^{k-\bar{j}}.$$

⑧

- Define the Wick power $:u_N^k:$

$$:u_N^k: = \sum_{\bar{j}=0}^k \binom{k}{\bar{j}} \underbrace{: \Phi_N^{\bar{j}} :}_{\downarrow} \underbrace{v_N^{k-\bar{j}}}_{\swarrow}$$

limit exists a.s.

postulate that
the remainder is
smoother

- defined only for u_N of the form

$$u_N = \Phi_N + v_N$$

- v_N satisfies

$$(\partial_t^2 + 1 - \Delta) v_N + \sum_{\bar{j}=0}^k \binom{k}{\bar{j}} \underbrace{: \Phi_N^{\bar{j}} :}_{\downarrow} v_N^{k-\bar{j}} = 0.$$

$: \Phi^{\bar{j}} :$

(9)

By taking $N \rightarrow \infty$, we arrive at

$$(SNLW_v) \quad (\partial_t^2 + 1 - \Delta) v + \sum_{j=0}^k \binom{k}{j} : \Phi^j : v^{k-j} = 0.$$

Then, the "solution" u to the renormalized SNLW:

$$(*) \quad (\partial_t^2 + 1 - \Delta) u + : u^k : = \xi$$

$$(**) \quad \text{is given by} \quad u = \Phi + v$$

• Here, the eqn $(*)$ for u is just a formal expression.
For example, when $k=3$, we have

$$(\partial_t^2 + 1 - \Delta) u + u^3 - 3 \cdot \underbrace{\infty}_{= \lim_{N \rightarrow \infty} \sigma_N} \cdot u = \xi$$

By $(*)$, we really mean $(SNLW_v)$ with the first order decomposition $(**)$.

Lec 10 09 / 04 / 21 (Wed)

①

First order decomposition: $u = \Psi + v$
 $\uparrow$ smoother

- Da Prato - Debussche '02.
- McKean '95, Bourgain '96.

Consider

$$\begin{aligned} (\text{SNLW}'_v) \quad & \left\{ \begin{aligned} & (\partial_t^2 + 1 - \Delta) v + \sum_{j=0}^k \binom{k}{j} \underline{\underline{\Xi_j}} v^{k-j} = 0 \\ & (v, \partial_t v)|_{t=0} = \underline{(u_0, u_1)}. \end{aligned} \right. \quad \Xi_0 = 1. \end{aligned}$$

Enhanced data set:

$$\begin{aligned} & \underline{\underline{\Xi}} // (u_0, u_1, \Xi_1, \dots, \Xi_k) \\ & \in \mathcal{H}^s \times \left(L^\infty([0,1]; W^{-\varepsilon, \infty}(\mathbb{T}^2)) \right)^{\otimes k} \end{aligned}$$

$$\|\Xi\|_{\chi^s} = \|(u_0, u_1)\|_{\chi^s} + \sum_{j=1}^k \|\Xi_j\|_{L^\infty([0, T]; W_x^{-\varepsilon, \infty})} \quad (2)$$

Prop: $\exists \varepsilon_k > 0$ small s.t. for $0 < \varepsilon < \varepsilon_k$,

$(SNLW'_\varepsilon)$ is locally well-posed:

- existence, uniqueness,
- $\Xi \mapsto v$ is continuous.
 $\cap$
 $C([0, T]; H^{1-\varepsilon})$.

$$T = T(\|\Xi\|_{\chi^s}) > 0$$

Case 1: $s = 1 - \varepsilon$.

Duhamel formulation

$$v(t) = \mathcal{P} v(t) = \mathcal{Z}_t S(t) u_0 + S(t) u_1 - \sum_{j=0}^k \binom{k}{j} \int_0^t S(t-t') \left(\Xi_j v^{t-j} \right)(t') dt'$$

$$S(t) = \frac{\sin t \langle \nabla \rangle}{\langle \nabla \rangle}$$

③

$$\underline{\bar{j} = 0}: \quad \left\| \int_0^t S(t-t') v^h(t') dt' \right\|_{C_T H_x^{1-\varepsilon}}$$

$$\lesssim T \| v^h \|_{C_T H_x^{-\varepsilon}}$$

$$\frac{s}{d} \geq \frac{1}{p} - \frac{1}{q}$$

$$\underline{T \leq 1}$$

$$\stackrel{\text{Sobolev}}{\lesssim} T \| v^h \|_{C_T L_x^{\frac{2}{1+\varepsilon}}}$$

$$\frac{\varepsilon}{2} = \frac{1+\varepsilon}{2} - \frac{1}{2}$$

$$= T \| v \|_{C_T L_x^{\frac{2h}{1+\varepsilon}}}^h$$

$$\stackrel{\text{Sobolev}}{\lesssim} T \| v \|_{C_T H_x^{1-\varepsilon}}^h$$

$$\frac{1-\varepsilon}{2} \geq \frac{1}{2} - \frac{1+\varepsilon}{2h} \quad \left(\Leftrightarrow \begin{matrix} \Leftarrow \\ \Rightarrow \end{matrix} 0 < \varepsilon \leq \frac{1}{h-1} \right)$$

$$\underline{1 \leq \bar{j} < h}: \quad \left\| \int_0^t S(t-t') \left(\sum_j v^{k-j} \right) (t') dt' \right\|_{C_T H_x^{1-\varepsilon}}$$

$$\lesssim T \left\| \sum_j v^{k-j} \right\|_{C_T H_x^{-\varepsilon}}$$

↑

only a distribution (in x)

④

$$= T \| \langle \nabla \rangle^{-\varepsilon} (\Xi_j v^{k-j}) \|_{C_T L_x^2}$$

$$\lesssim T \| \langle \nabla \rangle^{-\varepsilon} \Xi_j \|_{L_T^\infty L_x^{2/\varepsilon}} \| \langle \nabla \rangle^\varepsilon (v^{k-j}) \|_{L_T^\infty L_x^2}$$

frac. Leib. rule

$$\lesssim T \| \Xi \|_{\chi^{1-\varepsilon}} \| \langle \nabla \rangle^\varepsilon v \|_{L_T^\infty L_x^{2(k-j)}}^{k-j}$$

$$\frac{1}{p} + \frac{1}{2} \leq \frac{1}{2} + \frac{\varepsilon}{2}$$

Sobolev

$$\lesssim T \| \Xi \|_{\chi^{1-\varepsilon}} \| v \|_{C_T H_x^{1-\varepsilon}}^{k-j}$$

$$\frac{1-2\varepsilon}{2} \geq \frac{1}{2} - \frac{1}{2(k-j)}$$

$$\left(\Leftarrow 0 < \varepsilon \leq \frac{1}{2(k-1)} \right)$$

$j = k$:

$$\left\| \int_0^t S(t-t') \Xi_k(t') dt' \right\|_{C_T H_x^{1-\varepsilon}}$$

$$\lesssim T \| \Xi_k \|_{C_T H_x^{-\varepsilon}}$$

$$\lesssim T \| \Xi \|_{\chi^{1-\varepsilon}}.$$

(5)

$\Rightarrow$ Putting together, we get

$$\begin{aligned} \|\Gamma v\|_{C_T H_x^{1-\varepsilon}} &\leq C_1 \|(u_0, u_1)\|_{\mathcal{H}^{1-\varepsilon}} \\ &\quad + C_2 T \underbrace{\|\Xi\|_{\mathcal{H}^{1-\varepsilon}}}_k \left(1 + \|v\|_{C_T H_x^{1-\varepsilon}}\right)^{k-1} \\ &\quad + C_3 T \|v\|_{C_T H_x^{1-\varepsilon}}^k \end{aligned}$$

A similar estimate holds for the difference $\Gamma(v_1) - \Gamma(v_2)$.

$\Rightarrow$ Γ is a contraction on a ball of radius

$$\sim \|(u_0, u_1)\|_{\mathcal{H}^{1-\varepsilon}} + \|\Xi_k\|_{C([0,1]; W_x^{-\varepsilon, \infty})}$$

by choosing $T = T(\|\Xi\|_{\mathcal{H}^{1-\varepsilon}}) > 0$ sufficiently small.

(6)

Summary:

Known: 2-d NLW is ill-posed in negative Sobolev spaces
 (Christ - Colliander - Tao, '03; Oh - Okamoto - Tzvetkov, Forlano - Okamoto DPDE '20)

$\Rightarrow$ 2-d SNLW:

$(u_0, u_1, \mathbb{Z}) \longrightarrow u$ ill-posed.

Idea: Decompose the ill-defined soln map.

① Use stochastic analysis to construct an enhanced data set *only in this step.*

② use deterministic analysis to prove LWP for the remainder term $v = u - \Psi$. *continuous map*

$$(u_0, u_1, \mathbb{Z}) \xrightarrow{\textcircled{1}} \Xi = (u_0, u_1, \Psi, : \Psi^2 :, \dots, : \Psi^k :) \xrightarrow{\textcircled{2}} v \rightarrow u = \Psi + v$$

$\uparrow$
NOT CONT

Thm: $\exists \varepsilon_k > 0$ small s.t. for $0 < \varepsilon < \varepsilon_k$,

⑦

the renormalized SNLW on $\mathbb{T}^2$ is locally well-posed in $H^{1-\varepsilon}(\mathbb{T}^2)$.

Moreover, the soln u_N to the truncated renormalized

$$\text{SNLW} : (\partial_t^2 + 1 - \Delta) u_N + : u_N^k : = \xi_N$$

converges to $u = \Psi + v$ in $C([0, T_\infty]; H^{-\varepsilon}(\mathbb{T}^2))$, a.s.

• First order expansion $u = \Psi + v$.
 $\swarrow$ rough
 $\nwarrow$ smoother

• trick to solve the equation

• also gives a description

ex: in small scales, u "behaves like" Ψ .

• Without renormalization, no non-trivial limit exists.

($u_N \rightarrow 0$ or lin soln)

Oh-Okamoto-Robert
SPA '20.

$\Leftarrow$ Triviality

• Regularization on ξ :

• by $P \in N$

• by mollification $\xi_s = \eta_s * \xi$ (conv in prob.)
(spatial / space-time).

$\Rightarrow$ same limit

Q: Rougher initial data?

$$NLW: \partial_t^2 u - \Delta u + u^k = 0 \quad \text{on } \mathbb{R}^d.$$

• Scaling invariance

$\Rightarrow$ Scaling critical regularity

$$S_{\text{scaling}} = \frac{d}{2} - \frac{2}{k-1}.$$

• Lorentz invariance $\Rightarrow$
(Conformal)

$$S_{\text{conf}} = \frac{d+1}{4} - \frac{1}{k-1}$$

$$S_{crit} = \max(S_{scaling}, S_{conf}, 0)$$

⑨

$$d=2 \quad \max\left(1 - \frac{2}{k-1}, \frac{3}{4} - \frac{1}{k-1}, 0\right).$$

Thm: GKO TAMS '18

(i) $k=2,3$: $S > S_{crit}$

(ii) $k \geq 4$: $S \geq S_{crit}$.

The renormalized SNLW on $\mathbb{T}^2$ is locally well-posed in $\mathcal{H}^s(\mathbb{T}^2)$.

↑
for initial data

Strichartz estimates: $0 \leq s \leq 1$

(q, r) s -admissible

$(\tilde{q}, \tilde{r})$ dual s -admissible

$$\text{if } 1 \leq \tilde{q} \leq 2 \leq q \leq \infty \\ 1 \leq \tilde{r} \leq 2 \leq r \leq \infty$$

$$\text{scaling: } \frac{1}{q} + \frac{d}{r} = \frac{d}{2} - s = \frac{1}{\tilde{q}} + \frac{d}{\tilde{r}} - 2$$

$$(q, r, d) \neq (2, \infty, 3)$$

$$(\tilde{q}, \tilde{r}, d) \neq (2, 1, 3)$$

$$\text{admissibility: } \frac{1}{q} + \frac{d-1}{2r} \leq \frac{d-1}{4}, \quad \frac{1}{\tilde{q}} + \frac{d-1}{2\tilde{r}} \leq \frac{d-1}{4}$$

$(\tilde{q}, \tilde{r})$ dual s -admissible

$\Leftrightarrow (\tilde{q}', \tilde{r}')$ is $(1-s)$ -admissible

Then,
$$\begin{cases} (\partial_t^2 + 1 - \Delta) u = F \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases} \quad \text{on } \mathbb{T}^d.$$

$$\Rightarrow \| (u, \partial_t u) \|_{L_T^\infty H_x^s} + \| u \|_{L_T^q L_x^r} \quad 0 \leq T \leq 1$$

$$\lesssim \| (u_0, u_1) \|_{H_x^s} + \| F \|_{L_T^{\tilde{q}} L_x^{\tilde{r}}}$$

Also,
$$(\text{LHS}) \lesssim \| (u_0, u_1) \|_{H_x^s} + \| F \|_{L_T^1 H_x^{s-1}}$$

• Strichartz on $\mathbb{R}^d$: Visan's Oberwolfach note,
and finite speed of propagation

(11)

- We only consider $k=3$. $\Rightarrow$ Suit $= 1/4$.

(general case : GKO, Oh-Thomann, Toulouse '20.

$$(q, r) = \left(\frac{12}{1+4\delta}, \frac{3}{1-2\delta} \right), \quad \left(\frac{1}{4} + \delta \right) - \text{admiss.}$$

$$(\tilde{q}, \tilde{r}) = \left(\frac{12}{9+4\delta}, \frac{3}{3-2\delta} \right), \quad \text{dual } \left(\frac{1}{4} + \delta \right) - \text{admiss.}$$

nonlinearity: $\square_{\tilde{q}} v^{3-j}$ in $L_T^{\tilde{q}} L_x^{\tilde{r}} + L_T^1 H_x^{s-1}$

$$\|v^3\|_{L_T^{\tilde{q}} L_x^{\tilde{r}}} = \|v\|_{L_T^{\frac{36}{9+4\delta}} L_x^{\frac{9}{3-2\delta}}}^3 \lesssim T^\theta \|v\|_{L_T^q L_x^r}^3$$

$$\frac{36}{9+4\delta} < q = \frac{12}{1+4\delta}$$

$$\frac{9}{3-2\delta} < r = \frac{3}{1-2\delta}$$

$\uparrow$
deterministic theory.

LWP of cubic NLW
in $H^s(\mathbb{T}^2)$, $s > \frac{1}{4}$.

$$\cdot \quad \|\Xi_3\|_{L_T' H_x^{s-1}} \leq T \|\Xi_3\|_{L_T^\infty W_x^{-\varepsilon, \infty}} \quad \text{if } s-1 \leq -\varepsilon.$$

$$\leq T \|\Xi\|_{X^s}.$$

$$\boxed{s = \frac{1}{4} + \delta}$$

$$\cdot \quad \|\Xi_j v^{3-j}\|_{L_T' H_x^{s-1}}$$

$$s-1 = -\frac{3}{4} + \delta$$

Sobolev

$$\leq \|\langle \nabla \rangle^{-\delta} (\Xi_j v^{3-j})\|_{L_T' L_x^{\frac{8}{7-8\delta}}}$$

$$\frac{\frac{3}{4} - 2\delta}{2} = \frac{7-8\delta}{8} - \frac{1}{2}$$

$$\lesssim \|\langle \nabla \rangle^{-\delta} \Xi_j\|_{L_T^\infty L_x^\infty} \underbrace{\|\langle \nabla \rangle^\delta v^{3-j}\|_{L_T' L_x^{\frac{8}{7-8\delta}}}}_{\substack{\text{replace} \\ \text{by } L_x^\infty}}$$

$$\frac{1}{p} + \frac{1}{q} \leq \frac{1}{r} + \frac{s}{d}$$

$$\frac{\delta}{2} + \frac{7-8\delta}{2} \leq \frac{7-8\delta}{8} + \frac{\delta}{2}$$

g

$$\underline{\hat{q}=2:} \quad \lesssim T \|v\|_{L_T^\infty H_x^\delta} \leq T \|v\|_{L_T^\infty H_x^s}$$

$$\underline{\hat{q}=1:} \quad \lesssim T \|\langle \nabla \rangle^\delta v\|_{L_T^\infty L_x^{\frac{16}{7-8\delta}}} \stackrel{\text{Sobolev}}{\lesssim} T \|v\|_{L_T^\infty H_x^{\frac{1+16\delta}{8}}} \quad \leftarrow \text{frac Leib.}$$

$$\frac{1+8\delta}{8} / 2 = \frac{1}{2} - \frac{7-8\delta}{16} \quad \text{and add } \delta \text{ from } \langle \nabla \rangle^\delta$$

$$\text{Let } \|\vec{u}\|_{Y^S(T)} = \|(u, \partial_t u)\|_{L_T^p H_x^s} + \|u\|_{L_T^q L_x^r}$$

$$\begin{aligned} \Rightarrow \|\vec{\Gamma(v)}\|_{Y^S(T)} &= \|(\Gamma(v), \partial_t \Gamma(v))\|_{Y^S(T)} \\ &\leq C_1 \|(u_0, u_1)\|_{Y^S} + C_2 \left(\|v\|_{Y^S(T)} + \|\Xi\|_{\mathbb{Z}^{-\delta}} \right)^3 \end{aligned}$$

$$\text{where } \|\Xi\|_{\mathbb{Z}^{-\delta}} = \sum_{j=1}^3 \|\Xi_j\|_{L^p([0,1]; W_x^{-\delta, \infty})}$$

A similar estimate holds for the difference.

Lec 11: 14/04/21 (Wed)

①

(2.2) 2-d stoch. NLH $\rightarrow P = \text{polynomial.}$
= parabolic $P(\Phi)_2$ -model.
(or Φ_2^{k+1} -model):

$$\partial_t u + (1 - \Delta)u + u^k = \sqrt{2} \xi.$$

$$\left(\partial_t u + \frac{1}{2}(1 - \Delta)u + u^k = \xi \right).$$

Basic space: $C^s(\mathbb{T}^2) = B_{\infty, \infty}^s(\mathbb{T}^2).$

$\Rightarrow$ stoch convolution

$$\Psi = \Psi_{\text{heat}} = \int_0^t P(t-t') dW(t'), \quad P(t) = e^{t(\Delta-1)}$$

belongs to $C_t C_x^s(\mathbb{T}^2), \quad \forall s < 0, \text{ a.s.}$

(also in $L^p(\Omega).$)

As in the case of 2-d SNLW,

(2)

we use the first order expansion:

DaPrato-Debussche '03.

$$u = \Psi + v$$

and solve the equation for $v = u - \Psi$:

$$\partial_t v + (1 - \Delta) v + (v + \Psi)^k = 0$$

- Since Ψ is not a function (i.e. is only a distribution-valued function), $\Psi^{\hat{j}}$ does not make sense and thus we consider the renormalized version:

$$(SNLH_v) \quad \partial_t v + (1 - \Delta) v + \sum_{\hat{j}=0}^k \binom{k}{\hat{j}} \underbrace{\Psi^{\hat{j}}}_{\parallel} v^{k-\hat{j}} = 0$$

$$\lim_{N \rightarrow \infty} : (\mathbb{P} \llcorner_N \Psi)^{\hat{j}} :$$

③

In the following, we only consider $(SNLH_v)$.

but we can also show $v_N \rightarrow v$

where v_N satisfies

$$\partial_t v_N + (1 - \Delta) v_N + \sum_{j=0}^k \binom{k}{j} : \Psi_N^j : v_N^{k-j} = 0$$

$$\Psi_N = P_{\leq N} \Psi$$

and hence $u_N = \Psi_N + v_N \rightarrow u = \Psi + v$

Main tool: paraproduct decomposition (Bony '81)

f, g on $\mathbb{T}^d$ (or $\mathbb{R}^d$) of regularities s_1 and s_2

$$fg = \underline{f \odot g} + \underline{f \ominus g} + \underline{f \oslash g}$$

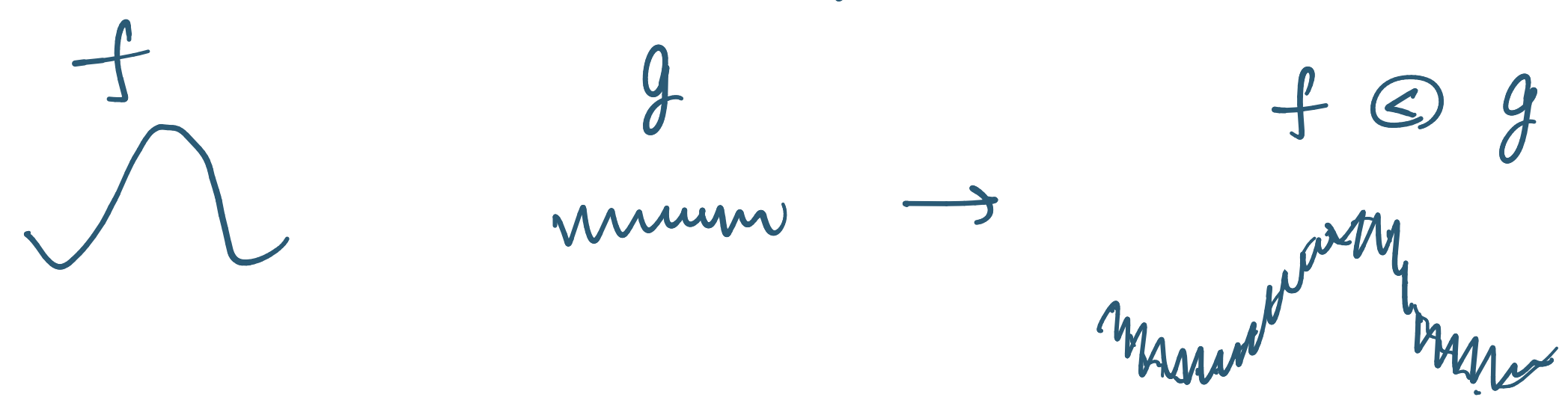
$$= \sum_{\underline{j < k-2}} P_j(f) P_k(g) + \sum_{\underline{|j-k| \leq 2}} P_j(f) P_k(g) + \sum_{\underline{k < j-2}} P_j(f) P_k(g)$$

$P_j = \text{LP projection onto } \{ |m| \sim 2^j \}.$
 $(\hat{j}=0 : \{ |m| \leq 1 \})$

• $f \otimes g = \text{paraproduct of } g \text{ by } f.$

$(f \prec g, \pi_f(g), \text{etc.})$

"freq of $g \gg$ freq of f ."



as a distribution. $\rightarrow$ - always makes sense (for any $s_1, s_2 \in \mathbb{R}$.)
 with regularity $\boxed{\min \left(\underset{\substack{\uparrow \\ \text{reg of } g}}{s_2}, s_1 + s_2 \right)}$

(5)

• same for $f \otimes g \sim \min(s_1, s_1 + s_2)$.

