REVISITING BOURGAIN'S PROBABILISTIC CONSTRUCTION OF SOLUTIONS TO THE 2-d CUBIC NLS

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Dedicated to Professor Yoshio Tsutsumi on the occasion of his seventieth birthday

Abstract. In a seminal paper (1996), Bourgain proved invariance of the Gibbs measure for the defocusing cubic nonlinear Schrödinger equation on the two-dimensional torus by constructing local-in-time solutions in a probabilistic manner. In this note, we revisit and streamline his argument, using the random tensor estimate developed by Deng, Nahmod, and Yue (2022).

Contents

1. Introduction 2
1.1. Overview 2
1.2. Gibbs measure 2
1.3. Renormalized NLS 4
2. Preliminaries 7
2.1. Notations and basic deterministic estimates 7
2.2. Tools from stochastic analysis 9
2.3. Random tensor estimate 9
2.4. Lattice point counting & tensor norm estimates 15
3. Resonant contribution 18
4. Purely stochastic term 20
5. Random tensor term I 22
6. Random tensor term II 26
7. Random tensor term III 29
8. Random tensor term IV 34
Appendix A. Auxiliary lemma 36
Appendix B. Declarations 37
References 38

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1. Introduction

1.1. **Overview.** In seminal works [5, 6], Bourgain initiated the probabilistic study of dispersive partial differential equations (PDEs) in the context of invariant Gibbs measures. In particular, in [6], he established invariance of the Gibbs measures associated with the following defocusing cubic nonlinear Schrödinger equation (NLS) on the two-dimensional torus $\mathbb{T}^2 = (\mathbb{R}/(2\pi\mathbb{Z}))^2$:

$$i\partial_t u + \Delta u = |u|^2 u, \qquad (x,t) \in \mathbb{T}^2 \times \mathbb{R}. \tag{1.1}$$

The key novelty of this breakthrough work is the probabilistic construction of local-in-time solutions to (1.1) with Gibbsian random initial data of low regularity. Bourgain's work has had a huge impact in the field, leading to the study of probabilistic well-posedness of dispersive PDEs, allowing us to go beyond the limit of deterministic analysis; see, for example, [12, 13, 19, 14, 39, 2, 56]. See also [3, 62] for (slightly outdated) surveys on the subject. In particular, over the recent years, we have witnessed a significant rapid progress in the study lying at the intersection of nonlinear dispersive PDEs and probability theory. See [51, 31, 32, 8, 23, 24, 44, 9, 25, 45, 10] for recent well-posedness studies on singular stochastic dispersive PDEs, broadly interpreted, with rough random initial data and / or stochastic forcing, where various tools and ideas have been introduced. Our main goal in this note is to revisit and streamline Bourgain's probabilistic construction of solutions in [6] by using tools and ideas from this recent development, in particular the random tensor estimate (Lemma 2.6) developed by Deng, Nahmod, and Yue [24].

1.2. **Gibbs measure.** Gibbs measures in statistical physics describe thermal equilibria and have been studied extensively. By drawing an analogy to finite-dimensional Hamiltonian systems, a Gibbs measure, formally given by

$$d\rho = Z^{-1} e^{-H(u)} du, \tag{1.2}$$

is expected to be invariant for a Hamiltonian PDE, where $H(u)$ denotes a Hamiltonian of the equation. The study of invariant Gibbs measures for Hamiltonian PDEs, initiated by Lebowitz, Rose, and Speer [37] and Bourgain [5, 6], consists of two disjoint problems:

(i) Construction of a Gibbs measure as a probability measure on functions / distributions.

(ii) Construction of dynamics on the support of a Gibbs measure.

In this subsection, we briefly discuss some aspects of the Gibbs measure construction for the defocusing cubic NLS (1.1) on $\mathbb{T}^2$ with the Hamiltonian:

$$H(u) = \frac{1}{2}\int_{\mathbb{T}^2} |\nabla u|^2 dx + \frac{1}{4}\int_{\mathbb{T}^2} |u|^4 dx.$$

In this case, the Gibbs measure ρ corresponds to (the complex-valued version of) the so-called Φ^4_2-measure, whose construction goes back to Nelson [41] (see also [21, 50]), and is given by[1]

$$d\rho = Z^{-1} e^{-\frac{1}{4}\int_{\mathbb{T}^2} |u|^4 dx} d\mu \tag{1.3}$$

[1]Namely, instead of ρ in (1.2), we consider a Gibbs measure of the form:

$$d\rho = Z^{-1} e^{-H(u) - \frac{1}{2}M(u)} du,$$

where $M(u) = \int_{\mathbb{T}^2} |u|^2 dx$ is the conserved mass.

(modulo a renormalization on the potential energy). Here, μ is the massive Gaussian free field on $\mathbb{T}^2$, formally given by

$$d\mu = Z^{-1} e^{-\frac{1}{2}\|u\|_{H^1}^2} du. \tag{1.4}$$

More precisely, μ is given as the induced probability measure under the map:[2]

$$\omega \in \Omega \longmapsto u(\omega) = \sum_{n\in\mathbb{Z}^2} \frac{g_n(\omega)}{\langle n\rangle} e_n, \tag{1.5}$$

where $\langle\,\cdot\,\rangle = (1+|\cdot|^2)^{\frac{1}{2}}$, $e_n(x) = e^{in\cdot x}$, and $\{g_n\}_{n\in\mathbb{Z}^2}$ is a sequence of mutually independent standard complex-valued Gaussian random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It is easy to see that u in (1.5) belongs almost surely to $H^s(\mathbb{T}^2)\setminus L^2(\mathbb{T}^2)$ for any $s<0$, and thus a renormalization is needed on the interaction potential in (1.3).