• $f \otimes g$: Resonant product of f and g .

($f \otimes g$, $R(f, g)$, etc.

- may not make sense as a distribution.

- In general,

$s_1 + s_2 > 0 \Rightarrow f \otimes g$ makes sense
 $\sim s_1 + s_2$.

(Recall the product estimate (ii) from GKO, where
 $f \otimes g$ makes sense for $s_1 + s_2 = 0$.
in terms of Sobolev spaces.

⑥

- In studying nonlinear PDEs, the main task is to make sense of the nonlinearity, say u^k
(or give a meaning.)

- Since the paraproduct $f \odot g$ and $f \oslash g$ always make sense, the main job is to make sense of the resonant product $f \ominus g$.
(more in the parabolic thinking.)

- When there is an issue in making sense of u^k , we overcome this issue by imposing a structure on u

- $u = \Psi + v$.

- $u \in X^{s,b}$

Lemma: (paraproduct estimates)

$$s_1, s_2 \in \mathbb{R}, \quad 1 \leq p, p_1, p_2, q \leq \infty, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$$

$$\textcircled{1} \quad \|f \otimes g\|_{B_{p,q}^{s_2}} \lesssim \|f\|_{L^{p_1}} \|g\|_{B_{p_2,q}^{s_2}}$$

↑
f is a function ($s_1 \geq 0$)

$s_1 < 0$

$$\textcircled{2} \quad \|f \otimes g\|_{B_{p,q}^{s_1+s_2}} \lesssim \|f\|_{B_{p_1,q}^{s_1}} \|g\|_{B_{p_2,q}^{s_2}}$$

$$\textcircled{1} \ \& \ \textcircled{2} : \quad f \otimes g \sim \min(s_2, s_1 + s_2)$$

$$\textcircled{3} \quad \underline{s_1 + s_2 > 0}$$

$$\|f \otimes g\|_{B_{p,q}^{s_1+s_2}} \lesssim \|f\|_{B_{p_1,q}^{s_1}} \|g\|_{B_{p_2,q}^{s_2}}$$

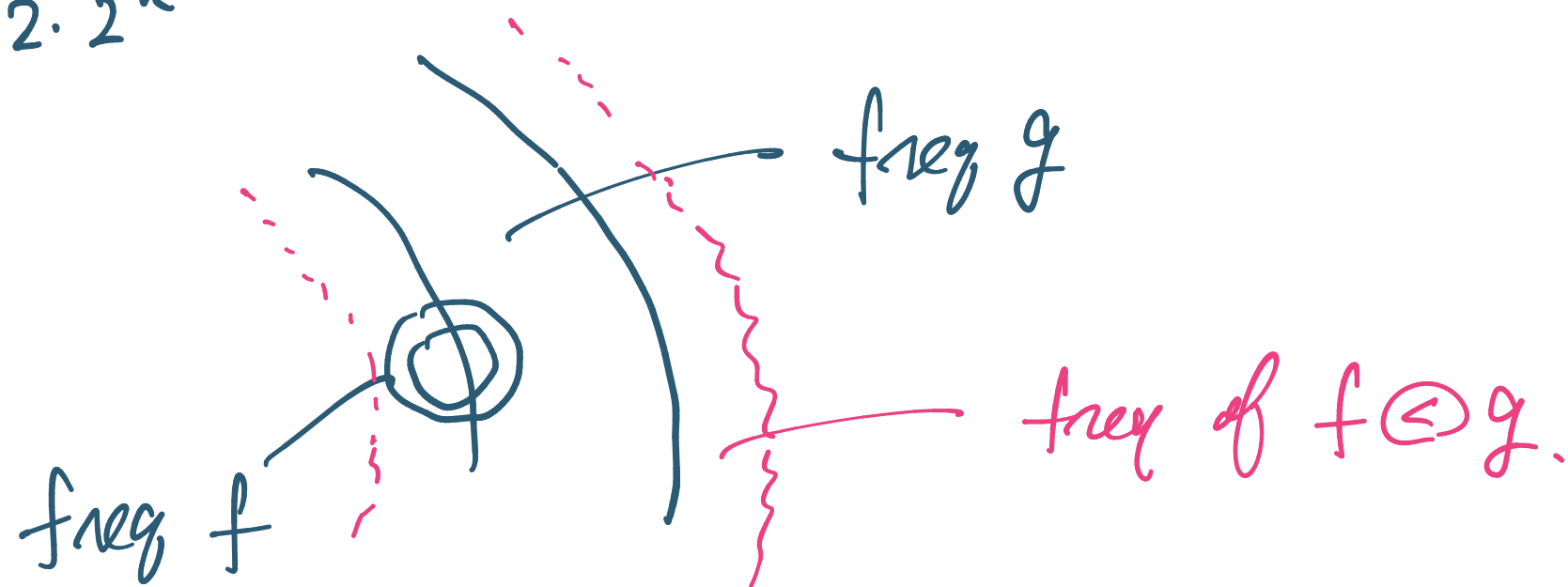
Pf: ① $\| f \otimes g \|_{B_{P, q}^{S_2}} \sim \| 2^{S_2 k} \| P_k (f \otimes g) \|_{L_x^p} \|_{l_k^q(\mathbb{Z}_{\geq 0})}$ ⑧

$$\underline{P_k (f \otimes g)} = \sum_{i=-2}^2 \sum_{j < k+i-2} P_j (f) \underline{P_{k+i} (g)}$$

freq $\sim 2^k$

$$\frac{1}{2} \cdot 2^k \leq \cdot \leq 2 \cdot 2^k$$

freq of $f \otimes g \sim$ freq of g



Let $S_k(f) = \sum_{j < k} P_j(f)$: f projected onto freq $\{|n| \leq 2^k\}$

$$\Rightarrow P_k(f \otimes g) = \sum_{i=-2}^2 S_{k+i-2}(f) P_{k+i}(g)$$

Take L^p_x -norm and apply Hölder's inequality.

⑨

$$\begin{aligned} \|P_h(f \otimes g)\| &\lesssim \sum_{i=-2}^2 \|S_{k+i-2}(f)\|_{L^{p_1}} \|P_{k+i}(g)\|_{L^{p_2}} \\ &\lesssim \|f\|_{L^{p_1}} \text{ unif in } k, i. \end{aligned}$$

$\nearrow 2^{S_2 k}$
 $\nwarrow \sim 2^{S_2(k+i)}$

• Take $l_h^q (\sim l_{k+i}^q)$

$$\|f \otimes g\|_{B_{p,q}^{S_2}} \lesssim \|f\|_{L^{p_1}} \|g\|_{B_{p_2,q}^{S_2}}.$$

(2)

(10)

$$P_k(f \otimes g) \sim \sum_{i=-2}^2 \sum_{0 \leq j < k+i-2} P_j(f) 2^{s_2 k} P_{k+i}(g)$$

$2^{(s_1+s_2)k}$ (pointing to $P_k(f \otimes g)$)
 $2^{-s_1 j} 2^{s_1 k}$ (under the sum)
 $2^{s_1 j}$ (pointing to $P_j(f)$)
 $2^{s_2 k}$ (pointing to $2^{s_2 k}$)

Take L^p_α -norm and apply Hölder's inequality:

$$\lesssim \sum_i \sum_{0 \leq j \leq k+i-2} 2^{\underline{s_1}(k-j)} \cdot \underbrace{\left(2^{s_1 j} \|P_j(f)\|_{L^{p_1}} \right)}_{l_j^q} \left(2^{s_2(k+i)} \|P_{k+i}(g)\|_{L^{p_2}} \right)$$

Hölder in j .

$s_1 < 0$

$$\lesssim \sum_i \|f\|_{B_{p_1, q}^{s_1}} \left(2^{s_2(k+i)} \|P_{k+i}(g)\|_{L^{p_2}} \right)$$

$\Rightarrow$ Now, take $l_h^q (\sim l_{h+i}^q)$.

③

$2^{(s_1+s_2)k}$

$$f_k(f \oplus g) = \sum_{i=-2}^2$$

$$\sum_{\bar{j} \geq k-10}$$

$$P_k(P_j(f) P_{j+i}(g))$$

$$2^{(s_1+s_2)(k-j)}$$

$$2^{s_1 j}$$

$$2^{s_2 j}$$

Take l_k^q -norm.

view this as a convolution of

$$2^{(s_1+s_2)j} \mathbb{1}_{j \leq 10} \text{ and } P_j(f) P_{j+i}(g) \cdot \mathbb{1}_{j \geq 0}$$

$$\in l_j^1(\mathbb{Z})$$

$\Leftarrow$ Need $s_1+s_2 > 0$

in $l_j^q(\mathbb{Z})$

Hölder in j .

$\Rightarrow$ By Young's inequality,

$$\|f \oplus g\|_{B_{p,q}^{s_1+s_2}}$$

$$\lesssim \|f\|_{B_{p_1,q}^{s_1}} \|g\|_{B_{p_2,\infty}^{s_2}}$$



11

Cor: Let $s > 0$. Then, $C^s(\mathbb{T}^d) = B_{\infty, \infty}^s(\mathbb{T}^d)$ is an algebra.

(12)

with $\|fg\|_{C^s} \lesssim \|f\|_{C^s} \|g\|_{C^s}$

Back to $(SNLH_V)$.

Duhamel formulation:

$$v(t) = \Gamma v(t) = P(t) u_0 - \sum_{j=0}^k \binom{k}{j} \int_0^t P(t-t') \underbrace{:\Psi^j:}_{\text{in } C_T C_x^{-\varepsilon}} v^{k-j}(t') dt'.$$

put this in $C^{-\varepsilon}$

in $C_T C_x^{-\varepsilon}$

(can replace this by
deterministic $\Xi_j \in C([0, T]; \tilde{C}_x^{-\varepsilon})$)

$s > 0$ $s = 2\varepsilon$.

$$\|\Gamma v\|_{C_T C_x^s} \lesssim \|u_0\|_{C^s} + \sum_{j=0}^k \binom{k}{j} \int_0^t \underbrace{(t-t')^{-\frac{3}{2}\varepsilon}}_{\text{Schauder}} \|:\Psi^j: v^{k-j}(t')\|_{C^{-\varepsilon}} dt'.$$

$$\| \cdot \Psi^j : v^{k-j} \|_{C^{-\varepsilon}}$$

$$\lesssim \| \cdot \Psi^j : \|_{C^{-\varepsilon}} \| v \|_{C^{2\varepsilon}}^{k-j}$$

$\Leftarrow$ Moral: In order to make sense of the product,
the sum of the regularities > 0 .

BUT: the resulting regularity of the product is
given by one of the para products.

ex) $s_1 < 0 < s_2$. Need $s_1 + s_2 > 0$.

but $fg \sim s$, coming from $f \otimes g$

$$\Rightarrow \|Pv\|_{C_T C_x^s} \lesssim \|u_0\|_{C^s} + \sum_{\hat{j}=0}^k \binom{k}{\hat{j}} T^\theta \|:\Psi^{\hat{j}}:\|_{C_T \bar{C}_x^{-\varepsilon}} \|v\|_{C_T C^s}^{k-\hat{j}} \quad (14)$$

for $s > 0$ (s.t. $s+\varepsilon > 0$)

$\Rightarrow$ LWP of $SNLH_v$ in $C^s(\mathbb{T}^2)$, $s > 0$.

• Rougher initial data? $u_0 \in C^s(\mathbb{T}^2)$, $s < 0$.

$$\|u\|_{Y(T)} = \sup_{0 < t \leq T} t^\theta \|u(t)\|_{C_x^\sigma} \quad \begin{matrix} \sigma > 0 \\ \theta > 0 \end{matrix}$$

$$\Rightarrow t^\theta \|Pv(t)\|_{C_x^\sigma} \lesssim t^\theta t^{\frac{s-\sigma}{2}} \|u_0\|_{C^s} + \sum_{\hat{j}=0}^k \binom{k}{\hat{j}} t^\theta \int_0^t \underbrace{(t-t')^{-\frac{3}{2}\varepsilon}}_{C_x^{2\varepsilon}} \|:\Psi^{\hat{j}}: v^{k-\hat{j}}(t')\|_{C_x^{-\varepsilon}} dt'$$

$\theta = \frac{\sigma-s}{2}$, $\sigma = 2\varepsilon$

$$\|\Psi^j: v^{k-j}(t')\|_{C_x^{-\varepsilon}}$$

$$\lesssim \|\Psi^j\|_{C_T C_x^{-\varepsilon}} \underbrace{\left((t')^\theta \|v(t')\|_{C_x^{2\varepsilon}} \right)^{k-j}}_{\leq \|v\|_{Y(T)}^{k-j}} \underbrace{(t')^{-(k-j)\theta}}_{\text{red line}}$$

Need to control

$$t^\theta \int_0^t (t-t')^{-\frac{3}{2}\varepsilon} \underbrace{(t')^{-(k-j)\theta}}_{\text{red line}} dt' \quad t \leq T.$$

↙ Beta function

FACT: $t^{\alpha_1} \int_0^t (t-t')^{\alpha_2} (t')^{\alpha_3} dt' = B(d_2+1, d_3+1) < \infty$

$$\alpha_1 + \alpha_2 + \alpha_3 = -1, \quad \alpha_2, \alpha_3 > -1$$

$$B(x, y) = \int_0^1 (1-t)^{x-1} t^{y-1} dt, \quad \operatorname{Re} x, \operatorname{Re} y > 0.$$

Need

$$\theta + \left(-\frac{3}{2}\varepsilon\right) + (-k\theta) \geq -1.$$

$$\Rightarrow s > -\frac{2}{k-1}.$$

$$\theta = \frac{\sigma-s}{2} = \varepsilon - \frac{s}{2}$$

$$\cdot d_2 > -1 \Rightarrow \varepsilon < 1$$

$$\cdot d_3 > -1 \quad -k\theta > -1 \Rightarrow \theta < \frac{1}{k}$$

$\parallel$
 $\varepsilon - \frac{s}{2}$

$$\Rightarrow \underline{s > -\frac{2}{k}} \left(> -\frac{2}{k-1} \right)$$

$\triangleq$ scaling with regularity

$$\left(\text{When } \hat{\gamma}=0: \quad t^\theta \int_0^t (t')^{-k\theta} dt' < \infty \Rightarrow -k\theta > -1. \right.$$

(17)

• By a contraction argument, construct $v \in Y(T)$

$$\Rightarrow v \in C([0, T]; C_x^\sigma(\mathbb{T}^2)), \quad \sigma = 2\varepsilon > 0.$$

• A posteriori, show $v \in C([0, T]; C_x^s(\mathbb{T}^2)), (s < 0)$

$$\Rightarrow \text{LWP of } (SNLH_v) \text{ in } C^s(\mathbb{T}^2), \quad s > -\frac{2}{k}.$$

Lec 12: 16 / 04 / 21 (Fri)

①

2.3 Global-in-time aspects:

2.3.i Parabolic $P(\mathbb{T})_2$ -model:

$$\cdot (\partial_t + 1 - \Delta) u + u^k = \sqrt{2} \xi \quad \text{on } \mathbb{T}^2, \quad k \in 2N+1.$$

↑
We renormalize the nonlinearity $\Rightarrow : u^{k+1} :$

Pathwise approach: Control the L^p -norm of $v = u - \Psi$.

· Why is this enough?

$$\begin{cases} (\partial_t + 1 - \Delta) v + \sum_{j=0}^k \binom{k}{j} : \Psi^j : v^{k-j} = 0 \\ v|_{t=0} = u_0. \end{cases}$$

$$\boxed{j = p(k)}$$

As in Lec 11, work with the $Y(T)$ -norm:

(2)

$$\|u\|_{Y(T)} = \sup_{0 < t \leq T} t^\theta \|u(t)\|_{C^\sigma}$$

$$\sigma = 2\varepsilon > 0$$

$$\theta = \varepsilon - \frac{s}{2}$$

$$\left(-\frac{2}{k} < s < 0\right)$$

$$t^\theta \|P(v)(t)\|_{C^\sigma} = B_{\infty, \infty}^\sigma$$

$$\lesssim \underbrace{t^\theta t^{-\frac{\sigma}{2} - \frac{d}{2}(\frac{1}{p} - \frac{1}{\infty})}}_{\lesssim 1 \text{ for } 0 < t \ll 1}$$

$$\lesssim 1$$

for $0 < t \ll 1$.

LP projection

↓

$$\sup_m \|P_m u_0\|_{L^p}$$

$$\lesssim \|u_0\|_{L^p}$$

+ Duhamel term.

$$\Rightarrow \text{Need } \theta - \frac{\sigma}{2} - \frac{1}{p} \geq 0 \Rightarrow \frac{1}{p} \leq -\frac{s}{2} \Rightarrow p \geq -\frac{2}{s} \gg 1$$

when $s \rightarrow 0^-$
i.e. $k \rightarrow \infty$

$\Rightarrow$ LWP of $(SNLH_\gamma)$ in $L^p(\mathbb{T}^2)$

and the local existence time $\sim \left(\|u_0\|_{L^p(\mathbb{T}^2)} + \text{Wick powers of } \Phi \right)^{-\sigma}$

③

For fixed $T \gg 1$,
this part may be large

BUT finite a.s

$$\sum_{j=1}^k \|\cdot \Phi^j \cdot\|_{C([0,T]; C^{-\varepsilon})} < C_0 < \infty$$

$\Rightarrow$ As long as we control $\sup_{0 \leq t \leq T} \|v(t)\|_{L^p}$

(for each $T \gg 1$), we obtain GWP.

→ compute $\partial_t \|v\|_{L^p}^p$ and use the equation and the control on the stochastic terms to get a bound on

$$\sup_{0 \leq t \leq T} \|v(t)\|_{L^p}^p.$$

/ Ginzburg - Landau.

See Trenberth '19(?) on SCGL
(= Schrödinger - heat)

$$\mathbb{T}^2. \quad \partial_t u = (a_1 + i a_2)(1 - |u|^2)u - (c_1 + i c_2)|u|^{k-1}u + \sqrt{2} \xi$$

$a_1 = \text{heat}$
 $a_2 = \text{Schrödinger}.$

With $r = \left| \frac{a_1}{a_2} \right|$, then GWP for $r \geq C(k)$.

i.e. heat part is suff. strong.

$\mathbb{R}^2$: Mourrat - Weber A. Prob '17.

GWP of the parabolic $P(\Phi)_2$ -model on $\mathbb{R}^2$.

$\Leftarrow$ weighted Besov spaces

Invariant measure argument:

Gibbs measure on $\mathbb{T}^2$:

$$d\rho = Z^{-1} e^{-\frac{1}{k+1} \int_{\mathbb{T}^2} u^{k+1} d\alpha} \quad k \in 2N+1$$

$\underbrace{d\mu_1}_{\uparrow} e^{-\|u\|_{H^1}^2} du.$

A typical function u under μ_1 is NOT a function.

$$\Rightarrow \int_{\mathbb{T}^2} u^{k+1} d\alpha = \infty, \text{ a.s.}$$

$\Rightarrow$ Need to renormalize the potential energy.

$$\text{We use } \int_{\mathbb{T}^2} : u^{k+1} : d\alpha = \lim_{N \rightarrow \infty} \int_{\mathbb{T}^2} : (P_{\leq N} u)^{k+1} : d\alpha$$

Recall : $: (P_{\leq N} u)^{k+1} : \rightarrow : u^{k+1} :$ in $W^{-\varepsilon, \infty}(\mathbb{T}^2)$ or $\mathcal{C}^{-\varepsilon}(\mathbb{T}^2)$
 a.s. / $L^p(\Omega)$.

- 70's : Euclidean quantum field theory.

⑥

$$e^{-\frac{1}{k+1} \int_{\mathbb{T}^2} : u^{k+1} : dx} \in L^p(d\mu), \quad \forall p < \infty.$$

- hypercontractivity of OV process / Wiener chaos estimate due to Nelson '65.
- Nelson's estimate.
- See my course from 2017 (Chap 3).

Also, Oh-Thomann

- Glimm-Jaffe, Simon, Da Prato-Tubaro '06

⇒ Use Bourgain's invariant measure argument (Bourgain '96, Oh-Robert-Tzvetkov for SNLW on 2-d mfd)

2.3. ii 2-d SNLW / sdNLW

⑦

pathwise approach: known only $k=3$. (GKOT)

$k \geq 5$: OPEN.

With $v = u - \Psi$, we have

$$(\partial_t^2 + 1 - \Delta)v + v^3 + 3v^2\Psi + 3v:\Psi^2: + :\Psi^3: = 0$$

rough perturbation

Two difficulties:

① $v(t) \in H^{1-\varepsilon}(\mathbb{T}^2) \setminus H^1(\mathbb{T}^2)$.

$\Rightarrow$ can not use the energy

$$E(\vec{v}) = \frac{1}{2} \int |\nabla v|^2 dx + \frac{1}{2} \int (\partial_t v)^2 + \frac{1}{4} \int v^4 dx$$

We need to smooth out v

$\Rightarrow$ I-method

② Even if v were in H^1 , v does not satisfy (deterministic) NLW. ⑧

$\Rightarrow E(\vec{v})$ is not conserved.

If the noise is a bit smoother, $\Psi_\varepsilon = (\partial_t^2 + 1 - \Delta)^{-1} \langle \nabla_x \rangle^{-\varepsilon} \Xi$
 $\in C_t L_x^\infty$
 $\Rightarrow$ GWP by the Gronwall argument (due to Burq - Tzvetkov '14)
 $\partial_t E(\vec{v})(t) \leq C(\Psi) E(t)$

• I-method (= method of almost conservation law).

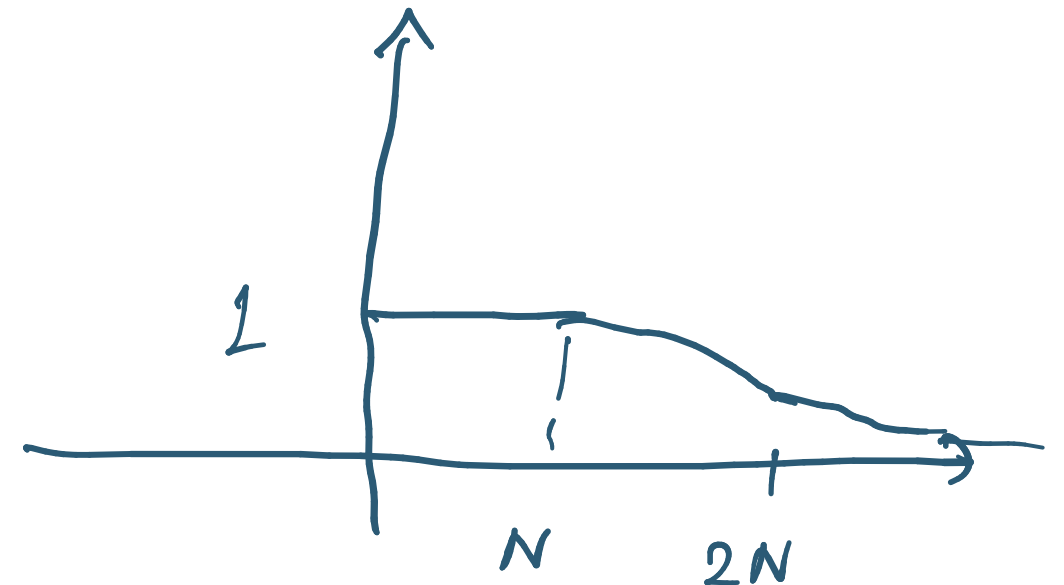
Colliander - Keel - Staffilani - Takaoka - Tao '02.

(after Bourgain's high-low method '98)

• $N \in \mathbb{N}$. Let

$0 < s < 1$

$$M_N(n) = \begin{cases} 1, & |n| \leq N \\ \frac{N^{1-s}}{|n|^{1-s}}, & |n| \geq 2N \end{cases}$$



low freq: identity, high freq: integration

⑨

$$\|If\|_{W^{a+\sigma,p}} \lesssim N^\sigma \|f\|_{W^{a,p}} \quad \forall \begin{matrix} 0 \leq \sigma \leq 1-s \\ 1 < p < \infty \end{matrix} \quad \Leftarrow \text{LP theory.}$$

$$\|f\|_{H^s} \lesssim \|If\|_{H^1} \lesssim \underline{N^{1-s}} \|f\|_{H^s}$$

$\Rightarrow$ Now, study the I-SNLW_V.

$$(\partial_t^2 + 1 - \Delta) Iv + I(v^3) + \underline{3I(v^2\Psi) + 3I(v:\Psi^2:) + I(:\Psi^3:)} = 0$$

$\Rightarrow E(\vec{Iv})$ is NOT conserved for two reasons:

① $I(v^3) \neq (Iv)^3 \Rightarrow$ Need a commutator estimate
 $I(v^3) - (Iv)^3$. (standard).

② perturbation terms for rough: Ψ^j :

Lemma: $p < \infty$

(10)

$$\| \| I\Psi \|_{L^p_{T,x}} \|_{L^p(\Omega)} \lesssim p^{1/2} T^{1/2 + 1/p} \underline{(\log N)^{1/2}}$$

$$I(v^2 \Psi) \rightarrow (Iv)^2 \cdot \underline{I\Psi} + \text{error.}$$

$\Rightarrow$ At the end, we obtain

$$E(I\vec{v}) \lesssim C + \int_0^t E(I\vec{v}) \underline{\log E(I\vec{v})(t')} dt'.$$

$\Rightarrow$ double exponential bound.

• Invariant measure argument for SdNLW

(11)

$$(\partial_t^2 + \partial_t + 1 - \Delta) u + u^k = \xi, \quad k \in 2\mathbb{N} + 1.$$

• Gibbs meas: $\vec{P}(du, d(\partial_t u)) = \overbrace{P(du)}^{\Phi_2^{k+1} \text{-measure}} \otimes \underbrace{\mu_0(d\partial_t u)}_{\text{white noise}}$
is formally invariant.

• Duhamel formulation for $v = u - \Psi$:

$$v(t) = \partial_t D(t) u_0 + D(t)(u_0 + u_1)$$

$$- \sum_{j=0}^k \binom{k}{j} \int_0^t D(t-t') (\Psi \hat{v} \cdot v^{k-j})(t') dt'.$$

where $D(t) = \frac{e^{-t/2} \sin\left(t \sqrt{\frac{3}{4} - \Delta}\right)}{\sqrt{\frac{3}{4} - \Delta}}$

$\leftarrow$ one deg of smoothing

• The same LWP argument (by Sobolev) as in SNLW works.

$\Rightarrow$ LWP $\Rightarrow$ a.s. GWP & invariance of $\vec{P}$.

Note: For damped NLW, the same Strichartz estimates hold locally in time.

$\uparrow$ on $\mathbb{T}^2$.

SPDE s.t. a given meas is invariant.

$\downarrow$

Rmk: • parabolic Φ_2^{k+1} -model = (parabolic) stochastic quantization equation (for Φ_2^{k+1} -measure)

$\hookrightarrow$ Parisi - Wu '81

• SdNLW = hyperbolic Φ_2^{k+1} -model
= canonical SQE.

$\leftarrow$ Ryang et al.

With $W = \partial_t U$,

$$\partial_t \begin{pmatrix} U \\ W \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial E}{\partial U} \\ \frac{\partial E}{\partial W} \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{\partial E}{\partial W} + \xi \end{pmatrix} \quad \text{Langvin eqn.}$$

(3) 3-d case:(3.1) parabolic Φ_3^4 -model:

$$(\partial_t + 1 - \Delta) u + u^3 = \xi.$$

• Recall on Π^3 :

$$\Psi = (\partial_t + 1 - \Delta)^{-1} \xi \sim -\frac{1}{2} -$$

$$-\frac{d}{2} + 1 -$$

$\Rightarrow \Psi^3$ (and hence u^3)

does NOT make sense

• Hairer '14

• Catellier - Chouk AP

($\Leftarrow$ Gubinelli - Imkeller
- Pukowski '15

• Mourrat - Weber GWP '17.

• Kupiainen AHP '16.

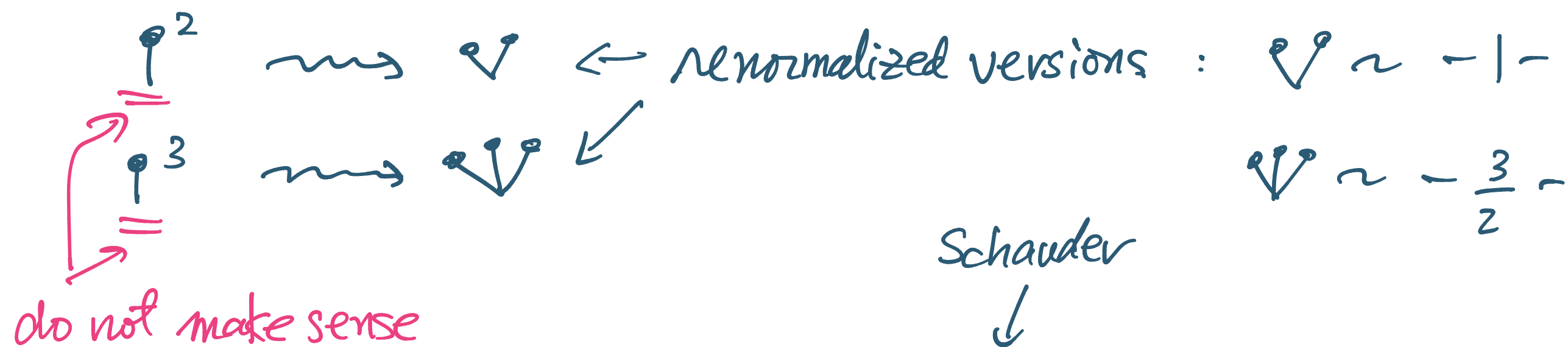
• Construction of such objects
Mourrat - Weber - Xu.

• Lecture note from my informal course.

1st order expansion: $U = V + \Gamma$

Tree notation: $\bullet = \Xi$
 $| = \text{edge} = (\partial_+ + 1 - \Delta)^{-1}$
ex: $\Gamma = \text{stoch conv}$

$$\begin{aligned} (\partial_+ + 1 - \Delta)V &= - (V + \Gamma)^3 \\ &= -V^3 - 3V^2\Gamma - 3\underline{V\Gamma} - \underline{\underline{V\Gamma}} \end{aligned} \quad \begin{matrix} (-\frac{1}{2}-) + (-\frac{1}{2}-) \\ \parallel \end{matrix}$$



• worst term: $V \sim -\frac{3}{2}- \Rightarrow V = (-\frac{3}{2}-) + \underset{\substack{\uparrow \\ \text{really } 2-}}{2} = \frac{1}{2}-$

$$\Rightarrow \nu \quad \nu$$

$$(\frac{1}{2}-) + (-1-) = -\frac{1}{2}- < 0 \Rightarrow \text{NOT well defined}$$

• 2nd order expansion: $u = \nu + \rho - \Psi$ ↖ Picard 2nd iterate

$$\Psi = (\partial_t + 1 - \Delta)^{-1} \nu \sim (-\frac{3}{2}-) + 2 = \frac{1}{2}-$$

$$v = u - \rho + \Psi$$

$$(\partial_t + 1 - \Delta) v = \nu - (\nu + \rho - \Psi)^3$$

$$= \cancel{\nu} - \cancel{\nu} - (\nu - \Psi)^3 - 3(\nu - \Psi)^2 \rho - \underbrace{3(\nu - \Psi) \nu}_{\rho^2}$$

$\uparrow \rho^3$
 $\downarrow$

WORST term: $(\nu - \Psi) \odot \nu \sim -1-$

$$\Rightarrow \nu \sim (-1-) + 2 = 1- \Rightarrow \nu \quad \nu \quad \text{NOT well defined.}$$

$$(1-) + (-1-) < 0$$

Note: higher order expansion would not help since the worst term involves the unknown.

- Idea: impose a structure on v .

Paracontrolled ansatz.

Lec 13: 21 / 04 / 21 (Wed)

①

2nd order expansion $u = 1 - \Psi + v$

$\quad \quad \quad -\frac{1}{2}- \quad \quad \frac{1}{2}-$

$$(\partial_t + 1 - \Delta) v = -v^3 - 3 \underbrace{(v - \Psi)}_{\sim -\frac{1}{2}-} \underbrace{v}_{\sim -\frac{1}{2}-} + Q(v)$$

$$Q(v) = b_0 + b_1 v + b_2 v^2$$

$$b_0 = (\Psi)^3 - 3 \underbrace{1 (\Psi)^2}$$

$$1 \odot \Psi \sim -\frac{1}{2}-$$

$$b_1 = 6 \underbrace{1 \Psi} - 3 (\Psi)^2$$

$$b_2 = -3 \underbrace{1} + 3 \Psi \sim -\frac{1}{2}-$$

all have
reg $\sim -\frac{1}{2}-$

Not well defined (via deterministic analysis)
 $\Leftarrow$ use the regularity lemma.

Issue: the product $v \cdot v$ does not make sense (2)

$$(1-) + (-1-) < 0.$$

$\Rightarrow$ Impose a structure.

• Use the paraproduct decomposition

$$v \cdot v = v \otimes v + v \oplus v + v \otimes v$$

$\sim -1-$
= worst term

ill-defined

But if we can make sense of it,
it will have a better reg
than $v \otimes v$

if $s_1 > 0 > s_2$
 $s_1 + s_2 > 0$
NOT true

Let $\oplus = \otimes + \ominus$

$\Rightarrow$ Write $v = X + Y$.

$$\begin{cases} (\partial_t + 1 - \Delta) X = -3(X + Y - \Psi) \otimes \nabla v \\ (\partial_t + 1 - \Delta) Y = -(X + Y)^3 - 3(X + Y - \Psi) \otimes \nabla v \\ \quad \quad \quad + Q(X + Y) \end{cases} \quad (3)$$

- X carries the rough regularity of v but Y is smoother.

$$\cdot \quad \boxed{X \sim (-1-) + 2 = 1-}$$

$\uparrow$
 $(\dots) \otimes \nabla v$

- For Y , we ignore the ill-defined $(\dots) \otimes \nabla v$.

$$\boxed{Y \sim \left(-\frac{1}{2}-\right) + 2 = \frac{3}{2}-}$$

$\Psi \otimes \nabla v$

We still need to make sense of the resonant product

(4)

$$-3(X+Y-\Psi)\ominus V$$

$$(i) \quad Y \ominus V \sim \left(\frac{3}{2}-\right) + (-1-) = \frac{1}{2}- (>0)$$

$\Rightarrow$ makes sense deterministically.

$$(ii) \quad \Psi \ominus V \rightsquigarrow \Psi \ominus V$$

$$\left(\frac{1}{2}-\right) + (-1-) < 0$$

Not Wick renormalization $\sigma_N = \mathbb{E}[I_N^2]$

$$\left(\Psi \ominus V\right)_N = \Psi_N \ominus V_N - 3\alpha_N I_N$$

$$\xrightarrow{N \rightarrow \infty} \Psi \ominus V \in C_t C_x^{-\frac{1}{2}-}$$

$$I_N = P_{\leq N} \Gamma$$

$$V_N = I_N^2 - \sigma_N$$

$$\alpha_N \sim \log N$$

$$\Psi_N = (\alpha_t + 1 - \Delta)^{-1} (I_N^3 - 3\sigma_N I_N)$$

Construction of Stoch objects : See Mourrat-Weber-Xu (5)
 Also, my note (taken by Justin Forleno)

Note:

$$\varphi(t) = \int_{-\infty}^t P(t-t') dW(t')$$

$$\Rightarrow \sigma_N = \mathbb{E}[\varphi(t,x)^2] = \sum_{|n| \leq N} \frac{1}{\langle n \rangle} \sim N, \text{ indep of } t, x.$$

$$\varphi(t) = \int_{-\infty}^t P(t-t') dW(t')$$

$$= \int_0^t P(t-t') dW(t') + \underbrace{P(t) \int_{-\infty}^0 P(-t') dW(t')}_{\text{random lin soln.}}$$

$\sim \mu_1 = \text{massive G.F.F.}$

invariant meas for
the linear flow.

$$(\partial_t + (1-\Delta))u = \tilde{z}.$$

Back to SQE:

(6)

Issue: $X \in \mathcal{V}$

$$(1-) + (-1-) < 0$$

$\Leftarrow$ use the structure of X i.e.

$$X(t) = P(t)X_0 - 3 \int_0^t P(t-t') [(X+Y-\dot{Y}) \in \mathcal{V}] dt'$$

• $P(t)X_0 \in \mathcal{V}$, well-defined.

• We "expect"

$$\int_0^t P(t-t') [(X+Y-\dot{Y}) \in \mathcal{V}] dt'$$

$$\approx (X+Y-\dot{Y}) \in \underline{\underline{Y}}$$

where $\dot{Y} = (\partial_t + 1 - \Delta)^{-1} \mathcal{V} \sim (-1-) + 2 = 1-$

⇒ We write

(7)

$$X = -3(X + Y - \Psi) \odot \Psi + \text{com}_1(X, Y)$$

$$\text{com}_1(X, Y) := P(t)X_0 - 3 \int_0^t P(t-s) [(X + Y - \Psi) \odot \Psi](s) ds + 3(X + Y - \Psi) \odot \Psi$$

↑
smoother $\sim 1 + \varepsilon$. (NOT true for dispersive PDEs.)

$$X \odot \Psi = -3 [(X + Y - \Psi) \odot \Psi] \odot \Psi$$

$$+ \underbrace{\text{com}_1(X, Y) \odot \Psi}_{\text{makes sense since } (1+) + (-1-) > 0.}$$

$-1 - \varepsilon, \forall \varepsilon.$
↓

↑
 $1 + \delta$ for some δ

$$(X + Y - \Psi) \odot Y \Leftarrow \text{high freq. behavior is given by } Y \quad (8)$$

$\Rightarrow$ We 'expect'

$$[(X + Y - \Psi) \odot Y] \oplus v \approx (X + Y - \Psi) \odot (Y \oplus v)$$

$$\left(\begin{array}{l} [f \odot g] \oplus h \\ n_1 \quad n_2 \quad n_3 \end{array} \right. \quad \begin{array}{l} |n_1| \ll |n_2| \\ |n_1 + n_2| \sim |n_3| \Rightarrow |n_2| \sim |n_3| \end{array} \quad \left. \begin{array}{l} \\ \approx f \odot (g \oplus h) \end{array} \right)$$

$$\begin{array}{c} \text{Y} \odot v \\ (1^-) + (-1^-) < 0. \end{array} \rightsquigarrow \begin{array}{c} \text{Y} \odot v \\ \oplus \end{array} \sim 0-$$

9

$$\begin{aligned} \text{Let } [\langle, \oplus] (f, g, h) \\ = (f \langle g) \oplus h - f (g \oplus h) \end{aligned}$$

$\uparrow$
NO $\langle$!

and set

$$\text{con}_2(X+Y) = [\langle, \oplus] (-3(x+y-\Psi), Y, v)$$

$$\begin{aligned} \Rightarrow X \oplus v &= -3 [(x+y-\Psi) \overset{\oplus}{\underset{\text{O-}}{\downarrow}} Y] + \text{con}_2(X+Y) \\ &\quad + \text{con}_1(X, Y) \oplus v \end{aligned}$$

Back to $Q(v)$: tree and tree^2 .

(10)

$$\text{tree} = \text{tree} \oplus \text{tree} + \underbrace{\text{tree} \oplus \text{tree} + \text{tree} \oplus \text{tree}}_{\text{use stoch analysis, i.e. the reg. lemma}}$$

use stoch analysis, i.e. the reg. lemma

$$\mathbb{E} \left[\left| \text{tree} \oplus \text{tree} (n) \right|^2 \right]$$

CHECK

$$\leq \sum_{n=n_1+n_2} \frac{1}{\langle n_1 \rangle^2} \frac{1}{\langle n_2 \rangle^4}$$

$$|n_1| \sim |n_2|$$

$$\sim \frac{1}{\langle n \rangle^3} \sim \langle n \rangle^{-3-2 \cdot 0}$$

$$\mathbb{E} \left[\left| \text{tree} (n) \right|^2 \right] \sim \underbrace{\langle n \rangle^{-3-1}}_{\text{II}} \sim \langle n \rangle^{-d-2s_0}$$

Summation

$$\sum_{n=n_1+n_2} \frac{1}{\langle n_1 \rangle^\alpha} \frac{1}{\langle n_2 \rangle^\beta} \lesssim \langle n \rangle^{d-\alpha-\beta}$$

① $\alpha + \beta > d$, $\alpha, \beta < d$

② If $|n_1| \sim |n_2|$, then we only need $\alpha + \beta > d$.

$$\Rightarrow \text{tree} = \text{tree} \oplus \text{tree} + \text{tree} \oplus \text{tree}$$

No renormalization needed

$$\underline{1(\Psi)^2}$$

Need to consider $1 \oplus (\Psi)^2$

(11)

$$1 \oplus (\Psi)^2 = 2 \underbrace{1 \oplus [\Psi \otimes \Psi]} + 1 \oplus [\Psi \oplus \Psi]$$

$$\left(\frac{1}{2}-\right) + \left(\frac{1}{2}-\right) = 1-$$

$$\left(-\frac{1}{2}-\right) + (1-) = \frac{1}{2}- > 0$$

$\Rightarrow$ makes sense

$$\Psi \Psi \oplus + [\otimes, \oplus](\Psi, \Psi, 1)$$

$$\downarrow \Psi \oplus 1$$

$$\left(\frac{1}{2}-\right) + (0-) = \frac{1}{2}- > 0$$

$$\Rightarrow 1(\Psi)^2 = 1 \oplus (\Psi)^2 + 1 \oplus [\Psi \oplus \Psi] + 2 \Psi \Psi \oplus + 2[\otimes, \oplus](\Psi, \Psi, 1)$$

Finally, we arrive at

(12)

$$(*) \left\{ \begin{aligned} (\partial_t + 1 - \Delta) X &= -3(X + Y - \Psi) \odot \nabla \\ (\partial_t + 1 - \Delta) Y &= -(X + Y)^3 - 3Y \odot \nabla + 3 \underbrace{\nabla \cdot \nabla}_{-\frac{1}{2}} \\ &\quad + 9[(X + Y - \Psi) \nabla \cdot \nabla] - 3 \text{com}_2(X + Y) \\ &\quad - 3 \text{com}_1(X, Y) \odot \nabla + \underbrace{Q(X + Y)}_{-\frac{1}{2}} \\ &\quad - 3(X + Y - \Psi) \odot \nabla \end{aligned} \right.$$

Enhanced data set

$$(u_0, \vec{z}) \mapsto (X_0, Y_0, \underbrace{1}_{-\frac{1}{2}}, \underbrace{\nabla}_{-1}, \underbrace{\Psi}_{\frac{1}{2}}, \underbrace{\nabla \cdot \nabla}_{0}, \underbrace{\nabla \cdot \nabla}_{-\frac{1}{2}}, \underbrace{\nabla \cdot \nabla}_{0})$$