Given $N\in\mathbb{N}$, let $\mathbf{P}_N$ be the frequency projector onto the frequencies $\{|n|\le N\}$ given by

$$\mathbf{P}_N f = \sum_{|n|\le N} \widehat{f}(n) e_n.$$

For each fixed $x\in\mathbb{T}^2$, $\mathbf{P}_N u(x)$ is a mean-zero complex-valued Gaussian random variable with variance:

$$\sigma_N = \mathbb{E}\big[|\mathbf{P}_N u(x)|^2\big] = \sum_{\substack{n\in\mathbb{Z}^2\\ |n|\le N}} \frac{1}{\langle n\rangle^2} \sim \log N \longrightarrow \infty, \tag{1.6}$$

as $N\to\infty$. In the current complex-valued setting, we define the Wick renormalized power $:|\mathbf{P}_N u|^4:$ by setting

$$\begin{aligned} :|\mathbf{P}_N u(x)|^4: &= 2\sigma_N^2 L_2(\sigma_N^{-1}|\mathbf{P}_N u(x)|^2)\\ &= |\mathbf{P}_N u(x)|^4 - 4\sigma_N |\mathbf{P}_N u(x)|^2 + 2\sigma_N^2 \end{aligned} \tag{1.7}$$

for $x\in\mathbb{T}^2$, where $L_2(z) = \frac{1}{2}z^2 - 2z + 1$ is the Laguerre polynomial of degree 2. We point out that, by writing

$$|\mathbf{P}_N u(x)|^4 = (\operatorname{Re}\mathbf{P}_N u(x))^4 + 2(\operatorname{Re}\mathbf{P}_N u(x))^2(\operatorname{Im}\mathbf{P}_N u(x))^2 + (\operatorname{Im}\mathbf{P}_N u(x))^4, \tag{1.8}$$

the renormalization in (1.7) is equivalent to separately applying the usual real-valued Wick renormalization via the Hermite polynomials to the powers of $\operatorname{Re}\mathbf{P}_N u(x)$ and $\operatorname{Im}\mathbf{P}_N u(x)$ in (1.8); see [50, Lemma 2.1]. See also [1, p. 2966] and [49, p. 107]. With this renormalization, we define the truncated Gibbs measure ρ_N by

$$d\rho_N = Z_N^{-1} \exp\bigg(-\frac{1}{4}\int_{\mathbb{T}^2} :|\mathbf{P}_N u|^4: dx\bigg) d\mu. \tag{1.9}$$

Then, by the Wiener chaos estimate (Lemma 2.4) and Nelson's estimate, we can show that, as $N\to\infty$, the truncated Gibbs measure ρ_N converges in total variation[3] to the (unique) limiting Gibbs measure ρ, which we write as

$$d\rho = Z^{-1} \exp\bigg(-\frac{1}{4}\int_{\mathbb{T}^2} :|u|^4: dx\bigg) d\mu. \tag{1.10}$$

[2]We endow $\mathbb{T}^2$ with the normalized Lebesgue measure $dx_{\mathbb{T}^2} = (2\pi)^{-2}dx$. With a slight abuse of notation, we still use dx to denote the normalized Lebesgue measure on $\mathbb{T}^2$.

[3]In fact, the density of the truncated Gibbs measure ρ_N converges in $L^p(\mu)$ for any finite $p\ge 1$.

See [50, Section 2] for details.

Remark 1.1. (i) Note that the limiting Gibbs measure ρ in (1.10) is equivalent to the Gaussian free field μ in (1.4). Namely, ρ and μ are mutually absolutely continuous with respect to each other. Moreover, given any $N \in \mathbb{N}$, ρ_N and ρ are equivalent.

(ii) The discussion above is restricted to the defocusing case. In fact, in the focusing case, it is known that the truncated Gibbs measure on $\mathbb{T}^2$ (with a Wick-ordered L^2-cutoff) does not converge to any limit in total variation, even up to a subsequence. See [11, 47, 29]. We also refer readers to [48, 45] for an overview of the Gibbs measure construction for NLS, the nonlinear wave equation, and the generalized KdV equation.

1.3. **Renormalized NLS.** We now turn to the well-posedness problem and discuss the construction of invariant Gibbs dynamics for NLS (1.1). Given $N \in \mathbb{N}$, it follows from (1.7) that the dynamical problem associated with the truncated Gibbs measure ρ_N in (1.9) is given by the following truncated Wick-ordered NLS on $\mathbb{T}^2$:

$$i\partial_t u_N + \Delta u_N = \mathbf{P}_N\big(|\mathbf{P}_N u_N|^2 - 2\sigma_N)\mathbf{P}_N u_N\big) \tag{1.11}$$

with $\text{Law}(u_N(0)) = \rho_N$, where σ_N is as in (1.6) and $\text{Law}(X)$ denotes the law of a random variable X. Then, it is easy to see that the truncated NLS (1.11) is ρ_N-almost surely globally well-posed and that ρ_N is invariant under the dynamics of (1.11); see [50, Lemma 5.1]. In view of Remark 1.1 (i), we see that (1.11) is also globally well-posed almost surely with respect to ρ. We recall the main statement in [6] regarding invariance of the limiting Gibbs measure ρ in (1.10) under the dynamics of the defocusing cubic NLS (1.11) on $\mathbb{T}^2$.

Theorem 1.2 (Bourgain [6])**.** *Given $N \in \mathbb{N}$, let Φ_N be the solution map for the truncated NLS (1.11), sending initial data $u_N(0)$ to the solution u_N. Then, there exists a set[4] $\Sigma \subset H^{-\varepsilon}(\mathbb{T}^2)$ with $\rho(\Sigma) = 1$ such that for each $u_0 \in \Sigma$, as $N \to \infty$, the solution $\Phi_N(u_0)$ to (1.11) converges to a non-trivial limit u with $u|_{t=0} = u_0$ in $C(\mathbb{R}; H^{-\varepsilon}(\mathbb{T}^2))$, endowed with the compact-open topology in time. Furthermore, we have $\text{Law}(u(t)) = \rho$ for any $t \in \mathbb{R}$.*

Formally speaking, Theorem 1.2 states that the Gibbs measure ρ in (1.10) is invariant under the following (formal) Wick-ordered NLS:

$$i\partial_t u + \Delta u = (|u|^2 - 2 \cdot \infty)u,$$

where $\infty = \lim_{N\to\infty} \sigma_N$.