$u_0 = X_0 + Y_0$

need 2nd renormalization

$$\mapsto (X, Y) \mapsto u = 1 - \Psi + X + Y$$

$$\sigma_N \sim N, \quad \alpha_N \sim \log N$$

$$\text{Diagram}_N = \text{Diagram}_N^2 - \sigma_N, \quad \text{Diagram}_N = \text{Diagram}_N^3 - 3\sigma_N \text{Diagram}_N$$

$$\text{Diagram}_N = \text{Diagram}_N \ominus \text{Diagram}_N - \alpha_N$$

$$\text{Diagram}_N = (\partial_t + 1 - \Delta)^{-1} \text{Diagram}_N$$

$$\left(\text{Diagram}_N \ominus \text{Diagram}_N \right) \sim \text{Diagram}_N \sim \log N$$

$\eta_1 + \eta_4 = 0$
 $\eta_2 + \eta_3 = 0$

$$\text{Diagram}_N = \text{Diagram}_N \ominus \text{Diagram}_N - 3\alpha_N \text{Diagram}_N$$

$$\text{Diagram}_N = \text{Diagram}_N \ominus \text{Diagram}_N \quad (\text{no renormalization})$$

Fix $N \in \mathbb{N}$ Start with $\bigotimes_N$

and get an eqn for $u_N = \varphi_N - \overset{\circ}{Y}_N + X_N + Y_N$

$$\begin{cases} (\partial_t + 1 - \Delta) u_N + \underbrace{u_N^3 - C_N u_N}_{\text{red}} = \xi_N & \xi_N = P_{\leq N} \xi \\ u_N|_{t=0} = (u_0)_N \end{cases}$$

where

$$C_N = 3 \frac{\sigma_N}{N^2} - 9 \frac{d_N}{(\log N)^2}$$

$$\Rightarrow (\partial_t + 1 - \Delta) u + u^3 - \infty \cdot u = \xi.$$

$L = \partial_t + 1 - \Delta$. We drop the subscript N

• $L\varphi = \xi$, $L\psi = \psi = \varphi^3 - 3\sigma_N\varphi$

• $Q(v) = -(v - \psi)^3 + v^3 - 3(v - \psi)\varphi$

$$Lu = L\varphi - L\psi + LX + LY$$

$$= \xi - \varphi^3 + 3\sigma_N\varphi - \underline{3(v - \psi) \odot v}$$

$$- v^3 - \underline{3(v - \psi) \odot v} + \varphi[(v - \psi) \odot \psi]$$

$$- \underline{3\cos_2(X + Y)} - 3\cos_1(X, Y) \odot v$$

$$- 3Y \odot v + 3\psi \odot v + Q(v)$$

$\psi \odot v - \sigma_N$
 ~~$\psi \odot v - \sigma_N$~~
 $\psi \odot v - \sigma_N$

$\psi \odot v - 3\sigma_N\varphi$

~~$-3[(X + Y - \psi) \odot Y] \odot v$
 $+ 3(X + Y - \psi)[\psi \odot v]$~~

$-9\sigma_N v$
 $+ 9\sigma_N \psi$

$$= \sum - \rho^3 - v^3 + (3\sigma_N - 9d_N) \rho - 9d_N(v - \psi) \quad (16)$$

$$+ Q(v) - 3(v - \psi) \oplus v - 3(Y - \psi) \oplus v$$

$$- 3 \left[-3[(v - \psi) \oplus Y] \oplus v + \omega_1(X, Y) \oplus v \right]$$

$$= X \oplus v$$

$$\rightarrow -3(v - \psi) \oplus v$$

$$= \sum - \rho^3 - v^3 + (3\sigma_N - 9d_N) \rho - 3(v - \psi) v$$

$$- 9d_N(v - \psi) + Q(v)$$

$$\parallel \rho^2 - \sigma_N$$

$$= \sum - \rho^3 - \cancel{v^3} + (3\sigma_N - 9d_N) (\rho + v - \psi)$$

$$- 3(v - \psi) \rho^2 + Q(v) = u$$

$$\parallel - (v - \psi)^3 + \cancel{v^3} - 3(v - \psi)^2 \rho$$

$$= \sum - (\rho + v - \psi)^3 + 3(\sigma_N - 9d_N) u$$

$$= \sum - u^3 + C_N u.$$

Prop (2nd commutator)

$$\alpha < 1, \beta, \gamma \in \mathbb{R}, \quad 1 \leq p, p_1, p_2, p_3 \leq \infty$$

s.t.

$$\beta + \gamma < 0, \quad \alpha + \beta + \gamma > 0, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}.$$

Then,

$$[\triangleleft, \oplus] : (f, g, h) \mapsto (f \triangleleft g) \oplus h - f(g \oplus h)$$

extends to a continuous trilinear map:

$$B_{p_1}^\alpha \times B_{p_2}^\beta \times B_{p_3}^\gamma \longrightarrow B_p^{\alpha + \beta + \gamma}$$

where $B_p^s = B_{p, \infty}^s$.

$$\alpha = \frac{1}{2} -$$

$$\beta = 1 - \quad \gamma = -1 -$$

$$\beta + \gamma < 0$$

Our case: $\text{com}_2(X+Y) = [\triangleleft, \oplus](-3(X+Y-\Psi), \Psi, \Psi).$

$$\alpha + \beta + \gamma = \frac{1}{2} - > 0.$$

$$2 \frac{1}{2} -$$

• Idea of the proof:

(18)

(**)

$$\| [\langle \cdot, \cdot \rangle] (f, g, h) \|_{B_p^{\alpha+\beta+\sigma}} \lesssim \| f \|_{B_{p_1}^\alpha} \| g \|_{B_{p_2}^\beta} \| h \|_{B_{p_3}^\sigma}$$

For proving (**), we instead consider

$$\| [\langle \cdot, \cdot \rangle] (\langle \nabla \rangle^\alpha f, \langle \nabla \rangle^\beta g, \langle \nabla \rangle^\sigma h) \|_{B_p^{\alpha+\beta+\sigma}}$$

$$\lesssim \| f \|_{B_{p_1}^0} \| g \|_{B_{p_2}^0} \| h \|_{B_{p_3}^0}$$

$\mathcal{F} ($

$) (n)$

$$= \sum_{|j-k| \leq 2} \sum_{n=n_1+n_2+n_3}$$

$$\frac{\langle n \rangle^{\alpha+\beta+\sigma}}{\langle n_1 \rangle^\alpha \langle n_2 \rangle^\beta \langle n_3 \rangle^\sigma}$$

$$\Psi_k(n_3)$$

$$= m(\overline{m})$$

$$\times \left(\sum_{l \leq q-2} \Psi_l(n_1) \Psi_l(n_2) \Psi_j(n_1+n_2) - \Psi_j(n_2) \right) \times \hat{f}(n_1) \hat{g}(n_2) \hat{h}(n_3)$$

Roughly speaking, we want to bound the weight

(19)

$$\sum_{|\hat{j}-k| \leq 2} m(\bar{n}) \varphi_k(m_3) \left(\sum_{l < q-2} \varphi_l(m_1) \varphi_q(m_2) \varphi_{\hat{j}}(m_1+m_2) - \varphi_{\hat{j}}(m_2) \right)$$

write it as (drop the summations in $\hat{j}, k$).

$$m(\bar{n}) \varphi_k(m_3) \sum_{l < q-2} \varphi_l(m_1) \varphi_q(m_2) \left(\varphi_{\hat{j}}(\underline{m_1+m_2}) - \underline{\varphi_{\hat{j}}(m_2)} \right)$$

$$+ m(\bar{n}) \varphi_k(m_3) \left(\sum_{l < q-2} \varphi_l(m_1) \varphi_q(m_2) - 1 \right) \varphi_{\hat{j}}(m_2)$$

$$=: \text{I} + \text{II}.$$

I:

$$|n_1| \ll |n_2| \sim |n_3|$$

(20)

$$|\varphi_j(n_1+n_2) - \varphi_j(n_2)| = \left| \varphi\left(\frac{n_1+n_2}{2^j}\right) - \varphi\left(\frac{n_2}{2^j}\right) \right|$$

$$\leq \left| \varphi'(n^*) \right| \frac{|n_1|}{2^j}$$

$$= \int_0^1 \varphi'\left(\frac{n_2 + n_1 t}{2^j}\right) \frac{n_1}{2^j} dt$$

or
MVT

$$|n_2| \sim 2^j$$

$$\lesssim \frac{|n_1|}{|n_2|}$$

$$\Rightarrow I \lesssim$$

$$\frac{m(\bar{n}) |n_1|}{|n_2|}$$

$$\Downarrow |n_1| \ll |n_2| \sim |n_3|$$

$$\sim \frac{\langle n \rangle^{\alpha+\beta+\gamma}}{\langle n_1 \rangle}$$

$$\Rightarrow |n| \lesssim \max(|n_2|, |n_3|)$$

$$\sim |n_2| \sim |n_3|$$

$$\alpha+\beta+\gamma > 0$$

$$\alpha < 1$$

$$\frac{\langle n_1 \rangle^\alpha \langle n_2 \rangle^{\beta+1} \langle n_3 \rangle^\gamma}{\langle n_2 \rangle^{\alpha+\beta+\gamma} \langle n_1 \rangle^{1-\alpha}}$$

$$\sim \frac{\langle n_2 \rangle^{\alpha+\beta+\gamma} \langle n_1 \rangle^{1-\alpha}}{\langle n_2 \rangle^{\beta+\gamma+1}}$$

$$\sim \left(\frac{\langle n_1 \rangle}{\langle n_2 \rangle} \right)^{1-\alpha}$$

$$\lesssim 2^{-\varepsilon j} = N_2^{-\varepsilon}$$

$$\underline{\alpha < 1}$$

II:

$$II = m(\bar{m}) \psi_h(m_3) \psi_{\bar{j}}(m_2) \left(\sum_{l < q-2} \psi_l(m_1) \psi_q(m_2) - 1 \right)$$

$\sum_{l, q} \psi_l(m_1) \psi_q(m_2)$
 $\sum_l \psi_l(3) = 1$

$\sum_{l \geq q-2}$
 $\Rightarrow |m_1| \gtrsim |m_2|$

$\Rightarrow$ We have $|m_1| \gtrsim |m_2| \sim |m_3|$

$\Rightarrow |m| \lesssim |m_1|$

$$\Rightarrow II \lesssim \frac{\langle m \rangle^{d+\beta+\gamma}}{\langle m_1 \rangle^\alpha \langle m_2 \rangle^{\beta+\gamma}} \stackrel{d+\beta+\gamma > 0}{\lesssim} \frac{\langle m_1 \rangle^{d+\beta+\gamma}}{\cancel{\langle m_1 \rangle^\alpha} \langle m_2 \rangle^{\beta+\gamma}} \sim \frac{\langle m_1 \rangle^{\beta+\gamma}}{\langle m_2 \rangle^{\beta+\gamma}} \lesssim 1$$

$\beta + \gamma < 0$
 $|m_1| \gtrsim |m_2|$



Next class, we prove a bound on $\text{com}_1(x, y)$

(22)

which comes down to $[P(t), \odot](f, g)$

Lec 14: 28/04/21 (Wed)

①

$$\begin{aligned} \text{com}_1(X, Y) &= P(t)X_0 - 3 \int_0^t P(t-t') [\underbrace{(X+Y)}_{\text{w}} - \underbrace{\dot{Y}}_{\text{w}}] \otimes \underbrace{v}_{-1-\varepsilon} dt' \\ &\quad + 3 \underbrace{(X+Y-\dot{Y})}_{\text{w}} \otimes \underbrace{Y}_{\text{w}} \end{aligned}$$

$1-\varepsilon$

Prop (1st commutator)

Let $\varepsilon > 0$, $\beta \in [4\varepsilon, 1+2\varepsilon]$, $p \in [1, \infty]$, $T > 0$.

Then,

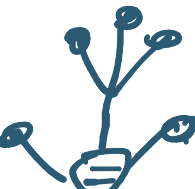
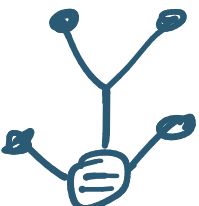
$$\begin{aligned} &\| \text{com}_1(X, Y)(t) - P(t)X_0 \|_{B_\infty^{1+2\varepsilon}} \\ &\lesssim \underbrace{K^2}_{\text{w}} + \int_0^t \frac{\underbrace{K}_{\text{w}}}{(t-t')^{1+2\varepsilon-\beta/2}} \| (X, Y)(t') \|_{B_p^\beta \times B_p^\beta} dt' \\ &\quad + \int_0^t \frac{\underbrace{K}_{\text{w}}}{(t-t')^{1+2\varepsilon}} \| \delta_{t', t}(X+Y) \|_{L^p} dt' \end{aligned}$$

$\frac{1}{2}+$ $1+$

$$(\delta_{t_1, t_2} f = f(t_2) - f(t_1)).$$

• $K = \text{bound on the (given) enhanced data set.}$

(2)

$\square_j$						
$\text{reg } s_j$	$-\frac{1}{2} - \varepsilon$	$-1 - \varepsilon$	$\frac{1}{2} - \varepsilon$	$-\varepsilon$	$-\frac{1}{2} - \varepsilon$	$-\varepsilon$

$$\sup_{0 \leq t \leq 1} \|\square_j(t)\|_{B_{\infty, \infty}^{s_j}} \leq K$$

Furthermore,

$$\sup_{0 \leq t_1 < t_2 \leq 1} \frac{\|\Psi(t_2) - \Psi(t_1)\|_{B_{\infty, \infty}^{\frac{1}{4} - \varepsilon}}}{|t_2 - t_1|^{\frac{1}{8}}} \leq K$$

③

Prop': $\alpha < 1$, $\beta \in \mathbb{R}$, $\underline{\gamma \geq \alpha + \beta}$, $1 \leq p, p_1, p_2 \leq \infty$
 $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$

Then,

(*) $\| [P(t), \odot] (f, g) \|_{B_p}^\gamma \lesssim t^{\underbrace{\frac{\alpha + \beta - \gamma}{2}}_{\leq 0}} \|f\|_{B_{p_1}}^\alpha \|g\|_{B_{p_2}}^\beta$

We assume Prop' and prove Prop on com_1 .

$$\begin{aligned} \text{com}_1 (X, Y)(t) - P(t) X_0 &= -3 \int_0^t P(t-t') [(X+Y-\Psi) \odot v](t') dt' \\ &\quad + 3(X+Y-\Psi)(t) \odot \underline{\Psi(t)}. \end{aligned}$$

$$= -3 \int_0^t [P(t-t'), \odot] (X+Y-\Psi, v)(t') dt'$$

$$- 3 \int_0^t (X+Y-\Psi)(t') \odot (P(t-t') v(t')) dt'$$

$$+ 3(X+Y-\Psi)(t) \odot \left[\int_0^t P(t-t') v(t') dt' \right]$$

Commutator
bet $\int_0^t \cdot dt'$
and $\odot$

$$= -3 \int_0^t \underbrace{[P(t-t'), \odot]}_{\text{red wavy line}} \underbrace{(X+Y-\dot{\Psi}, v)}_{\text{red underline}}(t') dt' \quad (4)$$

$$+ 3 \int_0^t \underbrace{[\delta_{t,t'}(X+Y-\dot{\Psi})]}_{\text{red underline}} \odot (P(t-t')v(t')) dt'$$

$$=: I + II. \quad \frac{1}{2} - \varepsilon$$

On I. Use Prop' on $[P(t), \odot]$.

Write $I = I_X + I_Y + I_{\dot{\Psi}}$.

$$\|I_{\dot{\Psi}}\|_{B_p^{1+2\varepsilon}} \lesssim \int_0^t \| [P(t-t'), \odot] (\dot{\Psi}(t'), v(t')) \|_{B_p^{1+2\varepsilon}} dt'.$$

$$\stackrel{\text{Prop'}}{\lesssim} K^2 \int_0^t \frac{1}{(t-t')^{\frac{3}{4}+2\varepsilon}} dt'$$

$$\lesssim K^2$$

$$\begin{cases} \gamma = 1+2\varepsilon \\ \alpha = \frac{1}{2} - \varepsilon \\ \beta = -1 - \varepsilon \end{cases}$$

$$\| I_X + I_Y \|_{B_p^{1+2\varepsilon}}$$

$$\lesssim K \int_0^t \frac{1}{(t-t')^{\frac{2+3\varepsilon-\beta}{2}}} \| (X+Y)(t') \|_{B_p^\beta} dt'$$

$$\begin{cases} \gamma = 1+2\varepsilon \\ \alpha = \beta \\ \beta = -1-\varepsilon \end{cases}$$

(5)

paraproduct est;

On II:

$$\| II \|_{B_p^{1+2\varepsilon}} \lesssim \int_0^t \| \delta_{t',t} (X+Y-\dot{Y}) \|_{L^p}$$

$$\times \| \underbrace{P(t-t')}_{-1-\varepsilon} \dot{Y}(t') \|_{B_\infty^{1+2\varepsilon}} dt'$$

Schauder

$$\lesssim K \int_0^t \frac{1}{(t-t')^{1+\frac{3}{2}\varepsilon}} \| \delta_{t',t} (X+Y-\dot{Y}) \|_{L^p} dt'$$

Not integrable near $t=0$

$$\| \delta_{t',t} \dot{Y} \|_{L^p} \lesssim |t-t'|^{1/8}$$

$$\lesssim K \int_0^t \frac{1}{(t-t')^{1+\frac{3}{2}\varepsilon}} \| \delta_{t',t} (X+Y) \|_{L^p} dt' + K^2$$



Before proving Prop' on $[P(t), \odot]$,
 we go over some basic estimates on $P(t) = e^{t(\Delta-1)} = e^{-t} e^{t\Delta}$ (6)

Lemma 1: $\| \mathcal{F}^{-1} \left(\varphi \left(\frac{\cdot}{2^j} \right) e^{-t|\cdot|^2} \right) \|_{L'_x} \lesssim e^{-ct2^{2j}}.$

(φ is supported on an annulus $A \sim \{|\xi| \sim 1\}$)

Pf: On $\mathbb{R}^d$: Let $g(t, x) = \int e^{ix \cdot \xi} \varphi(\xi) e^{-t|\xi|^2} d\xi$

$$= (1+|x|^2)^{-d} \int \underbrace{(1+|x|^2)^d e^{ix \cdot \xi}}_{= (1-\Delta_\xi)^d e^{ix \cdot \xi}} \cdot \varphi(\xi) e^{-t|\xi|^2} d\xi$$

$$= \underbrace{(1+|x|^2)^{-d}}_{\in L'_x} \int e^{ix \cdot \xi} \underbrace{(1-\Delta_\xi)^d}_{|\xi| \sim 1} \left(\varphi(\xi) e^{-t|\xi|^2} \right) d\xi$$

(7)

$$| (1 - \Delta_{\xi})^d (\varphi(\xi) e^{-t|\xi|^2}) |$$

$$\lesssim \underbrace{(1 + t|\xi|)^d}_{\stackrel{?}{\sim} t|\xi|^2} \underbrace{\varphi(\xi)}_{\substack{\uparrow \\ \text{support on } \{|\xi| \sim 1\} \\ \text{say } \frac{1}{2} \leq |\xi| \leq 2}} \underbrace{e^{-t|\xi|^2}}_{\sim \underline{\underline{e^{-ct}}}} \lesssim \underline{\underline{e^{-ct}}}$$

$$\left(\begin{array}{l} \text{use } x e^{-x} \leq e^{-cx} \quad \forall x \geq 0 \end{array} \right.$$

$$\begin{aligned} \bullet \quad g_{\lambda}(t, x) &= \int_{\mathbb{R}^d} e^{i x \cdot \xi} \varphi\left(\frac{\xi}{\lambda}\right) e^{-t|\xi|^2} d\xi \\ &= \lambda^d \int_{\mathbb{R}^d} e^{i \lambda x \cdot \xi} \varphi(\xi) e^{-t \lambda^2 |\xi|^2} d\xi \\ &= \lambda^d g(\lambda^2 t, \lambda x) \end{aligned}$$

$$\Rightarrow \|g_{\lambda}(t, x)\|_{L'_x} = \|g(t \lambda^2, x)\|_{L'_x} \lesssim e^{-c t \lambda^2}.$$

On $\mathbb{T}^d$: Use the Poisson summation formula.

⑧

$$g_\lambda^{\text{per}}(t, x) = \mathcal{F}_{\mathbb{T}^d}^{-1} \left(\varphi\left(\frac{\cdot}{\lambda}\right) e^{-t|\cdot|^2} \right).$$

$$\|g_\lambda^{\text{per}}(t, x)\|_{L'_x(\mathbb{T}^d)} = \left\| \sum_{n \in \mathbb{Z}^d} \underbrace{\varphi\left(\frac{n}{\lambda}\right) e^{-t|n|^2}}_{\hat{g}_\lambda(n)} e^{in \cdot x} \right\|_{L'_x(\mathbb{T}^d)}$$

$\hat{g}_\lambda(n)$

g_λ on $\mathbb{R}^d$

Poisson

$$= \left\| \sum_{n \in \mathbb{Z}^d} g_\lambda(x+n) \right\|_{L'(\mathbb{T}^d)}$$

$$= \|g_\lambda\|_{L'_x(\mathbb{R}^d)} \lesssim e^{-ct\lambda^2}$$



Lemma 2: $\| (1 - P(t))f \|_{B_{p,q}^\alpha} \lesssim t^{\underbrace{\frac{\beta-\alpha}{2}}_{\geq 0}} \| f \|_{B_{p,q}^\beta}$

for $0 \leq \beta - \alpha \leq 2$. ($\Rightarrow \beta \geq \alpha$)

pf: Claim: For $\hat{f}$ supported on $\{|m| \sim 2^j\}$,

$$\| (1 - P(t))f \|_{L^p} \lesssim (t 2^{2j} \wedge 1) \| f \|_{L^p}.$$

$\uparrow$
min.

Assume Claim:

$$2^{d\hat{j}} \| (1 - P(t)) P_{\hat{j}} f \|_{L^p} \stackrel{\text{Claim}}{\lesssim} 2^{d\hat{j}} (t 2^{2\hat{j}} \wedge 1) \| P_{\hat{j}} f \|_{L^p}$$

$$\lesssim t^{\frac{\beta-\alpha}{2}} \underbrace{\left((t 2^{2\hat{j}})^{\frac{\alpha-\beta}{2}} (t 2^{2\hat{j}} \wedge 1) \right)}_{\lesssim 1} 2^{\beta\hat{j}} \| P_{\hat{j}} f \|_{L^p}$$

□

$\lesssim 1 \quad \forall \hat{j}, t.$

⑨

Pf of Claim:

(10)

By Lemma 1, we have

$$\|P(t)P_j f\|_{L^p} \lesssim e^{-c t 2^{2j}} \|P_j f\|_{L^p}$$

↑
Young & Lemma 1

$\Rightarrow$ claim for $t 2^{2j} \geq 1$.

• Suffices to prove $\|(1-P(t))P_j f\|_{L^p} \lesssim \underline{t 2^{2j}} \|f\|_{L^p}$

On $\mathbb{R}^d$, let

(and $t 2^{2j} \ll 1$.)

$$g_\lambda(t, x) = \int_{\mathbb{R}^d} e^{ix \cdot \xi} \varphi\left(\frac{\xi}{\lambda}\right) (1 - e^{-t \langle \xi \rangle^2}) d\xi$$

$$= \lambda^d \int_{\mathbb{R}^d} e^{i\lambda x \cdot \xi} \varphi(\xi) (1 - e^{-t \lambda^2 \xi^2 - t}) d\xi.$$

$$\|g_\lambda(t, x)\|_{L'_x}$$

(11)

$$= \left\| \int_{|\xi| \sim 1} e^{ix \cdot \xi} \varphi(\xi) (1 - e^{-t\lambda^2|\xi|^2 - t}) d\xi \right\|_{L'_x(\mathbb{R}^d)}$$

$$|\xi| \sim 1$$

① On $Q = \{|x| \lesssim 1\}$.

$$(RHS) \leq \| \dots \|_{L^\infty_x(Q)}$$

$$\stackrel{H-Y}{\leq} \| \varphi(\xi) (1 - e^{-t\lambda^2|\xi|^2 - t}) \|_{L'_\xi}$$

$$\stackrel{MVT}{\leq} t(\lambda^2|\xi|^2 + 1)$$

$$\sim t(\lambda^2 + 1)$$

$$\sim t\lambda^2. \quad \lambda = 2^j \text{ for } j \geq 0.$$

② On Q^c

Multiply by $|x|^{-2d} \cdot \underline{|x|^{2d}}$

$$\uparrow$$

$$L'_x$$

(12)

$$|x|^{-2d} \underbrace{|x|^{2d}}_{\substack{\uparrow \\ L'_x}} \int e^{ix \cdot \vec{z}} \varphi(\vec{z}) (1 - e^{-t\lambda^2|\vec{z}|^2 - t}) d\vec{z}$$

$$-\Delta_{\vec{z}} e^{ix \cdot \vec{z}}$$

$$\stackrel{\text{IBP}}{=} \int_{|\vec{z}| \sim 1} e^{ix \cdot \vec{z}} (-\Delta_{\vec{z}})^d \left(\varphi(\vec{z}) \cdot \underline{(1 - e^{-t\lambda^2|\vec{z}|^2 - t})} \right) d\vec{z}$$

• all the derivatives hit $\varphi(\vec{z})$

$$1 - e^{-t\lambda^2|\vec{z}|^2 - t} \sim \underline{t\lambda^2} \text{ as before}$$

• If at least one derivative hits the 2nd factor,

$$\text{we get } t\lambda^2|\vec{z}| \sim \underline{t\lambda^2}$$

(and use the boundness of $e^{-t\lambda^2|\vec{z}|^2 - t}$.

By the Poisson summation formula as on page (8),
we obtain the same bd on $\mathbb{T}^d$.

(13)

□

Proof of Prop' on page (3):

$$\begin{aligned} [P(t), \odot](f, g) &= \sum_{k=0}^{\infty} \left[P(t) (P_{\leq k-2} f \cdot P_k g) \right. \\ &\quad \left. - \underbrace{P_{\leq k-2} f \cdot P_k P(t) g} \right] \\ &=: \sum_{k=0}^{\infty} h_k \end{aligned}$$

$P_{\leq k-2} = \sum_{j=0}^{k-2} P_j$

Note: $\hat{h}_k$ is supported on $C 2^k A$
 $\uparrow$
 annulus $\{|\beta| \sim |\beta|\}$.

Define $G_{h,t}$ by

(14)

$$G_{h,t} = \mathcal{F}^{-1} \left(\tilde{\varphi} \left(\frac{\cdot}{2^h} \right) e^{-t(\cdot)^2} \right)$$

$\uparrow$ wider support. $\varphi(\cdot) = \varphi(\frac{\cdot}{2}) + \varphi(1) + \varphi(2\cdot)$

$$\Rightarrow P(t) h_k = G_{h,t} * h_k$$

$$\Rightarrow h_k = G_{h,t} * (\underbrace{P_{\leq k-2} f}_{\text{red}} \cdot \underbrace{P_h g}_{\text{red}}) - \underbrace{P_{\leq k-2} f}_{\text{red}} (G_{h,t} * P_h g)$$

$$= - \int G_{h,t}(y) P_h g(x-y) \underbrace{\left[P_{\leq k-2} f(x) - P_{\leq k-2} f(x-y) \right]}_{\text{blue bracket}} dy$$

$$\text{FTC} = \int_0^1 \nabla P_{\leq k-2} f(x-sy) \cdot \underline{y} ds$$

$$= \int_0^1 \int P_h g(x-y) \underbrace{\tilde{G}_{h,t}(y)}_{= y G_{h,t}(y)} \cdot \nabla P_{\leq k-2} f(x-sy) dy ds$$

$$\text{Let } h_{k,s} = \int P_k g(x-y) \tilde{G}_{k,t}(y) \cdot \nabla P_{\leq k-2} f(x-sy) dy.$$

(15)

Claim: $\|h_{k,s}\|_{L_x^p} \lesssim \|\tilde{G}_{k,t}\|_{L^1} \|\nabla P_{\leq k-2} f\|_{L^{p_1}} \|P_k g\|_{L^{p_2}}$
unif in $s \in [0,1]$

Pf of Claim: Write $|\tilde{G}_{k,t}| = |\tilde{G}_{k,t}|^{\frac{1}{p'}} |\tilde{G}_{k,t}|^{\frac{1}{p}}$

$$\Rightarrow |h_{k,s}(x)| \lesssim \|\tilde{G}_{k,t}\|_{L^1}^{1-\frac{1}{p}} \left(\int |\tilde{G}_{k,t}(y)| |P_k g(x-y)|^p \times |\nabla P_{\leq k-2} f(x-sy)|^p dy \right)^{1/p}$$

$$\Rightarrow \|h_{k,s}\|_{L_x^p}^p \lesssim \|\tilde{G}_{k,t}\|_{L^1}^{p-1} \underbrace{\int |\tilde{G}_{k,t}(y)| \int |P_k g(x-y)|^p |\nabla P_{\leq k-2} f(x-sy)|^p dx dy}_{\text{Apply Hölder.}}$$

□

Apply Hölder. $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$

Now, we estimate $\|\tilde{G}_{k,t}\|_{L^1}$

$$\|y G_{k,t}\|$$

$$y = (y_1, \dots, y_d)$$

• Suffices to study $y_1 G_{k,t}(y)$

On the Fourier side,

$$y_1 G_{k,t}(\xi) \sim \partial_{\xi_1} \left(\varphi\left(\frac{\xi}{2^k}\right) e^{-t|\xi|^2} \right)$$

$$= \left(\frac{1}{2^k} \partial_{\xi_1} \varphi\left(\frac{\xi}{2^k}\right) - 2\xi_1 t \varphi\left(\frac{\xi}{2^k}\right) \right) e^{-t|\xi|^2}.$$

Lemma 1 on page (6)

$\Rightarrow$

$$\|\tilde{G}_{k,t}\|_{L^1} \lesssim 2^{-k} (1 + t 2^{2k}) e^{-ct 2^{2k}}$$

(**)

$$2^{-k} 2^{2k} t \overbrace{\frac{\xi_1}{2^k} \varphi\left(\frac{\xi}{2^k}\right)}^{\varphi}$$

$$\Rightarrow 2^{k\sigma} \|h_k\|_{L^p} \leq \int_0^1 2^{k\sigma} \|h_{k,s}\|_{L^p} ds$$

claim ~~claim~~ ~~(**)~~

$$\lesssim 2^{k(\sigma-\alpha-\beta)} (1+t2^{2k}) e^{-ct2^{2k}}$$

$$\times \underline{2^{\underline{k(\alpha-1)}}} \|\underline{\nabla} P_{\leq k-2} f\|_{L^{p_1}} \times 2^{k\beta} \|P_k g\|_{L^{p_2}}$$

$$\|\nabla P_{k \leq 2} f\|_{L^{p_1}} \leq \sum_{\substack{j \leq k-2 \\ 2^{j(1-\alpha)}}} \left(\|P_j \nabla f\|_{L^{p_1}} \right)_{2^{j(\alpha-1)}}$$

$$\alpha < 1$$

$$\lesssim \underline{2^{k(1-\alpha)}} \|\nabla f\|_{B_{p_1}^{d-1}} \quad \sup_j \dots = \|\nabla f\|_{B_{p_1}^{d-1}}$$

$$2^{k(\gamma-\alpha-\beta)} (1 + t 2^{2k}) e^{-c t 2^{2k}}$$

$$= t^{\frac{\alpha+\beta-\gamma}{2}}$$

$$(t 2^{2k})^{\frac{\gamma-\alpha-\beta}{2}}$$

$$\gtrsim t^{\frac{\gamma-\alpha-\beta}{2}} (1+x) e^{-cx} \gtrsim 1, \quad \forall x \geq 0$$

$$\text{if } \gamma \geq \alpha + \beta$$



Rmk: The commutator estimate such as $[P(t), \odot]$ in the wave case does not provide any smoothing.

Lec 15: 05/05/21 (Wed)

①

LWP of parabolic Φ_3^4 -model (for smoother initial data):

(SQE)

$$\begin{aligned}
 (\partial_t + 1 - \Delta) X &= -3 (X + Y - \Psi) \otimes \underbrace{v}_{-1-} \\
 (\partial_t + 1 - \Delta) Y &= -(X + Y)^3 - 3 Y \otimes \underbrace{v}_{-1-} + 3 \underbrace{\left(\begin{array}{c} \swarrow \downarrow \searrow \\ \otimes \end{array} \right)}_{-1/2-} \\
 &\quad - 3 (X + Y - \Psi) \otimes v \\
 &\quad - 3 \text{com}_1 (X, Y) \otimes v \\
 &\quad - 3 \text{com}_2 (X + Y) + \underbrace{Q(X + Y)}_{-\frac{1}{2}-}
 \end{aligned}$$

$$X \in C_T C_x^{\frac{1}{2} + 2\varepsilon} \cap C_T^{\frac{1}{8}} L_x^\infty =: \mathbb{Z}^X(T)$$

$$Y \in C_T C_x^{1 + 2\varepsilon} \cap C_T^{\frac{1}{8}} L_x^\infty =: \mathbb{Z}^Y(T)$$

$$\mathbb{Z}(T) = \mathbb{Z}^X(T) \times \mathbb{Z}^Y(T).$$

Write (SQE) in the Duhamel formulation:

(2)

$$X = \Gamma^X(x, y)(t) := \underline{P(t)} X_0 - 3 \int_0^t P(t-t') [(x+y-\psi) \odot v](t') dt'$$

$$Y = \Gamma^Y(x, y)(t) := \underline{P(t)} Y_0 - \int_0^t P(t-t') (x+y)^3(t') dt' + \dots$$

① $\Gamma^X \in C_T C_x^{\frac{1}{2}+2\varepsilon}$?

$\checkmark \int_0^t \frac{1}{(t-t')^{(\frac{3}{2}-\varepsilon)/2}} dt' < \infty$

$$\begin{aligned} \|\Gamma^X(x, y)\|_{C_T C_x^{\frac{1}{2}+2\varepsilon}} &\lesssim \|X_0\|_{C_x^{\frac{1}{2}+2\varepsilon}} + \underline{T^\theta} \|(x+y-\psi) \odot v\|_{C_T C_x^{-1-\varepsilon}} \\ &\lesssim \|X_0\|_{C_x^{\frac{1}{2}+2\varepsilon}} + \underline{T^\theta} \|x+y-\psi\|_{C_T L_x^\infty} \|v\|_{C_T C_x^{-1-\varepsilon}} \\ &\lesssim \|X_0\|_{C_x^{\frac{1}{2}+2\varepsilon}} + \underline{T^\theta} K (K + R) \end{aligned}$$

for $(x, y) \in \mathcal{B}_R \subset \mathcal{Z}(T)$

$$\textcircled{2} \quad \Gamma^X(x, y) \in C_T^{1/8} L_x^\infty?$$

③

$$\Gamma^X(x, y)(t_2) - \Gamma^X(x, y)(t_1)$$

$$t_2 > t_1$$

$$= (P(t_2 - t_1) - \text{Id}) P(t_1) X_0$$

$$+ (P(t_2 - t_1) - \text{Id}) \int_0^{t_1} P(t_1 - t') \square(t') dt'$$

$$+ \int_{t_1}^{t_2} P(t_2 - t') \square(t') dt'. \quad =: \text{I} + \text{II} + \text{III}.$$

use
Lemma 2
from Lec 14

$$\begin{aligned} \| \text{I} \|_{L_x^\infty} &\lesssim \| \text{I} \|_{B_\infty^\varepsilon} \\ &\lesssim (t_2 - t_1)^{1/8} \| X_0 \|_{C_x^{1/4+\varepsilon}}. \\ &\quad \uparrow \\ &\text{Lemma 2.} \end{aligned}$$

Lemma 2

$$\| (1 - P_{\leq \Sigma}) f \|_{B_p^\alpha} \lesssim t^{\frac{\beta-\alpha}{\Sigma}} \| f \|_{B_p^\beta}$$

$0 \leq \beta - \alpha \leq 2$

$$\cdot \quad \| \text{II} \|_{L_x^\infty} \lesssim (t_2 - t_1)^{\frac{1}{8}} \left\| \int_0^{t_1} p(t_1 - t') \underbrace{\square(t')}_{(X+Y-\Psi) \otimes v \in C^{-1-\varepsilon}} dt' \right\|_{C^{\frac{1}{4}+\varepsilon}} \quad (4)$$

$$\lesssim \underline{(t_2 - t_1)^{\frac{1}{8}}} T^\theta K(K+R).$$

$$\cdot \quad \| \text{III} \|_{L_x^\infty} \lesssim \| \text{III} \|_{C_x^\varepsilon} \lesssim \int_{t_1}^{t_2} \frac{1}{(t_2 - t')^{\frac{1}{2}+\varepsilon}} \| \square \|_{C_T C_x^{-1-\varepsilon}} dt'$$

$$\lesssim (t_2 - t_1)^{\frac{1}{8}} T^\theta K(K+R).$$

$$(0 < t_1 < t_2 \leq T.)$$

$$\Rightarrow \boxed{\| \Gamma^X(X, \Psi) \|_{C_T^{\frac{1}{8}} L_x^\infty} \lesssim \| X_0 \|_{C^{\frac{1}{2}+2\varepsilon}} + T^\theta K(K+R).}$$

$$\textcircled{3} \quad \Gamma^Y(X, Y) \in C_T C_x^{1+2\varepsilon} ?$$

⑤

$$\begin{aligned} \cdot \quad \| I(t) \|_{C_x^{1+2\varepsilon}} &\lesssim \int_0^t \frac{1}{(t-t')^{\frac{1}{4}}} \| (X+Y)^3 \|_{C_T C_x^{\frac{1}{2}+2\varepsilon}} dt' \\ &\lesssim T^\theta R^3 \quad \text{by algebra property of } C_x^s, s \geq 0 \end{aligned}$$

• Similarly,

$$\begin{aligned} \| \text{III} + \text{VII}(t) \|_{C_x^{1+2\varepsilon}} &\lesssim \int_0^t \frac{1}{(t-t')^{\frac{3}{4}+\frac{3}{2}\varepsilon}} \| \Psi + Q(X+Y) \|_{C_T C_x^{-\frac{1}{2}-\varepsilon}} dt' \\ &\lesssim T^\theta K(1 + (K+R)^2). \end{aligned}$$

$$\cdot \quad \| \underline{II}(t) \|_{C_x^{1+2\varepsilon}} \lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{1+\varepsilon}{2}}} \| Y \oplus v \|_{C_T C_x^\varepsilon} d\tau \quad (6)$$

$$\lesssim \| Y \|_{C_T C_x^{1+2\varepsilon}} \| v \|_{C_T C_x^{-1-\varepsilon}}$$

$$\lesssim T^\theta K R.$$

$$\cdot \quad \| \underline{IV}(t) \|_{C_x^{1+2\varepsilon}} \lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{3}{4}+2\varepsilon}} \| (X+Y-\Psi) \oplus v \|_{C_T C_x^{-\frac{1}{2}-2\varepsilon}} d\tau$$

$$\lesssim \| X+Y-\Psi \|_{C_T C_x^{\frac{1}{2}-\varepsilon}} \| v \|_{C_T C_x^{-1-\varepsilon}}$$

$$\lesssim T^\theta K (K+R).$$

$$\cdot \quad \| \nabla(t) \|_{C_x^{1+2\varepsilon}} \lesssim \int_0^t \frac{1}{(t-t')^{\frac{1+\varepsilon}{2}}} \| \operatorname{com}_1(X, Y) \ominus \nabla \|_{C_T C_x^\varepsilon} dt' \quad (7)$$

$$\lesssim T^\theta (K^2 + KR).$$

$$\left(\Leftarrow \int_0^t \frac{K}{(t-t')^{1+2\varepsilon}} \| \delta_{t',t}(X+Y) \|_{L_x^\infty} dt' \lesssim T^\theta KR \right.$$

$\underbrace{\frac{1}{(t-t')^{\frac{7}{8}+2\varepsilon}}}_{\text{locally integrable.}} \quad \frac{1}{(t-t')^{1/8}}$

$\Leftarrow$ In this step, we needed the Hölder in-time regularity of Ψ

$$\cdot \quad \| \nabla(t) \|_{C^{1+2\varepsilon}} \lesssim \int_0^t \frac{1}{(t-t')^{\frac{3}{4}+\frac{5\varepsilon}{2}}} \| \operatorname{com}_2(X+Y) \|_{C_T C_x^{\frac{1}{2}-3\varepsilon}} dt'$$

$$\lesssim T^\theta \| X+Y-\Psi \|_{C_T C_x^{\frac{1}{2}-\varepsilon}} \| Y \|_{C_T C_x^{1-\varepsilon}} \| \nabla \|_{C_T C_x^{-1-\varepsilon}}$$

$$\lesssim T^\theta K^2 (K + R).$$

Hence, we obtain

$$\|\Gamma^Y(X, Y)\|_{C_T C_x^{1+2\varepsilon}} \lesssim \|Y_0\|_{C_x^{1+2\varepsilon}} + T^\theta (K+R)^3$$

⑧

④ $\Gamma^Y(X, Y) \in C_T^{1/8} L_x^\infty$?

Proceed as in Case ② for $\Gamma^X(X, Y)$.

Write

$$\Gamma^Y(X, Y)(t_2) - \Gamma^Y(X, Y)(t_1) =: \text{I} + \text{II} + \text{III}$$

as on page ③

Lemma 2

$$\cdot \|\text{I}\|_{L_x^\infty} \lesssim \|\text{I}\|_{B_x^\varepsilon} \lesssim (t_2 - t_1)^{1/8} \|Y_0\|_{C_x^{1/4+\varepsilon}}$$

$$\begin{aligned} \cdot \|\text{II}\|_{L_x^\infty} &\lesssim (t_2 - t_1)^{1/8} \left\| \int_0^{t_1} P(t_1 - t') \boxed{} dt' \right\|_{C_x^{1/4+\varepsilon}} \\ &\lesssim (t_2 - t_1)^{1/8} T^\theta (K+R)^3 \end{aligned}$$

↑
Apply the result from
Case ③.

$$\begin{aligned} \cdot \quad \| \underline{III} \|_{L_x^\infty} &\lesssim \| \underline{III} \|_{C_x^\varepsilon} \lesssim \int_{t_1}^{t_2} \frac{1}{(t_2 - t')^{\frac{1}{4} + \frac{3\varepsilon}{2}}} \| \square \|_{C_T C_x^{-\frac{1}{2} - 2\varepsilon}} dt'. \quad (9) \\ &\lesssim (t_2 - t_1)^{\frac{1}{8}} T^\theta (K + R)^3. \end{aligned}$$

By similar computations, we can obtain difference estimates

$$\text{on } \Gamma(X_1, Y_1) - \Gamma(X_2, Y_2)$$

$$\text{where } \Gamma = (\Gamma^X, \Gamma^Y)$$

$\Rightarrow \Gamma$ is a contraction on $B_R \subset Z(T)$

$$R \sim \|X_0\|_{C^{\frac{1}{2} + 2\varepsilon}} + \|Y_0\|_{C^{1 + 2\varepsilon}}$$

by choosing $T = T(R, K) > 0$ suff. small.

As in the 1-d, 2-d cases, we can also handle rougher initial data in $C^{-3/5} \times C^{-3/5}$

(point: $-\frac{3}{5} < -\frac{1}{2}$ ~ regularity of the Gibbs meas on $\mathbb{T}^3$)

Catellier - Chouk, Mourrat - Weber CMP '17.