The proof of Theorem 1.2 in [6] is based on studying the following gauge-equivalent formulation. Given $N \in \mathbb{N}$, let u_N be a solution to the truncated NLS (1.11) and define α_N by setting

$$\alpha_N = \|\mathbf{P}_N u_N(0)\|_{L^2}^2 - \sigma_N. \tag{1.12}$$

Then, by setting

$$v_N(t) = \mathbf{P}_N e^{2it\alpha_N} u_N(t) + \mathbf{P}_N^\perp u_N(t), \tag{1.13}$$

where $\mathbf{P}_N^\perp = \text{Id} - \mathbf{P}_N$, we see that v_N satisfies the following truncated NLS:

$$\begin{cases} i\partial_t v_N + \Delta v_N = \mathbf{P}_N\big(|\mathbf{P}_N v_N|^2 - 2\int_{\mathbb{T}^2} |\mathbf{P}_N v_N|^2 dx)\mathbf{P}_N v_N\big) \\ v_N|_{t=0} = u_N(0). \end{cases} \tag{1.14}$$

[4]We fix a sufficiently small constant $\varepsilon > 0$ in the rest of this note.

Note from (1.13) that ρ_N is invariant under (1.14). By formally taking $N \to \infty$, we obtain the following renormalized NLS:[5]

$$i\partial_t v + \Delta v = \mathfrak{N}(v) \stackrel{\text{def}}{=} \big(|v|^2 - 2\fint_{\mathbb{T}^2} |v|^2 dx\big)v. \tag{1.15}$$

Theorem 1.3. (i) *The renormalized NLS* (1.15) *is μ-almost surely locally well-posed, where μ is the Gaussian free field defined in* (1.4). *More precisely, there exists a set $\Sigma \subset H^{-\varepsilon}(\mathbb{T}^2)$ with $\mu(\Sigma) = 1$ such that for each $u_0 \in \Sigma$, there exists a unique local-in-time solution v to* (1.15) *with $v|_{t=0} = u_0$. Furthermore, for each $u_0 \in \Sigma$, by possibly shrinking the local existence time $T = T(u_0) > 0$, the solution v_N to* (1.14) *with $v_N|_{t=0} = u_0$ converges to v in $C([-T, T]; H^{-\varepsilon}(\mathbb{T}^2))$ as $N \to \infty$.*

(ii) *The renormalized NLS* (1.15) *is ρ-almost surely globally well-posed and the Gibbs measure ρ is invariant under the resulting dynamics. More precisely, there exists a set $\Sigma \subset H^{-\varepsilon}(\mathbb{T}^2)$ with $\rho(\Sigma) = 1$ such that for each $u_0 \in \Sigma$, there exists a unique global-in-time solution v to* (1.15) *with $v|_{t=0} = u_0$. Furthermore, for each $u_0 \in \Sigma$, the global-in-time solution v_N to* (1.14) *with $v_N|_{t=0} = u_0$ converges to v in $C(\mathbb{R}; H^{-\varepsilon}(\mathbb{T}^2))$, endowed with the compact-open topology in time, as $N \to \infty$.*

We first note that Theorem 1.3 (ii) follows from the local-in-time result (Theorem 1.3 (i)) and Bourgain's invariant measure argument [5, 6] together with the invariance of the truncated Gibbs measure ρ_N for (1.14); see [13, 46] for details of such an argument. Moreover, by noting that α_N in (1.12) converges μ-almost surely (and hence ρ-almost surely) to a finite limit as $N \to \infty$, Theorem 1.2 follows from Theorem 1.3 (ii) and (1.13).

Our main goal in this note is to revisit the proof of the μ-almost sure local well-posedness of the renormalized NLS (1.15) by using tools and ideas from the recent development in the field, in particular the random tensor estimate (Lemma 2.6) developed by Deng, Nahmod, and Yue [24]. Note that the local-in-time convergence claimed in Theorem 1.3 (i) follows from a straightforward modification of such an argument, and hence we omit details.

Before proceeding further, we first recall the known deterministic results for (1.1). In [4], Bourgain introduced the Fourier restriction norm method, based on the $X^{s,b}$-space defined via the norm:

$$\|u\|_{X^{s,b}} = \|\langle n\rangle^s \langle \tau + |n|^2\rangle^b \widehat{u}(n,\tau)\|_{\ell^2_n L^2_\tau}, \tag{1.16}$$

and proved local well-posedness of (1.1) in $H^s(\mathbb{T}^2)$ for any $s > 0$ by establishing the following trilinear estimate; given any $s > 0$ and $b_1 > b_2 > \frac{1}{2}$ sufficiently close to $\frac{1}{2}$, we have

$$\|u_1 \overline{u}_2 u_3\|_{X^{s,b_1-1}_T} \lesssim \prod_{j=1}^3 \|u_j\|_{X^{s,b_2}_T} \tag{1.17}$$

for any $0 < T \leq 1$, where $X^{s,b}_T$ denotes the local-in-time version of the $X^{s,b}$-space defined in (2.4). This local well-posedness is essentially sharp in the sense that, when $s < 0$, (1.1) is known to be ill-posed in $H^s(\mathbb{T}^2)$; see [43]. A key ingredient for the trilinear estimate (1.17) is the following L^4-Strichartz estimate due to Bourgain [4]:

$$\|e^{it\Delta}\mathbf{P}_N f\|_{L^4([0,1];L^4(\mathbb{T}^2))} \lesssim N^s \|\mathbf{P}_N f\|_{L^2} \tag{1.18}$$

[5]The equation (1.15) is often referred to as the Wick-ordered NLS, since its frequency-truncated version (1.14) is equivalent to the (truncated) Wick-ordered NLS (1.11). See, for example, [18, 30, 33, 27, 53, 54].

for any $s > 0$. We point out that (1.18) fails when $s = 0$ and the well-posedness issue of the cubic NLS in the critical space $L^2(\mathbb{T}^2)$ remains a challenging open problem. See [42, Remark 4.3] and [36, Section 3] (see also [59]) for a sharp lower bound and a recent breakthrough work [34] for a sharp upper bound for the L^4-Strichartz estimate on $\mathbb{T}^2$.

Let us turn our attention to the renormalized NLS (1.15). Define the trilinear operators $\mathcal{N}$ and $\mathcal{R}$ by setting

$$\begin{aligned} \mathcal{F}_x\big(\mathcal{N}(v_1, v_2, v_3)\big)(n) &= \sum_{\substack{n = n_1 - n_2 + n_3 \\ n \neq n_1, n_3}} \widehat{v}_1(n_1)\overline{\widehat{v}_2(n_2)}\widehat{v}_3(n_3), \\ \mathcal{F}_x\big(\mathcal{R}(v_1, v_2, v_3)\big)(n) &= \widehat{v}_1(n)\overline{\widehat{v}_2(n)}\widehat{v}_3(n), \end{aligned} \tag{1.19}$$

where $\mathcal{F}_x$ denotes the Fourier transform on $\mathbb{T}^2$. For simplicity, we set $\mathcal{N}(v) = \mathcal{N}(v, v, v)$ and $\mathcal{R}(v) = \mathcal{R}(v, v, v)$. Then, from (1.15) and (1.19), we have

$$\mathfrak{N}(v) = \mathcal{N}(v) - \mathcal{R}(v). \tag{1.20}$$

The first term $\mathcal{N}(v)$ represents the non-resonant contribution, while the second term $\mathcal{R}(v)$ represents the (doubly) resonant contribution. We point out that the renormalization in $\mathfrak{N}(v)$ essentially removes the resonant contribution from $n = n_1$ or n_3 in the usual cubic nonlinearity (at the expense of adding the doubly resonant term $\mathcal{R}(v)$) but that the well- / ill-posedness results for (1.1) mentioned above also apply to the renormalized NLS (1.15). In particular, while the Gaussian free field μ is supported on $H^s(\mathbb{T}^2) \setminus L^2(\mathbb{T}^2)$, $s < 0$, the renormalized NLS (1.15) is ill-posed in $H^s(\mathbb{T}^2)$ for any $s < 0$.

In order to overcome this difficulty, Bourgain [6] considered the following first order expansion:

$$v(t) = z(t) + w(t),$$

where

$$z(t) = z^\omega(t) = e^{it\Delta}u_0^\omega = \sum_{n \in \mathbb{Z}^2} \frac{g_n(\omega)}{\langle n \rangle} e^{-it|n|^2} e_n \tag{1.21}$$

denotes the random linear solution with $\mathrm{Law}(u_0^\omega) = \mu$, and studied the following equation satisfied by the residual term w:

$$\begin{cases} i\partial_t w + \Delta w = \mathfrak{N}(w + z) \\ w|_{t=0} = 0. \end{cases} \tag{1.22}$$

See also [40, 20]. By writing (1.22) in the Duhamel formulation, we have

$$w(t) = -i \int_0^t e^{i(t-t')\Delta} \mathfrak{N}(w + z)(t') dt'.$$

Then, in view of the nonhomogeneous standard linear estimate (see [4, 35, 60]):

$$\bigg\| \int_0^t e^{i(t-t')\Delta} F(t') dt' \bigg\|_{X_T^{s, \frac{1}{2}+\varepsilon}} \lesssim T^\varepsilon \|F\|_{X_T^{s, -\frac{1}{2}+2\varepsilon}}$$

for any sufficiently small $\varepsilon > 0$, almost sure local well-posedness of (1.22) (and hence Theorem 1.3 (i)) follows once we prove the following proposition.[6]

[6]We also need a multilinear version of (1.23). Since it follows from a straightforward modification of the proof of (1.23), we omit details.

Proposition 1.4. *Let $\mathfrak{N}(v)$ be as in (1.20). Given small $s > 0$ and $\varepsilon > 0$, there exist $c_1, c_2 > 0$ with the following property; given $0 < T \ll 1$, there exists $\Sigma_T \subset H^{-\varepsilon}(\mathbb{T}^2)$ with $\mu(\Sigma_T^c) < e^{-\frac{c_1}{T^{c_2}}}$ such that we have*

$$\|\mathfrak{N}(w+z)\|_{X_T^{s,-\frac{1}{2}+2\varepsilon}} \lesssim 1 \tag{1.23}$$

for any w with $\|w\|_{X_T^{s,\frac{1}{2}+\varepsilon}} \leq 1$ and $u_0 \in \Sigma_T$, where $z(t) = e^{it\Delta}u_0$.

In view of (1.20), the multilinearity of $\mathcal{N}$ and $\mathcal{R}$, and the symmetry in the first and third arguments in (1.19), it suffices to estimate the following terms:

(i) resonant term: $\mathcal{R}(w+z)$; see Section 3,
(ii) "deterministic" term: $\mathcal{N}(w,w,w)$; see (1.17),
(iii) purely stochastic term: $\mathcal{N}(z,z,z)$; see Section 4,
(iv) random tensor term I: $\mathcal{N}(z,w,z)$; see Section 5,
(v) random tensor term II: $\mathcal{N}(w,z,z)$; see Section 6,
(vi) random tensor term III: $\mathcal{N}(z,w,w)$; see Section 7,
(vii) random tensor term IV: $\mathcal{N}(w,z,w)$; see Section 8,

where, with a slight abuse of notation, we refer to the terms (iv) - (vii) as random tensor terms. Since the contribution from the "deterministic" term $\mathcal{N}(w,w,w)$ is already handled in (1.17), the rest of this note is devoted to estimating the contributions from the other terms.