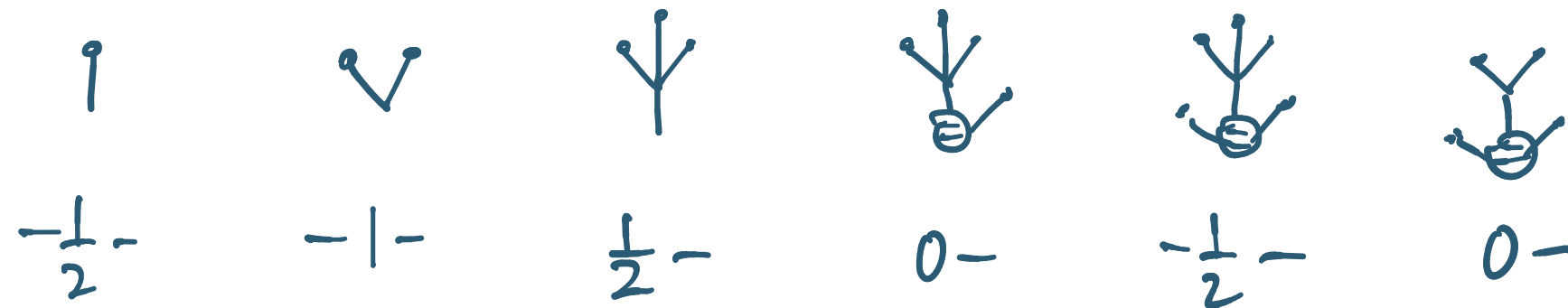
$\Leftarrow$ Need to use the $V(T)$ -type norm. (X beta function)

$$\| (X, Y) \|_{Z(T)} = \max \left\{ \| X \|_{G_T C_x^{-3/5}}, \sup_{0 < t \leq T} \underline{t^{3/5}} \| X(t) \|_{C_x^{\frac{1}{2} + 2\varepsilon}}, \right. \\ \left. \sup_{0 < t_1 < t_2 \leq T} \frac{\underline{t_1^{1/2}} \| X(t_2) - X(t_1) \|_{L_x^\infty}}{|t_2 - t_1|^{1/8}}, \right. \\ \left. \| Y \|_{G_T C_x^{-3/5}}, \sup_{0 < t \leq T} \underline{t^{17/20}} \| Y(t) \|_{C^{1+2\varepsilon}}, \sup_{0 < t_1 < t_2 \leq T} \frac{\underline{t_1^{1/2}} \| Y(t_2) - Y(t_1) \|_{L_x^\infty}}{|t_2 - t_1|^{1/8}} \right\}.$$

On the construction of the stoch. objects

(11)

- Mourrat - Weber - Xu
- Note by Justin.



same proof as 2-d case

· Ψ : $\mathbb{E}[|\hat{\Psi}(t, n)|^2] = 3! \times (t, n) \longleftrightarrow (t, -n)$

$$= 6 \sum_{n=n_1+n_2+n_3} \prod_{j=1}^3 \frac{1}{\langle n_j \rangle^2}$$

2 sums

divergent.

$$\lesssim \sum_{n_1} \frac{1}{\langle n_1 \rangle^2} \frac{1}{\langle n-n_1 \rangle^2} = \infty$$

$$\left(\begin{array}{l} p = \int_{-\infty}^t p(t-t') dW(t') \\ \mathbb{E}[|\hat{p}(t, n)|^2] = \frac{1}{\langle n \rangle^2} \end{array} \right.$$

(12)

$$\begin{aligned}
 \mathbb{E} \left[\hat{\rho}(t_1, m) \overline{\hat{\rho}(t_2, m)} \right] & \quad t_2 < t_1 \\
 &= \int_{-\infty}^{t_2} e^{-(t_1+t_2-2t')\langle m \rangle^2} dt' \\
 &= \frac{e^{-|t_1-t_2|\langle m \rangle^2}}{\langle m \rangle^2} \lesssim \frac{1}{\langle m \rangle^2} \left(\frac{1}{|t_1-t_2|\langle m \rangle^2} \right)^r, \forall r \geq 0
 \end{aligned}$$

$$\Rightarrow \mathbb{E} \left[\hat{\psi}(t_1, m) \hat{\psi}(t_2, -n) \right] = 6 \times (t_1, m) \begin{array}{c} \xrightarrow{n_1} \xleftarrow{-n_1} \\ \xrightarrow{n_2} \xleftarrow{-n_2} \\ \xrightarrow{n_3} \xleftarrow{-n_3} \end{array} (t_2, -n)$$

$$\lesssim \frac{1}{|t_1-t_2|^r} \sum_{n=n_1+n_2+n_3} \frac{1}{\langle n_1 \rangle^{2+2r}} \frac{1}{\langle n_2 \rangle^2} \frac{1}{\langle n_3 \rangle^2}$$

$$\lesssim \frac{1}{|t_1-t_2|^r} \sum_{n_1} \frac{1}{\langle n_1 \rangle^{2+2r}} \frac{1}{\langle m-n_1 \rangle}$$

$$\lesssim \frac{1}{|t_1-t_2|^r} \frac{1}{\langle m \rangle^{2r}}$$

$$\left(\sum \frac{1}{\langle n_1 \rangle^\alpha} \frac{1}{\langle m-n_1 \rangle^\beta} \lesssim \frac{1}{\langle m \rangle^{\alpha+\beta-d}} \right. \\
 \left. \text{for } \alpha+\beta > d \right. \\
 \left. \alpha, \beta < d. \right)$$

$$\mathbb{E} [|\hat{\Psi}(t, n)|^2] = 6 \times (t, n) \quad \begin{array}{c} t_1 \quad t_2 \\ \text{---} \quad \text{---} \end{array} (t, -n)$$

already computed

(13)

$$\approx \int \hat{P}_{t-t_1}(n) \hat{P}_{t-t_2}(n) \frac{1}{|t_1-t_2|^\sigma} \frac{1}{\langle n \rangle^{2\sigma}} dt_2 dt_1$$

↓

$$\hat{P}_t(n) = e^{-t\langle n \rangle^2} \mathbb{1}_{t \geq 0}$$

$$\stackrel{C-S}{\leq} \frac{1}{\langle n \rangle^{2\sigma}} \|\hat{P}_{t-t_1}(n)\|_{L^2_{t_1}} \|\hat{P}_{t-t_2}(n) * \frac{1}{|\cdot|^\sigma(t_1)}\|_{L^2_{t_1}}$$

$$\lesssim \frac{1}{\langle n \rangle^{1+2\sigma}} \frac{1}{\langle n \rangle^{3-2\sigma}}$$

$$= \langle n \rangle^{-3-2(\frac{1}{2})}$$

$$\lesssim \|\hat{P}_{t-t_2}(n)\|_{L^q_{t_1}}$$

$$\lesssim \frac{1}{\langle n \rangle^{2/q}}$$

H-L-S inequality

$$\frac{1}{2} + 1 = \frac{1}{1/\sigma} + \frac{1}{q}$$

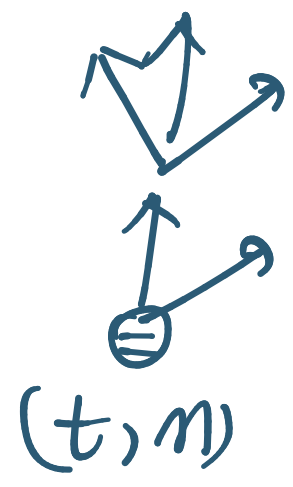
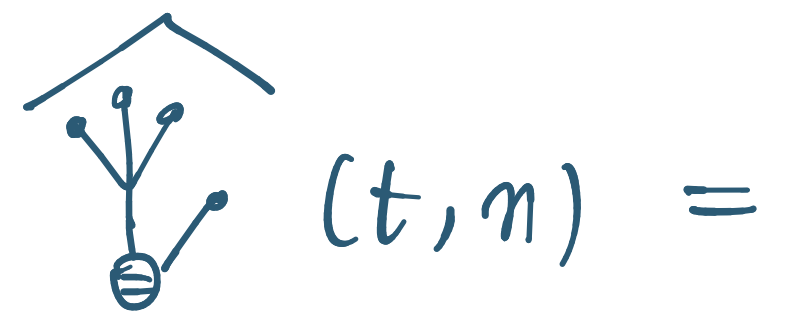
$$\Rightarrow \frac{1}{q} = \frac{3}{2} - \sigma$$

$$\Rightarrow \Psi(t) \in C_x^{\frac{1}{2}-\varepsilon}(\mathbb{T}^3).$$

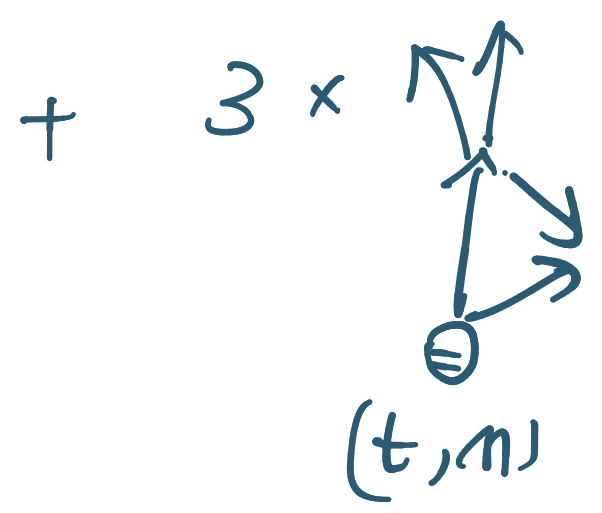
For time difference, we have terms like

$$\begin{aligned} & \hat{P}_{t+h-t_1}(n) - \hat{P}_{t-t_1}(n) \\ &= \underbrace{(e^{-h\langle n^2 \rangle} - 1)}_{\substack{\sim \\ \text{MRT}} \quad h^\sigma \langle n^2 \rangle^\sigma} \hat{P}_{t-t_1}(n) \end{aligned}$$

(P.125)
See the last part of Justin's note.

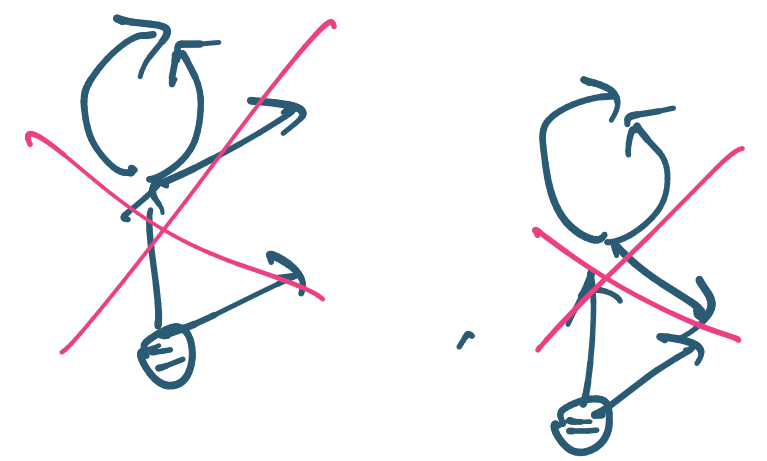


$\uparrow$
 $\mathcal{H}_4$

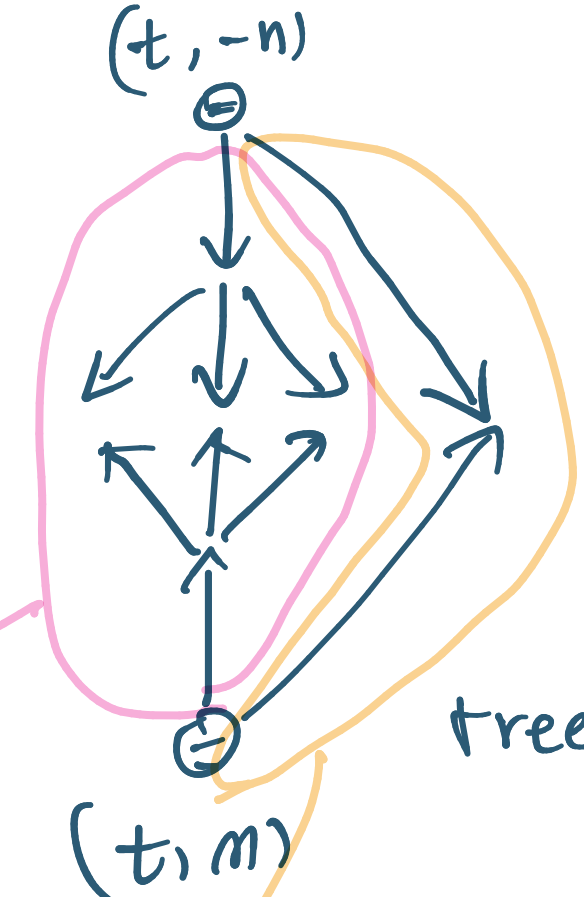


$\uparrow$
 $\mathcal{H}_2$

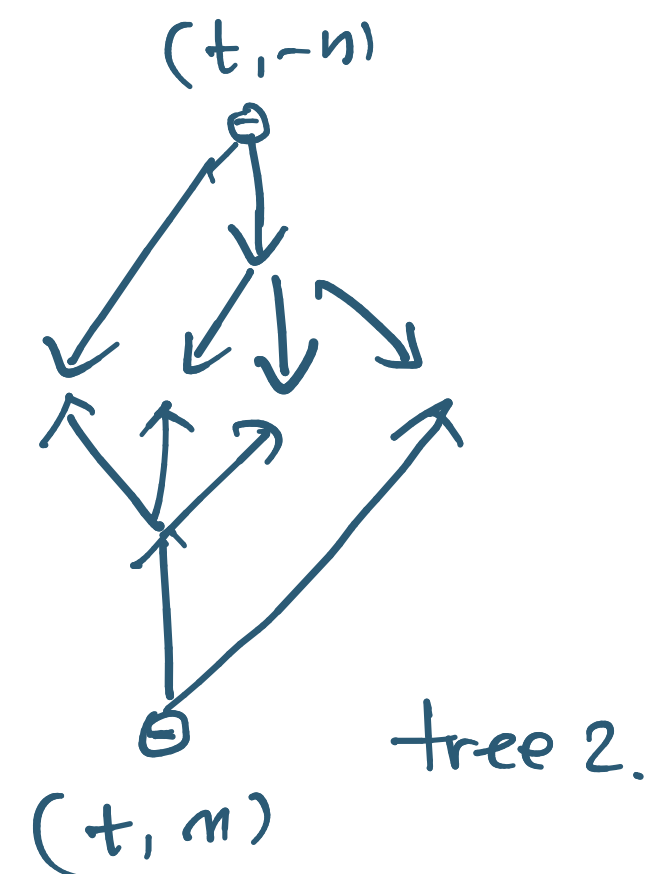
$=: I + II$



$\Leftarrow$ removed by the Wick renormalization
 $q^3 \rightarrow$

$$\mathbb{E}[|I(t, m)|^2] = 6 \times \text{tree 1} + 3 \cdot 3! \times \text{tree 2}$$


tree 1



tree 2 $\leq$ tree 1 by Jensen's inequality
(in this case, by C-S)
see Lec 9 in Justin's note.

$$\sum_{n=n_1+n_2} \frac{1}{\langle n_1 \rangle^4} \frac{1}{\langle n_2 \rangle^2} \leq \langle n \rangle^{-3} = \langle n \rangle^{-3-2 \cdot 0}.$$

$$\underline{\underline{|n_1| \sim |n_2|}}$$

drop the condition $\alpha, \beta < d$

$$\Rightarrow \text{tree}(t) \in C_x^\alpha(\pi^3)$$

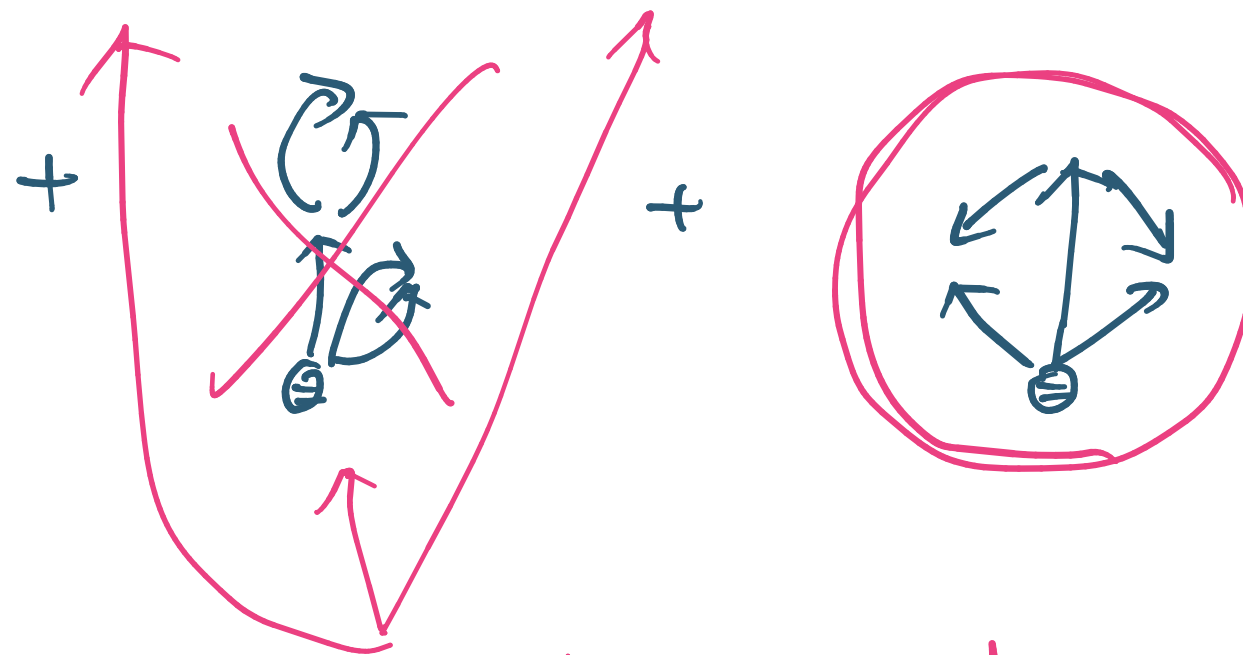
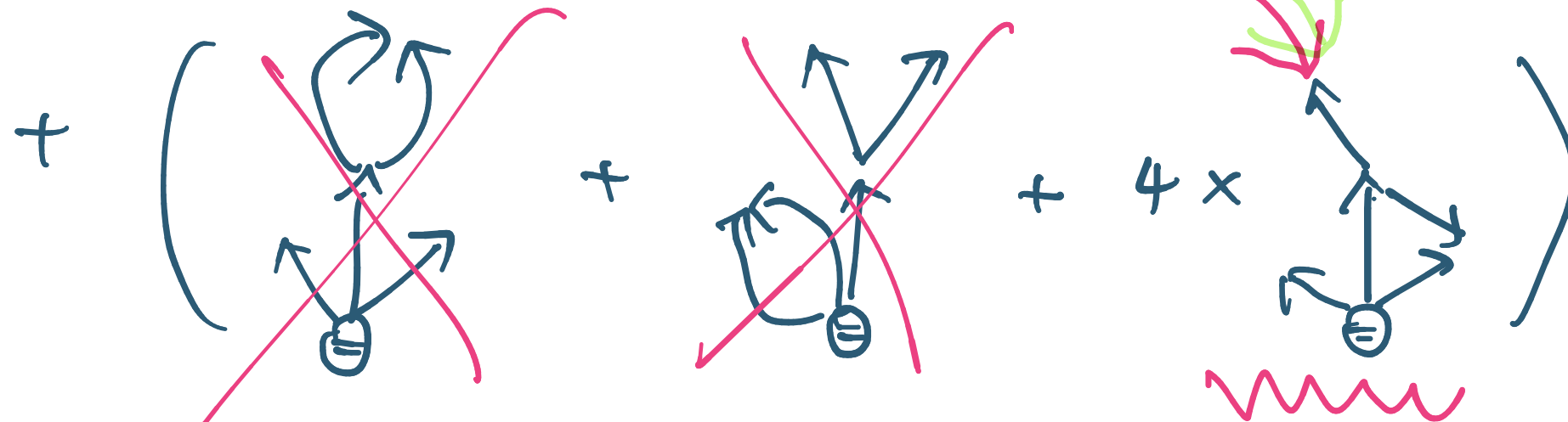
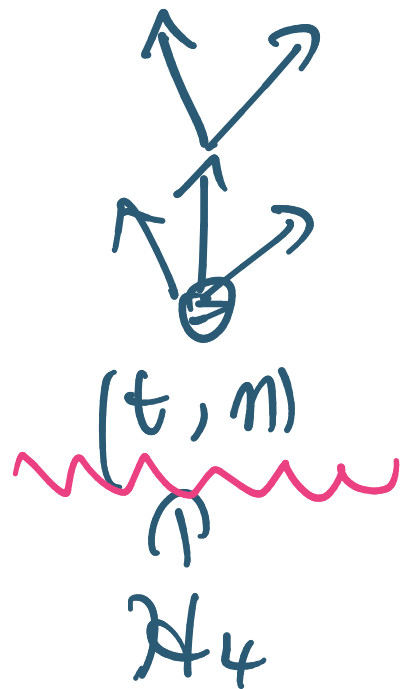
Let's consider

$$I(p_N^2) p_N^2 \quad (\text{without the Wick renormalization})$$

$$\cap \\ N \leq 4$$

$$\in \mathcal{H}_2$$

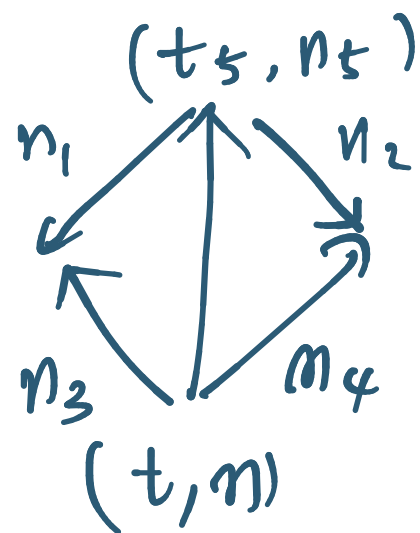
F.T $\ln x$



logarithmically divergent.
must be removed
 $\Leftarrow$ 2nd renormalization

removed by the Wick renormalization.

(diverges $\sim N$)



$$= \sum_{n_1, \dots, n_5} \int \hat{P}_{t-t_5}(n_5) \left(\int \hat{P}_{t_5-t_1}(n_1) \hat{P}_{t-t_1}(-n_1) dt_1 \right)$$

$$n_1 + n_3 = 0$$

$$n_2 + n_4 = 0$$

$$n_5 = n_1 + n_2$$

$$n = n_3 + n_4 + n_5$$

$$\Rightarrow n = 0$$

$$\times \left(\int \hat{P}_{t_5-t_2}(n_2) \hat{P}_{t-t_2}(-n_2) dt_2 \right) dt_5$$

$$= \mathbb{1}_{\{n=0\}} \sum_{\substack{n_1, n_2, n_5 \\ n_5 = n_1 + n_2}}$$

$$\int \hat{P}_{t-t_5}(n_5) \frac{e^{-|t-t_5| \langle n_1 \rangle^2}}{2 \langle n_1 \rangle^2} \cdot \frac{e^{-|t-t_5| \langle n_2 \rangle^2}}{2 \langle n_2 \rangle^2} dt_5$$

2 summations

$$\lesssim \frac{1}{\langle n_1 \rangle^2} \frac{1}{\langle n_2 \rangle^2} \frac{1}{\langle n_1 \rangle^2 + \langle n_2 \rangle^2 + \langle n_5 \rangle^2}$$

deg 6.

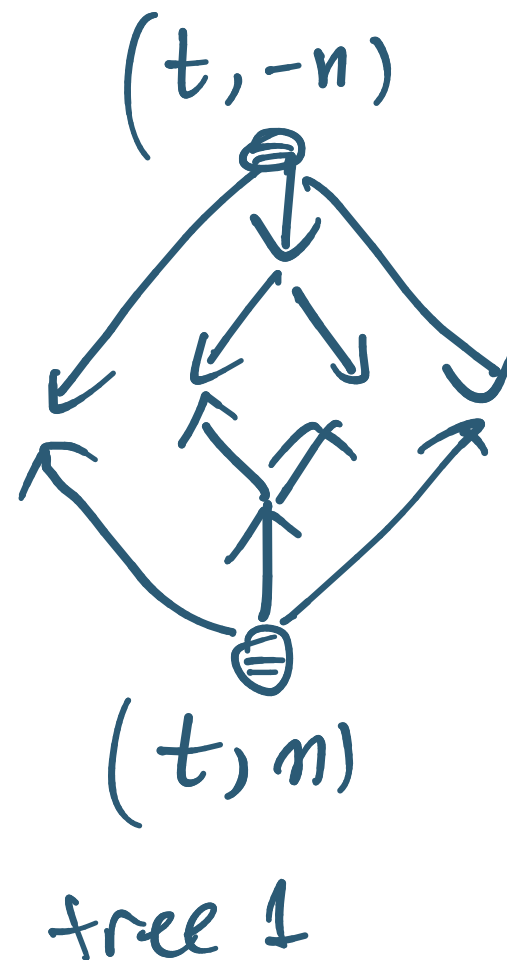
$\Rightarrow$ log divergent!!

• We only consider the $\mathcal{H}_4$ -contribution.

(18)

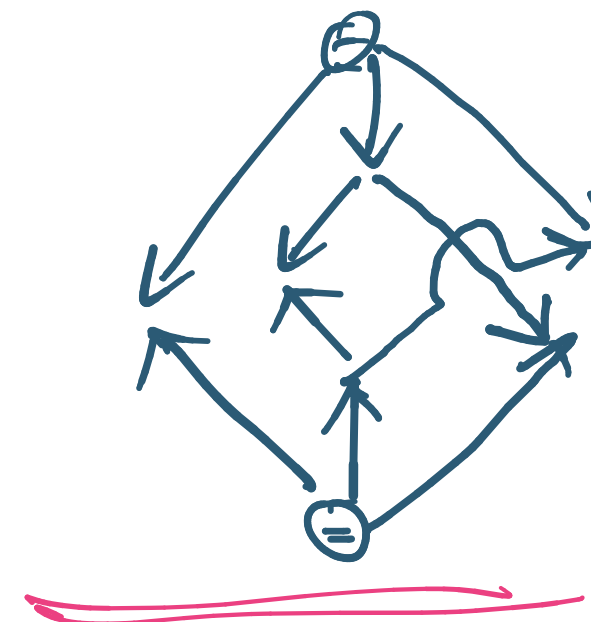
$$\mathbb{E}[|I(t, n)|^2]$$

$$= 2 \times 2$$

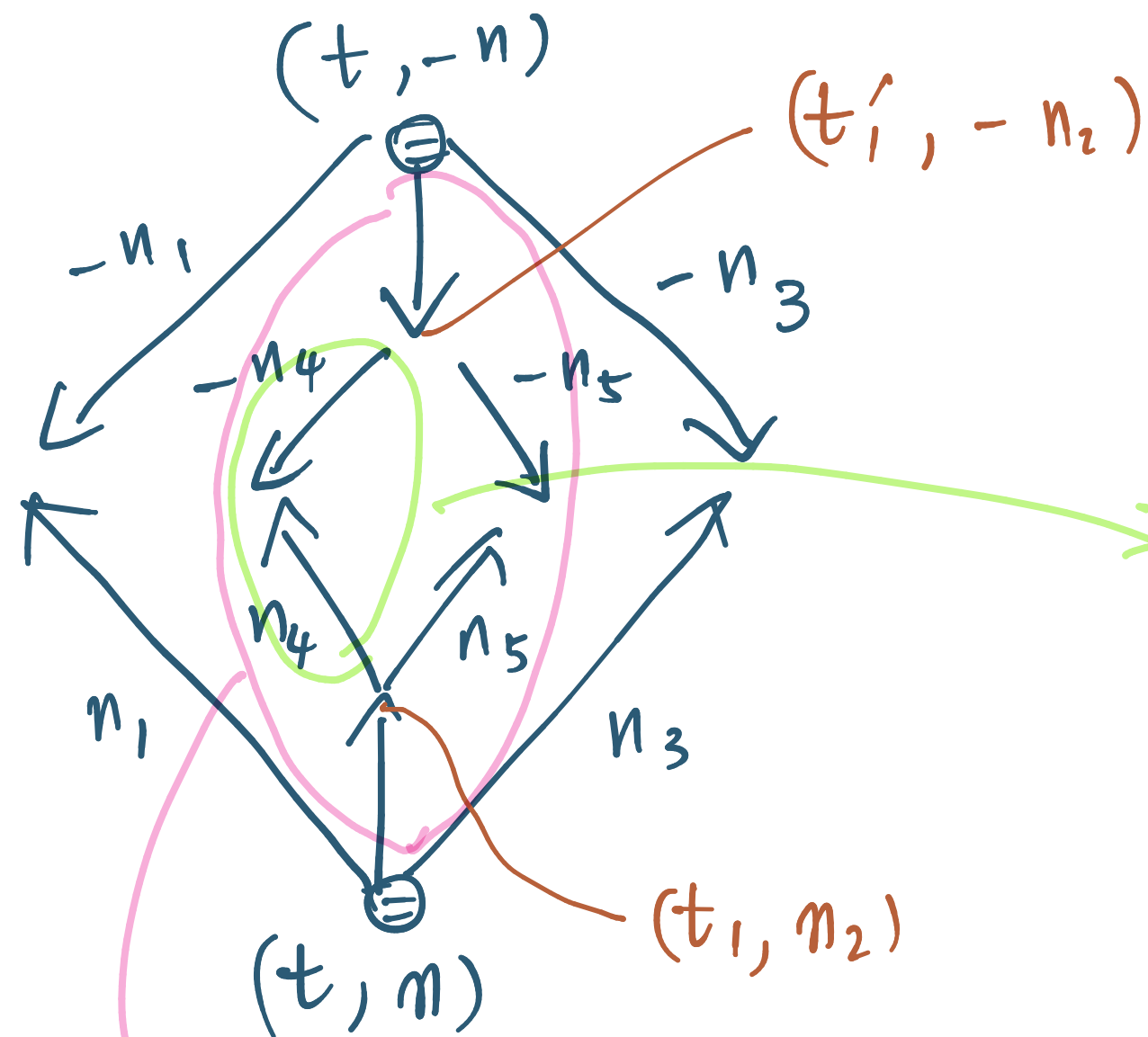


$$I(t, n) = \begin{array}{c} \nearrow \nwarrow \\ \nwarrow \nearrow \\ \ominus \\ (t, n) \end{array} \in \mathcal{H}_4$$

+ terms like



dominated by tree 1
via Jensen's ineq.



$$M = n_1 + n_2 + n_3$$

$$n_2 = n_4 + n_5$$

$$|n_1 + n_3| \sim |n_2|$$

$$\mathbb{E} [\hat{\varphi}(t_1, n_4) \hat{\varphi}(t'_1, -n_4)]$$

$$= \frac{e^{-|t_1 - t'_1| \langle M_4 \rangle^2}}{2 \langle M_4 \rangle^2}$$

$$\int \hat{P}_{t-t_1}(n_2) \hat{P}_{t-t'_1}(-n_2) \sum_{n_2 = n_4 + n_5} \frac{1}{|t_1 - t'_1|^\sigma} \frac{1}{\langle n_4 \rangle^{2+2\sigma}} \times \frac{1}{\langle n_5 \rangle^2} dt'_1 dt_1$$

$$\lesssim \frac{1}{\langle n_2 \rangle^{1+2\sigma}} \|\hat{P}_{t-t_1}(n_2)\|_{L_{t_1}^2}$$

$$\times \left\| \left(\frac{1}{|\cdot|^\sigma} \times \hat{P}_{t-\cdot}(n_2) \right)(t_1) \right\|_{L_{t_1}^2}$$

$$\lesssim \frac{1}{\langle n_2 \rangle^{1+2\sigma}}$$

Apply H-L-S

$$\frac{1}{2} + 1 = \frac{1}{1-\sigma} + \frac{1}{q}$$

$$\Rightarrow \frac{2}{q} = 3 - 2\sigma$$

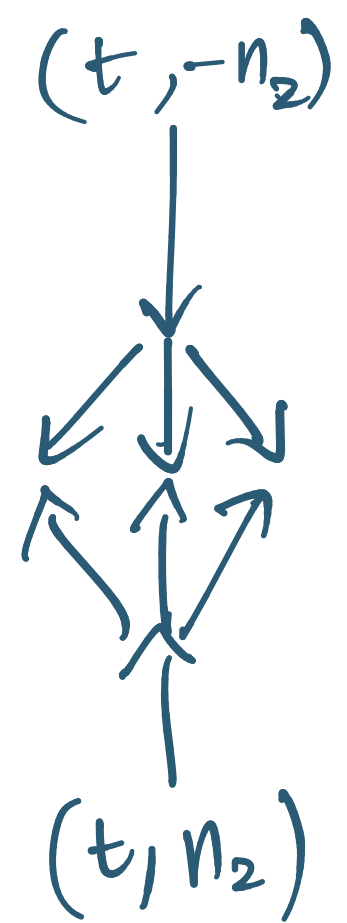
$$\approx \frac{1}{\langle n_2 \rangle^{1+2r}} \frac{1}{\langle n_2 \rangle} \frac{1}{\langle n_2 \rangle^{2/9}} = 3 - 2r$$

$$= \frac{1}{\langle n_2 \rangle^5}$$

$$\Rightarrow \mathbb{E}[|I(t, n)|^2] \lesssim \sum_{\substack{n = n_1 + n_2 + n_3 \\ |n_1 + n_3| \sim |n_2|}} \frac{1}{\langle n_2 \rangle^5} \frac{1}{\langle n_1 \rangle^2} \frac{1}{\langle n_3 \rangle^2}$$

$$\lesssim \sum_{\substack{n_2 \\ |n - n_2| \sim |n_2|}} \frac{1}{\langle n_2 \rangle^5} \frac{1}{\langle n - n_2 \rangle}$$

$$\sim \langle n \rangle^{-3} = \langle n \rangle^{-3-2 \cdot 0} \Rightarrow \underline{S < 0}$$



$$\approx \frac{1}{\langle n_2 \rangle^4} \quad \text{as in } \Psi$$

$$\Rightarrow \mathbb{E} [| \dots (t, m) |^2] \approx \sum_{n = n_1 + n_2 + n_3} \frac{1}{\langle n_1 \rangle^2} \frac{1}{\langle n_3 \rangle^2} \frac{1}{\langle n_2 \rangle^4}$$

$$|n_1 + n_3| \sim |n_2|$$

$$\approx \sum_{n_2} \frac{1}{\langle n - n_2 \rangle} \frac{1}{\langle n_2 \rangle^4}$$

$$|n - n_2| \sim |n_2|$$

$$\approx \langle n \rangle^{-2} = \langle n \rangle^{-3 - 2(-1/2)}$$

$$\Rightarrow \boxed{S < -1/2}$$

Paracontrolled distribution:

We say f is paracontrolled by g if we have

$$f = \underline{f'} \odot \underline{g} + R$$

$f' \sim$ Grubinelli derivative

for some f' and smoother R .

$$\left(\begin{array}{l} \text{GIP '15: } f \in C^\alpha, g \in C^\alpha, f' \in C^\alpha \\ R \in C^\beta, \beta > \alpha. \end{array} \right.$$

Parabolic Φ_3^4 -model: $X = (\dots) \odot \underline{Y} + \text{com}, (X, Y)$

$$(\Leftarrow (\partial_t + 1 - A) X = (\dots) \odot v$$

$$\Rightarrow \underline{v = X + Y = (\dots) \odot Y + R} \leftarrow \text{smoother}$$

· Controlled rough path (Grubinelli JFA '04)

(2)

We say that Y is controlled by X if

$$\underline{Y_{s,t} = Y'_s X_{s,t} + R_{s,t}} \quad Y_{s,t} = Y_t - Y_s$$

for some Y' and smoother R

$\nwarrow$ Grubinelli derivative

ex. $Y, X, Y' \in C_t^\alpha, R \in C^{2\alpha}.$

Moral: local fluctuation of $Y \approx \cdot$ of X .

(f paracontrolled by g
 $\Rightarrow$ small scale behavior of $f \approx \cdot$ of g .

3.2 quadratic SNLW on $\mathbb{T}^3$

3

$$(SNLW) \quad \begin{cases} (\partial_t^2 + 1 - \Delta) u + u^2 = \xi & \text{on } \mathbb{R}_+ \times \mathbb{T}^3 \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases}$$

$d=3$: ξ has spatial regularity $-\frac{3}{2} - \varepsilon$

$$\Rightarrow \rho = \mathcal{I}(\xi) \sim -\frac{1}{2} - \varepsilon.$$

• parabolic case:

$$(SNLH) \quad (\partial_t + 1 - \Delta) u + u^2 = \xi$$

$$\Rightarrow \rho = \mathcal{I}_{\text{heat}}(\xi) \sim -\frac{1}{2} - \varepsilon.$$

Da Prato - De Bussche : $u = \varphi + v$.

(4)

$$(SNLH) \Rightarrow (\partial_t + 1 - \Delta) v = - (v + \varphi)^2$$

$$'' = '' - v^2 - \underline{2v\varphi} - \underset{-1-}{v}$$

$\uparrow$
Wick renormalization

$$\Rightarrow v \sim 1- = (-1-) + 2$$

$\Rightarrow v \varphi \leftarrow$ The product is well defined.

$$(1-) + (-\frac{1}{2}-) > 0$$

$\Rightarrow$ can perform a contraction argument in $C_T C_x^{1-}$

(5)

Back to the wave case:

$$\varphi = I(\xi) \sim -\frac{1}{2} -$$

$$\varphi = \lim_{N \rightarrow \infty} (\underbrace{\varphi_N^2}_{= V_N} - \sigma_N)$$

$$\varphi_N = P_{\leq N} \varphi$$

$$\sigma_N = \mathbb{E}[\varphi^2(t, x)] \sim tN$$

$$\Rightarrow V_N \rightarrow \varphi \text{ in } C_T W_x^{-1-, \infty}, \text{ a.s.}$$

2nd order object: $\underline{\gamma} = I(\varphi) = \int_0^t \frac{\sin((t-t')\langle \nabla \rangle)}{\langle \nabla \rangle} \varphi(t') dL'$

A naive "parabolic thinking" gives

$$0- = (-\frac{1}{2}-) + (-\frac{1}{2}-) + 1 \quad \Leftarrow \text{Not good.}$$

(6)

• Multilinear dispersive smoothing gives
extra $\frac{1}{2}$ -smoothing.

Prop: $Y_N = I(V_N) \rightarrow Y$

in $C_T W_x^{\frac{1}{2}-\varepsilon, \infty} \cap C_T' W_x^{-1-\varepsilon, \infty}$, a.s.

• Second order expansion: $u = 1 - Y + v$

$$(SNLW) \Rightarrow (\partial_t^2 + 1 - \Delta) v = - (v + 1 - Y)^2 + v$$

$$= - (v - Y)^2 - \underbrace{2v}_{{-\frac{1}{2}-}} + \underbrace{2Y}_{{-\frac{1}{2}-}}$$

$\Rightarrow$ expect $v \sim \frac{1}{2}-$

$$\Rightarrow v \text{ : } (\frac{1}{2}-) + (-\frac{1}{2}-) < 0$$

$\nwarrow$ NOT well defined.

Paracontrolled ansatz : $\mathcal{V} = \underset{S_1}{X} + \underset{S_2}{Y}$

$0 < S_1 < S_2.$

(7)

$$(\partial_t^2 + 1 - \Delta) X = -2(X + Y - Y') \underline{\otimes} 1$$

$$(\partial_t^2 + 1 - \Delta) Y = - (X + Y - Y')^2 - 2(X + Y - Y') \underline{\otimes} 1$$

$$\begin{array}{c} \uparrow \\ \otimes + \otimes \end{array}$$

$$\left(\begin{array}{l} \square \mathcal{V} = (\dots) \underline{\otimes} 1 + R \\ \uparrow \text{ paracontrolled.} \end{array} \right.$$

Expect : $\underline{X \sim \frac{1}{2}-} = (-\frac{1}{2}-) + 1$

For now, ignore $\otimes$ in the Y -eqn.

$$(X + Y - Y') \otimes 1 \sim (\frac{1}{2}-) + (-\frac{1}{2}-) \sim 0-$$

$$\Rightarrow \underline{Y \sim 1-}$$

⑧

- Need to make sense of the resonant product

$$(X + Y - Y') \in \mathcal{I}$$

① $Y \in \mathcal{I}$

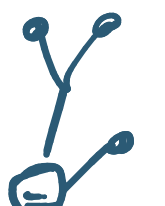
$$(1-) + (-\frac{1}{2}-) = \frac{1}{2}- > 0$$

good as long as $s_2 > \frac{1}{2}$

② $\begin{array}{c} \bullet \\ \diagup \\ \ominus \\ \diagdown \\ \bullet \end{array} = \underline{Y \in \mathcal{I}}$

$$0- = (\frac{1}{2}-) + (-\frac{1}{2}-)$$

- Once again, we need to exploit multilinear dispersive smoothing.
- without renormalization
 - $\Leftarrow$ there was NO extra renormalization in the parabolic case

Prop:  = $Y_N \ominus I_N \rightarrow$  in $C_T W_{21}^{-\varepsilon, \infty}$, a.s. ⑨

③ $X \ominus 1$

$$\left(\frac{1}{2}-\right) + \left(-\frac{1}{2}-\right) < 0$$

$\Rightarrow$ NOT well defined.

· Use the Duhamel formulation for X :

$$X(t) = \underbrace{\partial_t S(t) X_0 + S(t) X_1}_{=:\vec{S}(t)(X_0, X_1)} - 2 \underline{\underline{I((X + Y - Y) \ominus 1)}}$$

$$=:\vec{S}(t)(X_0, X_1)$$

NOT smoother!!

$$S(t) = \frac{\chi_m(t \langle \nabla \rangle)}{\langle \nabla \rangle}$$

Lemma: $s_1 > 0$. $(X_0, X_1) \in \mathcal{X}^{s_1}(\mathbb{T}^3)$

(10)

$$Z_N = (\vec{S}(t)(X_0, X_1)) \ominus \mathbb{I}_N$$

↓

$$Z = (\vec{S}(t)(X_0, X_1)) \ominus \mathbb{I} \text{ in } \underline{C_T H_x^{s_1 - \frac{1}{2} - \varepsilon}}, \underline{\text{a.s.}}$$

depends on (X_0, X_1)

↑ need to include this in the enhanced data set.

Note: set of prob 1 depends on the initial data (X_0, X_1) .

-
- Duhamel $\ominus \mathbb{I}$: In the parabolic case, we introduced two commutators.

$$\text{com}_1 = [\text{Duhamel integral}, \trianglelefteq], \text{com}_2 = [\trianglelefteq, \ominus]$$

Bad news: In the wave case,

(11)

$\left[\int_0^+ S(t-t') \dots dt', \textcircled{<} \right]$ does NOT provide any smoothing

Main idea: directly study the following paracontrolled operators

Given $w \in C(\mathbb{R}_+; H^s(\pi^3))$, $0 < s_1 < 1/2$,

define

$$\mathcal{I} \textcircled{<} (w)(t) = \mathcal{I}(w \textcircled{<} 1)(t)$$

$$= \sum_{n \in \mathbb{Z}^3} e_n \sum_{n=n_1+n_2} \int_0^t \frac{\sin(t-t') \langle n \rangle}{\langle n \rangle} \hat{w}(t', n_1) \hat{1}(t', n_2) dt'$$

$$\boxed{|n_1| \ll |n_2|}$$

↑
signifies the para product (formal)

Goal: Make sense of $f \otimes (w) \in I$

(12)

(with $w = X + Y - Y^\bullet$.)