Remark 1.5. We conclude this introduction by mentioning several works related to Theorem 1.2. In [26], Fan, Ou, Staffilani, and Wang extended Theorem 1.2 to the irrational torus case. In [63], Zine established convergence of invariant Gibbs dynamics for the stochastic complex Ginzburg-Landau equation on $\mathbb{T}^2$ with the defocusing cubic nonlinearity to that for the defocusing cubic NLS on $\mathbb{T}^2$ as the dissipation and the noise vanish in an appropriate manner.

2. Preliminaries

2.1. **Notations and basic deterministic estimates.** Let $A \lesssim B$ denote an estimate of the form $A \leq CB$ for some constant $C > 0$. We write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$, while $A \ll B$ denotes $A \leq cB$ for some small constant $c > 0$. We may write $\lesssim_\alpha$ and $\sim_\alpha$ to emphasize the dependence on an external parameter α. We use $C > 0$ to denote various constants, which may vary line by line.

Given dyadic $N \geq 1$, we use $\mathbf{Q}_N$ to denote the Littlewood-Paley projector onto the frequencies $\{|n| \sim N\}$ for $N \geq 2$ (and $\{|n| \lesssim 1\}$ when $N = 1$). We then set

$$\boldsymbol{\mathcal{S}}_N f = \sum_{\substack{1 \leq M \leq N \\ \text{dyadic}}} \mathbf{Q}_M f \qquad \text{and} \qquad \boldsymbol{\mathcal{S}}_N^\perp f = f - \boldsymbol{\mathcal{S}}_N f. \tag{2.1}$$

Given dyadic numbers $N_1, N_2, N_3 \geq 1$, we use $N_{\max}$, N_{med}, and $N_{\min}$ to denote their decreasing rearrangement:

$$N_{\max} \geq N_{\text{med}} \geq N_{\min}. \tag{2.2}$$

We use $\eta \in C_c^\infty(\mathbb{R})$ to denote a smooth cutoff function supported on $[-2,2]$ with $\eta \equiv 1$ on $[-1,1]$, and let $\eta_T(t) = \eta(T^{-1}t)$. Then, we have

$$\|\widehat{\eta}_T\|_{L^q_\tau} \sim T^{\frac{q-1}{q}} \|\widehat{\eta}\|_{L^q_\tau} \sim T^{\frac{q-1}{q}} \tag{2.3}$$

for any $1 \leq q \leq \infty$ and $T > 0$.

Given $T > 0$, we define the space $X^{s,b}_T$ to be the restriction of the $X^{s,b}$-space defined in (1.16) onto the time interval $[-T,T]$ via the norm:

$$\|u\|_{X^{s,b}_T} := \inf\big\{\|v\|_{X^{s,b}} : v|_{[-T,T]} = u\big\}. \tag{2.4}$$

Note that $X^{s,b}_T$ is complete. Given any $s \in \mathbb{R}$ and $b > \frac{1}{2}$, we have $X^{s,b}_T \subset C([-T,T]; H^s(\mathbb{T}^2))$. Given any $s, b \in \mathbb{R}$, the following homogeneous linear estimate holds:

$$\|e^{it\Delta} f\|_{X^{s,b}_T} \lesssim_b \|f\|_{H^s} \tag{2.5}$$

for any $0 < T \leq 1$. We also recall the following lemma; see [60, Lemma 2.11].

Lemma 2.1. *Let $s \in \mathbb{R}$ and $-\frac{1}{2} < b_1 \leq b_2 < \frac{1}{2}$. Then, we have*

$$\|u\|_{X^{s,b_1}_T} \leq C T^{b_2 - b_1} \|u\|_{X^{s,b_2}_T}$$

for any $0 < T \leq 1$.

In the remaining part of this note, we fix a sufficiently small constant $\varepsilon > 0$ and set

$$b = \frac{1}{2} + \varepsilon \qquad \text{and} \qquad b' = \frac{1}{2} - 2\varepsilon \tag{2.6}$$

(unless otherwise specified). With this notation, we state a slightly improved trilinear estimate. Given u_j on $\mathbb{T}^2 \times \mathbb{R}$, $j = 1,2,3$, suppose that there exists $j_* \in \{1,2,3\}$ such that, given any $t \in \mathbb{R}$, we have

$$|n_{j_*}| \gtrsim \max_{j \neq j_*} |n_j|$$

for any $n_k \in \operatorname{supp} \widehat{u}_k(\cdot, t)$, $k = 1,2,3$. Then, by a slight modification of the proof of the trilinear estimate (1.17), we have

$$\|\mathcal{N}(u_1,u_2,u_3)\|_{X^{s,-b'+2\varepsilon_1}} \lesssim \|u_{j_*}\|_{X^{s,b-\varepsilon_1}} \prod_{\substack{j=1\\ j\neq j_*}}^{3} \|u_j\|_{X^{\varepsilon_2, b-\varepsilon_1}} \tag{2.7}$$

for any $s > 0$ and $\varepsilon_2 > 0$, provided that $\varepsilon_1 \geq 0$ is sufficiently small.

Lastly, we recall the following elementary calculus lemma; see, for example, [28, Lemma 4.2].

Lemma 2.2. *Let $\alpha, \beta \in \mathbb{R}$ satisfy*

$$\alpha \geq \beta \geq 0 \qquad \text{and} \qquad \alpha + \beta > 1.$$

Then, we have

$$\int_{\mathbb{R}} \frac{dx}{\langle x\rangle^\alpha \langle x-a\rangle^\beta} \lesssim \frac{1}{\langle a\rangle^{\beta-\lambda}},$$
$$\sum_{n\in\mathbb{Z}} \frac{1}{\langle n\rangle^\alpha \langle n-a\rangle^\beta} \lesssim \frac{1}{\langle a\rangle^{\beta-\lambda}}$$

for any $a \in \mathbb{R}$, where $\lambda = \max(1-\alpha, 0)$ when $\alpha \neq 1$ and $\lambda = \varepsilon$ when $\alpha = 1$ for any $\varepsilon > 0$.