Divide $f \otimes$ into good and bad parts

Fix $\theta > 0$ small.

$$f \otimes = f^{(1)} \otimes + f^{(2)} \otimes$$

$\nearrow$ restriction of $f \otimes$ onto $\{|n_1| \geq |n_2|^\theta\}$
" $|n_1|$ is NOT too small."

want $\sim \frac{1}{2} + 2\varepsilon$.

$$\langle n \rangle^{\frac{1}{2} + 2\varepsilon} \frac{1}{\langle n \rangle} = \langle n \rangle^{-\frac{1}{2} + 2\varepsilon} \lesssim \langle n_1 \rangle^{\frac{4\varepsilon}{\theta}} \underbrace{\langle n_2 \rangle^{-\frac{1}{2} - 2\varepsilon}}_{\substack{\text{put on } \uparrow(t', n_2)}}$$
$$\lesssim \langle n_1 \rangle^{s_1 - \varepsilon} \langle n_2 \rangle^{-\frac{1}{2} - 2\varepsilon}$$

by choosing $\varepsilon = \varepsilon(s_1, \theta) > 0$ suff. small.

Lemma : $0 < s_1 < 1/2$ Given small $\theta > 0$,

$\exists$ small $\varepsilon = \varepsilon(s_1, \theta) > 0$ s.t.

given any $\Xi \in C(\mathbb{R}_+, W^{-\frac{1}{2}-\varepsilon, \infty}(\mathbb{T}^3))$,
the paracontrolled operator

$$f_{\odot}^{(1), \Xi} = f_{\odot}^{(1)} \text{ with } \eta \text{ replaced by } \Xi$$

belongs to $\mathcal{L}_2 = \mathcal{L}(C_T H_x^{s_1}; C_T H_x^{\frac{1}{2}+2\varepsilon})$.

(i.e. No stochastic analysis is needed)

• As for $f_{\odot}^{(2)}$, i.e. $|m_1| \ll |m_2|^\theta$, the positive regularity of w does not help.

⇒ We use stochastic analysis to directly study $J_{\odot, \ominus}$

(14)

$$J_{\odot, \ominus}(w)(t) = J_{\odot}^{(2)}(w)(t) \ominus \mathfrak{I}(t)$$

$$= \mathcal{I}(w \odot \mathfrak{I})(t) \ominus \mathfrak{I}(t).$$

$$= \sum_{n \in \mathbb{Z}^3} e_n \int_0^t \sum_{n_1} \widehat{w}(t, n_1) \underline{A_{n, n_1}(t, t')} dt'.$$

$$A_{n, n_1}(t, t') = \mathbb{1}_{[0, t]}^{(t')} \sum_{\substack{n - n_1 = n_2 + n_3 \\ |n_1| \ll |n_2|^\theta}} \frac{\sin((t - t') \langle n_1 + n_2 \rangle)}{\langle n_1 + n_2 \rangle} \underbrace{\widehat{\mathfrak{I}}(t', n_2)}_{=} \underbrace{\widehat{\mathfrak{I}}(t, n_3)}_{=}$$

$$|n_1 + n_2| \sim |n_3| \leftarrow \text{signifies } \ominus$$

$$= \mathbb{1}_{[0,t]}(t') \sum \frac{\sin((t-t')\langle n_1+n_2 \rangle)}{\langle n_1+n_2 \rangle} \left(\hat{\rho}_{(t',n_2)} \hat{\rho}_{(t,n_3)} - \mathbb{1}_{n_2+n_3=0} \sigma_{n_2}(t,t') \right)$$

$$+ \mathbb{1}_{[0,t]}(t) \cdot \mathbb{1}_{n=n_1} \sum_{\substack{n_2 \\ |n| \ll |n_2|^\theta}} \frac{\sin((t-t')\langle n+n_2 \rangle)}{\langle n+n_2 \rangle} \sigma_{n_2}(t,t')$$

$$=: A_{n,n_1}^{(1)}(t,t') + A_n^{(2)}(t,t')$$

$\uparrow$ NOT abs conv. $\uparrow$ $\frac{1}{\langle n+n_2 \rangle} \frac{1}{\langle n_2 \rangle^2}$
 $\nwarrow$ deterministic counter term

$$0 \leq t_2 \leq t_1$$

$$\sigma_n(t_1, t_2) = \mathbb{E}[\hat{\rho}(t_1, n) \hat{\rho}(t_2, n)]$$

$$= \frac{\cos((t_1 - t_2)\langle n \rangle)}{2 \langle n \rangle^2} t_2 + O\left(\frac{1}{\langle n \rangle^3}\right)$$

(16)

exploit dispersion, stationary phase,
symmetrization ($n_2 \leftrightarrow -n_2$).

i.e. the order of summations matter.
 $\Rightarrow$ only conditionally convergent.

Prop: $S_2 < 1$. $\exists$ small $\theta(S_2) > 0$ and $\varepsilon > 0$ s.t.

$$f(\otimes, \otimes) \in L_1 = L(C_T L_x^2 \cap \underline{C_T' H_x^{-1-\varepsilon}}; C_T H_x^{S_2-1}) \text{ a.s.}$$

$f_{\otimes, \otimes}^N$ (with $\mathbf{I}$ replaced by $\mathbf{I}_N$) $\rightarrow f(\otimes, \otimes)$ in L_1 , a.s.

GKD, JEMS

can drop this assumption

Oh-Okamoto-Tolomeo '20.

We arrive at the following system:

$$(\partial_t^2 + 1 - \Delta) X = -2(X + Y - Y') \otimes 1$$

$$(\partial_t^2 + 1 - \Delta) Y = - (X + Y - Y')^2 - 2(X + Y - Y') \otimes 1$$

(SNLW')

$$- 2Y \otimes 1 + 2 \underline{Y'} - 2 \underline{Z}$$

$$+ 4 f^{(1)} \otimes (X + Y - Y') \otimes 1$$

$$+ 4 \underline{f \otimes \otimes} (X + Y - Y')$$

$$(X, \partial_t X, Y, \partial_t Y) /_{t=0} = (X_0, X_1, Y_0, Y_1)$$

Thm: $\frac{1}{4} < s_1 < \frac{1}{2} < s_2 \leq s_1 + \frac{1}{4}$.

Then, $(SNLW')$ is locally well-posed in $\mathcal{H}^{s_1} \times \mathcal{H}^{s_2}$.

$$(X, Y) \in \mathbb{Z}_T^{s_1, s_2} = X_T^{s_1} \times Y_T^{s_2}$$

$$X_T^{s_1} = C_T H_x^{s_1} \cap C_T' H_x^{s_1-1} \cap L_T^{\frac{\delta}{2}} W_x^{s_1-\frac{1}{4}, \frac{\delta}{3}} \quad (\frac{\delta}{2}, \frac{\delta}{3}), \frac{1}{4}\text{-admiss.}$$

$$Y_T^{s_2} = C_T H_x^{s_2} \cap C_T' H_x^{s_2-1} \cap L_T^4 W_x^{s_2-\frac{1}{2}, 4} \quad (4, 4), \frac{1}{2}\text{-admiss.}$$

enhanced data set

$$\mathbb{E} = (X_0, X_1, Y_0, Y_1, \overset{\vec{S}(X_0, X_1) \ominus \tau}{\underset{''}{\tau}}, \underline{Y}, \underline{Y}, \underline{Z}, f \ominus, \ominus)$$

$$\in \mathcal{H}^{s_1} \times \mathcal{H}^{s_2} \times C_T W_x^{-\frac{1}{2}-\varepsilon, \infty} \times \left(C_T W_x^{\frac{1}{2}-\varepsilon} \cap C_T' W_x^{-1-\varepsilon, \infty} \right)$$

$$\times C_T W_x^{-\varepsilon, \infty} \times C_T H_x^{s_1-\frac{1}{2}-\varepsilon} \times L_1 =: X_T^{s_1, s_2, \varepsilon}$$

• soln map is continuous from $X_T^{S_1, S_2, \varepsilon} \rightarrow Z_T^{S_1, S_2}$

$$\square \longrightarrow (X, Y)$$

• Proof is done by the Strichartz esti & a contraction argument.

$$u_N = I_N - \dot{Y}_N + \underbrace{X_N + Y_N}$$

$$\downarrow$$

$$\downarrow$$

$$\downarrow$$

$$u = I - \dot{Y} + X + Y \quad \text{in } C_T H_x^{-\frac{1}{2} - \varepsilon}$$

• Remarks: • quadratic nonlin $\Leftarrow$ neither defocusing or focusing. (20)

but the Gibbs measure

$$d\rho = Z^{-1} e^{-\sigma \int :u^3: dx - A \left(\int :u^2: dx \right)^\gamma} du$$

can be constructed in the weakly nonlinear regime ($0 < \sigma \ll 1$)

(When $\sigma \gg 1$, ρ is non-normalizable.)

$\Rightarrow$ a.s GWP. (O-Ok-Tol '21)

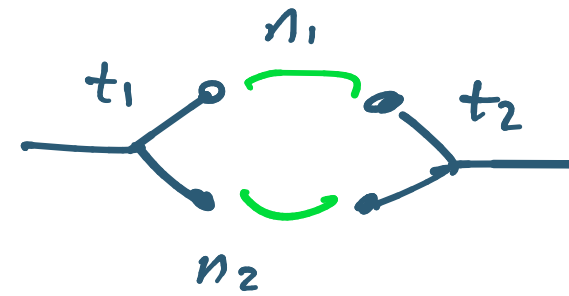
• The real question is the hyperbolic Φ_3^V -model.

Bringmann '20: Hartree nonlinearity $(V * u^2) u$

$$\widehat{V}(n) \sim \langle n \rangle^{-\varepsilon}, \quad \forall \varepsilon > 0.$$

Key point on $\dot{Y}$: By the regularity lemma, we study (2)

$$\mathbb{E}[|\hat{Y}(t, n)|^2]$$



$$= 4 \sum_{\substack{n=n_1+n_2 \\ n_1 \neq \pm n_2}} \int_0^t \frac{\sin(t-t_1)\langle n \rangle}{\langle n \rangle} \int_0^{t_1} \frac{\sin(t-t_2)\langle n \rangle}{\langle n \rangle}$$

$$\times \sigma_{n_1}(t_1, t_2) \sigma_{n_2}(t_1, t_2) dt_2 dt_1$$

$$+ \dots_{n_1=n_2}$$

$$\frac{\cos(t_1-t_2)\langle n \rangle}{2\langle n \rangle^2} t_2 + O\left(\frac{1}{\langle n \rangle^3}\right)$$

$$0 \leq t_2 \leq t_1$$

$\Leftarrow$ expand sines and cosines
in complex exponentials

$$\Rightarrow \sum_{\substack{n=n_1+n_2 \\ n_1 \neq \pm n_2}} \frac{e^{i(\varepsilon_1 + \varepsilon_2)t \langle n \rangle}}{\langle n \rangle^2 \langle n_1 \rangle^2 \langle n_2 \rangle^2} \int_0^t e^{-it_1 K_1(\bar{n})} \int_0^{t_1} t_2^2 e^{-it_2 K_2(\bar{n})} dt_2 dt_1$$

$$K_1(\bar{n}) = \varepsilon_1 \langle n \rangle - \varepsilon_3 \langle n_1 \rangle - \varepsilon_4 \langle n_2 \rangle$$

$$K_2(\bar{n}) = \varepsilon_2 \langle n \rangle + \varepsilon_3 \langle n_1 \rangle + \varepsilon_4 \langle n_2 \rangle$$

$$\varepsilon_j \in \{\pm 1\}.$$

Integrate in t_1 first.

$$\left| \int_0^t t_2^2 e^{-it_2 K_2(\bar{n})} \frac{e^{-it K_1(\bar{n})} - e^{-it_2 K_1(\bar{n})}}{-i K_1(\bar{n})} dt \right|$$

$$\lesssim \frac{C(T)}{\underline{1 + |K_1(\bar{n})|}}$$

Need to bound

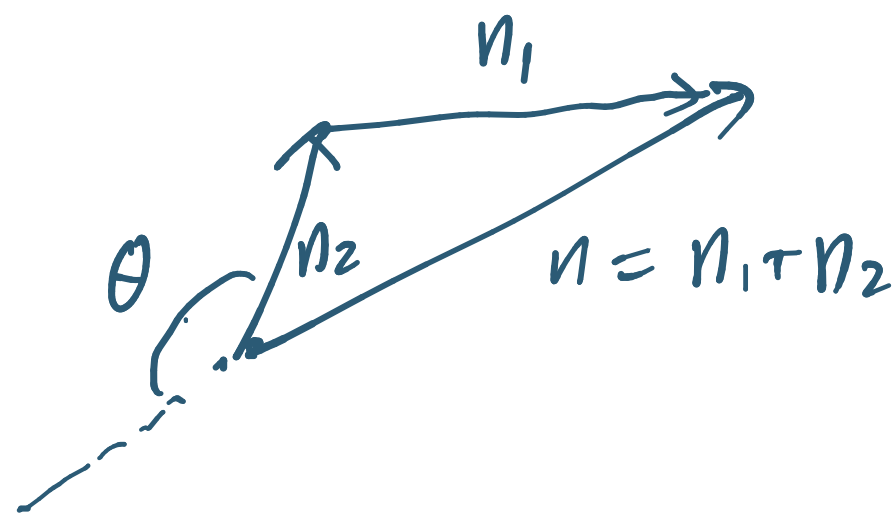
(23)

$$I = \sum_{n=n_1+n_2} \frac{1}{\langle n \rangle^2 \langle n_1 \rangle^2 \langle n_2 \rangle^2} \frac{1}{1 + |K_1(\vec{n})|} \left(\stackrel{\text{WTS}}{\sim} \langle n \rangle^{-4+} \right) \Rightarrow s < 1/2$$

Bad case $(\varepsilon_1, \varepsilon_3, \varepsilon_4) = (\pm 1, \pm 1, \mp 1)$ $\left(\begin{array}{l} \text{Assume} \\ |n| \gg 1 \\ |n_1| \geq |n_2| \end{array} \right)$

$$|K_1(\vec{n})| = \langle n \rangle + \langle n_2 \rangle - \langle n_1 \rangle$$

$$\Rightarrow \langle n_1 \rangle \sim \langle n \rangle + \langle n_2 \rangle$$



law of cosines

$$|n|^2 + |n_2|^2 - |n_1|^2 = 2|n||n_2|\cos(\angle(n, n_2))$$

$$\Rightarrow |K_1(\vec{n})| = \frac{(\langle n \rangle + \langle n_2 \rangle)^2 - \langle n_1 \rangle^2}{\langle n \rangle + \langle n_1 \rangle + \langle n_2 \rangle} = \frac{2\langle n \rangle \langle n_2 \rangle + \boxed{|n|^2 + |n_2|^2 - |n_1|^2} + 1}{\langle n \rangle + \langle n_1 \rangle + \langle n_2 \rangle}$$

$$\Rightarrow |K_1(\bar{n})| \gtrsim \frac{|n_1| |n_2| (1 - \cos \theta)}{\langle n_1 \rangle}$$

$$\theta = \angle(n_2, -n)$$

Case 1: $1 - \cos \theta \gtrsim 1$. (large angle)

$$I \lesssim \sum \frac{1}{\langle n \rangle^3 \langle n_1 \rangle \langle n_2 \rangle^3}$$

$$\langle n_1 \rangle \sim \max(\langle n \rangle, \langle n_2 \rangle)$$

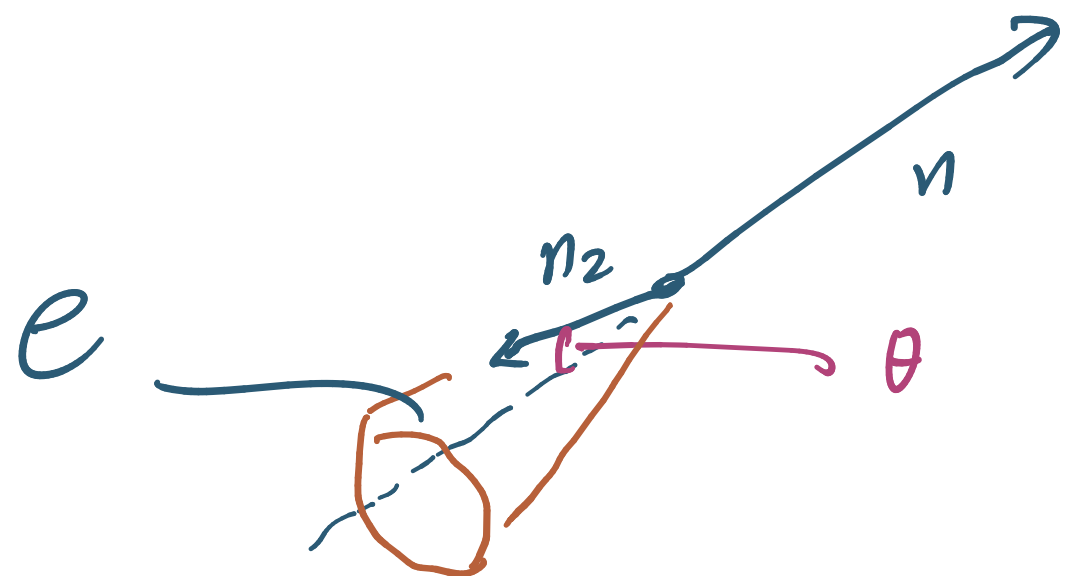
$$\lesssim \langle n \rangle^{-4+}$$

Case 2: $1 - \cos \theta \ll 1$

(nearly) resonant.

$$\Rightarrow 0 \leq \theta \ll 1$$

$$\Rightarrow 1 - \cos \theta \sim \theta^2 \ll 1$$



Dyadically decompose $[N_2) \sim N_2$

$N_2 \geq 1$, dyadic

For fixed $n \in \mathbb{Z}^3$,

$n_2 \in \text{cone } \mathcal{C}$ height $\sim N_2 \cos \theta \sim N_2$

$|n_2| \sim N_2$

radius of the base

$\sim N_2 \sin \theta \sim N_2 \theta$

$$\Rightarrow \text{vol}(\mathcal{C}) \sim N_2^3 \theta^2$$

$$\Rightarrow I \lesssim \sum_{\substack{N_2 \geq 1 \\ \text{dyadic}}} \frac{1}{\langle n \rangle^3 \max(\langle n \rangle, N_2)} \cancel{N_2^3} \theta^2 \times \cancel{N_2^3} \theta^2$$

$\uparrow$
 $\sum_{N_2}$

$\lesssim \langle n \rangle^{-4+}$

Actually →

$$\sum_{n_2} 1 \approx \underline{\underline{1 + \text{vol}(\mathcal{C})}}$$

(26)

So, to be correct, we go back to

$$I = \sum_{n=n_1+n_2} \frac{1}{\langle n \rangle^2 \langle n_1 \rangle^2 \langle n_2 \rangle^2} \frac{1}{\underline{\underline{1 + |K_1(n)|}}} \mathbb{1}_{0 \leq \theta \ll 1}.$$

↑

$$\sum_{\substack{N_2 \geq 1 \\ \text{dyadic}}} \left(\sum_{|n_2| \sim N_2} \right) \approx \underline{\underline{1 + \text{vol}(\mathcal{C})}}$$

$$\sum_{\substack{N_2 \geq 1 \\ \text{dyadic}}} \frac{1}{\langle n \rangle^2 \max(\langle n \rangle^2, N_2^2)} N_2^2$$

$$\approx \langle n \rangle^{-4}.$$

Key point on $\gamma = \gamma \ominus 1$

(27)

New difficult term

$0 < \delta \ll 1$

$$\sum_{n_2 \in \mathbb{Z}^3} \frac{\sin(t-t') (\langle n+n_2 \rangle - \langle n_2 \rangle)}{\langle n \rangle^2 \langle n+n_2 \rangle \langle n_2 \rangle^2} \quad \left(\begin{array}{l} \text{WTS} \\ \sum \langle n \rangle^{-3} \end{array} \right)$$

$$|n+n_2| \sim |n_2|$$

$$|n| \ll |n_2|^\delta$$

can drop $|n+n_2| \sim |n_2|$

$$\odot^+ + \frac{\langle n, n_2 \rangle}{\langle n_2 \rangle}$$

$$\sum_{n_2 \in \mathbb{Z}^3} |n| \ll |n_2|^\delta$$

$\Leftrightarrow$ symmetrize $n_2 \leftrightarrow -n_2$.

$$\text{Let } \odot^\pm(n, n_2) = \underbrace{\langle n \pm n_2 \rangle - \langle n_2 \rangle} \mp \frac{\langle n, n_2 \rangle}{\langle n_2 \rangle} = \mathcal{O}\left(\frac{\langle n \rangle^2}{\langle n_2 \rangle}\right)$$

$$S_{nm} = \sum_{\substack{n_2 \in "Z^{3/2}" \\ |n| \ll |n_2|^\delta}} \frac{1}{\langle n \rangle^2 \langle n+n_2 \rangle \langle n_2 \rangle^2} \\ \times \left[\sin(t-t_1) \left(\frac{\langle n, n_2 \rangle}{\langle n_2 \rangle} + \Theta^+(n, n_2) \right) \right. \\ \left. - \sin(t-t_1) \left(\frac{\langle n, n_2 \rangle}{\langle n_2 \rangle} - \Theta^-(n, n_2) \right) \right]$$

$$\stackrel{MVT}{\lesssim} \sum_{|n| \ll |n_2|^\delta} \frac{1}{\langle n \rangle^2 \langle n_2 \rangle^3} \left(\frac{\langle n \rangle^2}{\langle n_2 \rangle} \right)^\delta \text{ for any } \delta \in [0, 1]$$

$$\lesssim \langle n \rangle^{-3+}$$

- On $\int \Theta, \ominus$
 - symmetrization: $n_2 \leftrightarrow -n_2$
 - integration by parts in time to handle $\sin(t-t') (\langle n+n_2 \rangle + \langle n_2 \rangle)$