2.2. **Tools from stochastic analysis.** We first recall a basic bound on Gaussian random variables; see [19, Lemma 3.4] for a proof.

Lemma 2.3. *Let $\{g_n\}_{n\in\mathbb{Z}^d}$ be a sequence of independent standard complex-valued Gaussian random variables. Given any $\delta_1, \delta_2 > 0$, there exist $c_1, c_2 > 0$ such that*

$$|g_n(\omega)| \lesssim T^{-\delta_1}\langle n\rangle^{\delta_2}$$

for any $n \in \mathbb{Z}^d$ and $0 < T \ll 1$, outside an exceptional set of probability $< e^{-\frac{c_1}{T^{c_2}}}$.

Next, we recall the Wiener chaos estimate ([58, Theorem I.22]); see [52, Lemma 3.2] for the following particular version.

Lemma 2.4. *Let $\{g_n\}_{n\in\mathbb{Z}^d}$ be a sequence of independent standard real-valued Gaussian random variables. Given $k \in \mathbb{N}$, let $\{P_j\}_{j\in\mathbb{N}}$ be a sequence of polynomials in $\bar{g} = \{g_n\}_{n\in\mathbb{Z}^d}$ of degree at most k. Then, for any finite $p \geq 1$, we have*

$$\Big\| \sum_{j\in\mathbb{N}} P_j(\bar{g}) \Big\|_{L^p(\Omega)} \leq (p-1)^{\frac{k}{2}} \Big\| \sum_{j\in\mathbb{N}} P_j(\bar{g}) \Big\|_{L^2(\Omega)}.$$

2.3. **Random tensor estimate.** In this section, we provide the basic definition of tensors and state the random tensor estimate (Lemma 2.6). See [24, Sections 2 and 4], [9, Section 4], and [55, Appendix C] for further discussion.

Definition 2.5. Let A be a finite index set. We denote by n_A the tuple $\{n_j\}_{j\in A}$. A tensor $h = h_{n_A}$ is a function: $(\mathbb{Z}^d)^A \to \mathbb{C}$ with the input variables n_A. Note that the tensor h may also depend on $\omega \in \Omega$. The support of a tensor h is the set of n_A such that $h_{n_A} \neq 0$.

Given a finite index set A, let (B, C) be a partition of A. We define the norms $\|\cdot\|_{n_A}$ and $\|\cdot\|_{n_B\to n_C}$ by

$$\|h\|_{n_A} = \|h\|_{\ell^2_{n_A}} = \Big(\sum_{n_A} |h_{n_A}|^2 \Big)^{\frac{1}{2}}$$

and

$$\|h\|^2_{n_B\to n_C} = \sup\Big\{ \sum_{n_C} \Big| \sum_{n_B} h_{n_A} f^{\text{in}}_{n_B} \Big|^2 : \|f^{\text{in}}\|_{\ell^2_{n_B}} = 1 \Big\}, \tag{2.8}$$

where we used the short-hand notation $\sum_{n_Z} = \sum_{n_Z\in(\mathbb{Z}^d)^Z}$ for a finite index set Z. Note that, by duality, we have $\|h\|_{n_B\to n_C} = \|h\|_{n_C\to n_B} = \|\overline{h}\|_{n_B\to n_C}$ for any tensor $h = h_{n_A}$. If $B = \varnothing$ or $C = \varnothing$, then we have $\|h\|_{n_B\to n_C} = \|h\|_{n_A}$.

Let (B, C) be a partition of A. Then, by duality, we can write (2.8) as

$$\|h\|_{n_B\to n_C} = \sup\Big\{ \Big| \sum_{n_B, n_C} h_{n_A} f^{\text{in}}_{n_B} f^{\text{out}}_{n_C} \Big| : \|f^{\text{in}}\|_{\ell^2_{n_B}} = \|f^{\text{out}}\|_{\ell^2_{n_C}} = 1 \Big\}. \tag{2.9}$$

For example, when $A = \{1, 2\}$, the norm $\|h\|_{n_1\to n_2}$ denotes the usual operator norm $\|h\|_{\ell^2_{n_1}\to\ell^2_{n_2}}$ for an infinite-dimensional matrix operator $\{h_{n_1 n_2}\}_{n_1,n_2\in\mathbb{Z}^d}$. By bounding the matrix operator norm by the Hilbert-Schmidt norm (= the Frobenius norm), we have

$$\|h\|_{\ell^2_{n_1}\to\ell^2_{n_2}} \leq \|h\|_{\ell^2_{n_1,n_2}}. \tag{2.10}$$

With the above notation, we have the following random tensor estimate, originally introduced in [24, Propositions 4.14 and 4.15]; see also [9, Proposition 4.50] and [55, Lemma C.3] for similar formulations as ours.

Lemma 2.6. *Let A be a finite index set with $k = |A| \geq 1$. Given a (deterministic) tensor $h_{bcn_A} \in \ell^2_{bcn_A}$ such that $n_a \in \mathbb{Z}^d$ for each $a \in A$ and $(b, c) \in (\mathbb{Z}^d)^q$ for some integer $q \geq 2$, define a random tensor $H = H_{bc}$ by*

$$H_{bc} = \sum_{n_A} h_{bcn_A} \prod_{n\in\mathbb{Z}^d} \mathfrak{H}_{k_n}(g_n), \tag{2.11}$$

where $\{g_n\}_{n\in\mathbb{Z}^d}$ is a family of independent standard real-valued Gaussian random variables and $\mathfrak{H}_{k_a}$ denotes the Hermite polynomial of degree[7]

$$k_n = \#\{a \in A : n_a = n\} \tag{2.12}$$

such that $\sum_{n\in\mathbb{Z}^d} k_n = k$.

(i) *Given $N \in \mathbb{N}$, suppose that*

$$\operatorname{supp} h \subset \big\{|b - b^*|, |c - c^*|, |n_a| \lesssim N \text{ for each } a \in A\big\}$$

for some fixed $(b^*, c^*) \in (\mathbb{Z}^d)^q$. Then, for any $\theta > 0$ and finite $p \geq 1$, we have

$$\big\| \|H_{bc}\|_{b\to c} \big\|_{L^p(\Omega)} \lesssim p^{\frac{k}{2}} N^\theta \max_{(B,C)} \|h\|_{bn_B\to cn_C},$$

where the maximum is taken over all partitions (B, C) of A and the implicit constant is independent of $N \in \mathbb{N}$ and $(b^*, c^*) \in (\mathbb{Z}^d)^q$.

(ii) *Write $b = (b_0, b_1) \in \mathbb{Z}^d \times (\mathbb{Z}^d)^{q_1}$ and $c = (c_0, c_1) \in \mathbb{Z}^d \times (\mathbb{Z}^d)^{q_2}$ with $q_1 + q_2 + 2 = q$. Given $N \in \mathbb{N}$, suppose that*

- (ii.a) $\operatorname{supp} h \subset \big\{|b_1 - b^*|, |c_1 - c^*|, |n_a| \lesssim N \text{ for each } a \in A\big\}$ *for some fixed $(b^*, c^*) \in (\mathbb{Z}^d)^{q_1+q_2}$,*
- (ii.b) *$|b_0 - c_0| \leq C_1 N$ and $\big||b_0|^2 - |c_0|^2\big| \lesssim N^{a_1}$ for some $C_1, a_1 \geq 2$,*
- (ii.c) *the tensor $h_{bcn_A} = h_{(b_0,b_1),(c_0,c_1),n_A}$ depends only on $b_0 - c_0$, $|b_0|^2 - |c_0|^2$, b_1, c_1, and n_A.*

Given $m \in \mathbb{Z}^d$, define a tensor $h^m_{(b_0',b_1),(c_0',c_1),n_A}$ by

$$h^m_{(b_0',b_1),(c_0',c_1),n_A} = h_{(b_0'+m,b_1),(c_0'+m,c_1),n_A} \cdot \mathbf{1}_{|b_0'|,|c_0'|\leq C_1 N}. \tag{2.13}$$

Then, for any $\theta > 0$ and finite $p \geq 1$, we have

$$\big\| \|H_{bc}\|_{b\to c} \big\|_{L^p(\Omega)} \lesssim p^{\frac{k}{2}} N^\theta \sup_{m\in\mathbb{Z}^d} \max_{(B,C)} \|h^m_{(b_0',b_1),(c_0',c_1),n_A}\|_{(b_0',b_1),n_B\to(c_0',c_1),n_C}, \tag{2.14}$$

where the maximum is taken over all partitions (B, C) of A and the implicit constant is independent of $N \in \mathbb{N}$ and $(b^*, c^*) \in (\mathbb{Z}^d)^{q_1+q_2}$.

(iii) *Let h be as in Part (ii), and set*

$$\widetilde{h}_{(b_0,b_1),(c_0,c_1),n_A} = h_{(b_0,b_1),(c_0,c_1),n_A} \cdot \mathbf{1}_{|b_0|,|c_0|\geq 10C_1 N},$$

[7] Since A is a finite set, we have $k_n = 0$ except for finitely many n's. Recalling that $\mathfrak{H}_0(x) = 1$, we see that the product in (2.11) is indeed over a finite set $\{n \in \mathbb{Z}^d : \text{given } n_A \in (\mathbb{Z}^d)^A, \text{ there exists } a \in A \text{ such that } n_a = n\}$.

where $C_1 \geq 1$ is as in (ii.b) above, and define a random tensor $\widetilde{H} = \widetilde{H}_{bc}$ by

$$\widetilde{H}_{bc} = \sum_{n_A} \widetilde{h}_{bcn_A} \prod_{n\in\mathbb{Z}^d} \mathfrak{H}_{k_n}(g_n),$$

where $\{g_n\}_{n\in\mathbb{Z}^d}$ and k_n are as in (2.11). Given $m \in \mathbb{Z}^d$, define a tensor $\widetilde{h}^m_{(b_0',b_1),(c_0',c_1),n_A}$ by

$$\begin{aligned}\widetilde{h}^m_{(b_0',b_1),(c_0',c_1),n_A} &= \widetilde{h}_{(b_0'+m,b_1),(c_0'+m,c_1),n_A} \cdot \mathbf{1}_{|b_0'|,|c_0'|\leq C_1N}\\ &= h_{(b_0'+m,b_1),(c_0'+m,c_1),n_A} \cdot \mathbf{1}_{|b_0'+m|,|c_0'+m|\geq 10C_1N} \cdot \mathbf{1}_{|b_0'|,|c_0'|\leq C_1N}.\end{aligned} \tag{2.15}$$

Then, for any $\theta > 0$ and finite $p \geq 1$, we have

$$\big\| \|\widetilde{H}_{bc}\|_{b\to c} \big\|_{L^p(\Omega)} \lesssim p^{\frac{k}{2}} N^{\theta} \sup_{m\in\mathbb{Z}^d} \max_{(B,C)} \|\widetilde{h}^m_{(b_0',b_1),(c_0',c_1),n_A}\|_{(b_0',b_1),n_B\to(c_0',c_1),n_C}, \tag{2.16}$$

where the maximum is taken over all partitions (B, C) of A and the implicit constant is independent of $N \in \mathbb{N}$ and $(b^*, c^*) \in (\mathbb{Z}^d)^{q_1+q_2}$.

In [55], the random tensor estimate was extended to multiple stochastic integrals (with respect to white-in-time noises). In [16, 17], the random tensor estimate was further extended for multiple stochastic integrals with respect to fractional-in-time noises. These extensions have played a crucial role in establishing pathwise well-posedness of stochastic dispersive PDEs with multiplicative noises; see also [15].

See [55, Lemma C.3 and Remark C.4 (ii)] for a proof of Lemma 2.6 (i). When $b = b_0$ and $c = c_0$, Lemma 2.6 (ii) is essentially proven in [24, Proposition 4.15]. While the general case follows from a straightforward modification, we include its proof for readers' convenience. For this purpose, we need the following compactness lemma. See also [23, Claim 5.2], [22, Claim 3.7], and [24, Claim 4.16].

Lemma 2.7. *Let $a_1 \geq 1$. Given $N \gg 1$, let $\{f_m\}_{m\in\mathbb{Z}^d}$ be the family of functions defined by*

$$f_m(n) = m \cdot n$$

with the domain given by

$$D_m = \big\{n \in \mathbb{Z}^d : |n| \lesssim N \text{ and } |m \cdot n| \lesssim N^{a_1}\big\}. \tag{2.17}$$

Then, there exists $a_2 \gg 1$, independent of N, and $J = J(a_1, a_2, N) \subset \mathbb{Z}^d$ with $|J| = O(N^{a_2})$ such that, given any $m \in \mathbb{Z}^d$, there exists $m_0 \in J$ such that

$$f_{m_0} = f_m \qquad \text{and} \qquad D_{m_0} = D_m. \tag{2.18}$$

Proof. We first consider the one-dimensional case. For $|m| \lesssim N^{a_1}$, we include such m in J. If $|m| \gg N^{a_1}$, it follows from (2.17) that $D_m = \{0\}$ and thus $f_m \equiv 0$ on D_m. Thus, by including one such m_0 with $|m_0| \gg N^{a_1}$ in J, we have $f_m = f_{m_0} \equiv 0$ with $D_m = D_{m_0} = \{0\}$ for any $|m| \gg N^{a_1}$. By construction, we have $|J| = O(N^{a_1})$ such that (2.18) holds.

Next, we consider the higher-dimensional case. Fix $m \in \mathbb{Z}^d$, and write

$$D_m = \bigcup_{|p|\lesssim N^{a_1}} D_{m,p}, \quad \text{where } \ D_{m,p} = D_m \cap \{m \cdot n = p\}. \tag{2.19}$$

Fix p with $|p| \lesssim N^{a_1}$ such that $D_{m,p} \neq \varnothing$, and let $\{n_i\}_{i=0}^r \subset D_{m,p}$ (for some $0 \leq r \leq d$) be a maximal affine independent[8] set in $D_{m,p}$. Let L be the lattice generated by $\{n_i - n_0\}_{i=0}^r$, and

[8]Namely, there exists no $(\alpha^0, \alpha^1, \dots, \alpha^r) \subset \mathbb{R}^{r+1} \setminus \{0\}$ with $\sum_{i=0}^r \alpha^i = 0$ such that $\sum_{i=0}^r \alpha^i n_i = 0$.

fix an LLL reduced basis; see, for example, [38] and [7, Chapter 4] for the definition of an LLL reduced basis $\{\ell_i\}_{i=1}^r$. Then, it follows from [7, Theorem 4.8] that

$$|\ell_i| \lesssim \max_{k=1,\dots,r} |n_k - n_0| \lesssim N.$$

Then, given any $n \in D_{m,p}$, there is a unique vector $\alpha = \alpha(n) = (\alpha^1, \dots, \alpha^r) \in \mathbb{Z}^r$ such that $|\alpha| \lesssim N$,[9] $n - n_0 = \sum_{i=1}^r \alpha^i \ell_i$, and

$$m \cdot n = m \cdot \bigg(\sum_{i=1}^r \alpha^i \ell_i \bigg) + m \cdot n_0 = y \cdot \alpha + m \cdot n_0, \tag{2.20}$$

where $y = (y^1, \cdots, y^r) \in \mathbb{Z}^r$ with $y^i = m \cdot \ell_i \in \mathbb{Z}$. Since $n, n_0 \in D_{m,p} \subset D_m$, it follows from (2.17) that $|m \cdot n|, |m \cdot n_0| \lesssim N^{a_1}$. Then, using (2.20), we see that $|y \cdot \alpha(n)| \lesssim N^{a_1}$ for any $n \in D_{m,p}$. For each $i = 1, \dots, r$, define $\alpha_i = (\alpha_i^1, \dots, \alpha_i^r)$ such that $n_i - n_0 = \sum_{i=1}^r \alpha_i^k \ell_k$. Since $\{\alpha_i\}_{i=1}^r$ is a set of linearly independent vectors in $\mathbb{Z}^r$ with $|\alpha_i| = O(N)$, we conclude that $|y| \lesssim N^{Ca_1}$; see Lemma A.1 and Remark A.2.

Given $m \in \mathbb{Z}^d$ and p with $|p| \lesssim N^{a_1}$, we define $F_{m,p}$ on $D_{m,p}$ by

$$F_{m,p}(n) = m \cdot n = y \cdot \alpha + m \cdot n_0, \quad n \in D_{m,p}, \tag{2.21}$$

where y and $\alpha = \alpha(n)$ are as in (2.20). Since $D_{m,p}$ is generated by $\{n_i\}_{i=0}^r$, it follows from the linearity that the function $F_{m,p}$ is completely determined by its action on $\{n_i\}_{i=0}^r$. Hence, by recalling that $|y| \lesssim N^{Ca_1}$, $|\alpha(n)| \lesssim N$ for any $n \in D_{m,p}$, and $|m \cdot n_0| \lesssim N^{a_1}$, it follows from (2.21) that the collection $\{F_{m,p}\}_{m \in \mathbb{Z}^d}$ consists of $O(N^{a_1'})$ many elements for some $a_1' > a_1$. Now, we define F_m on D_m by setting $F_m = F_{m,p}$ on $D_{m,p}$ for each p with $|p| \lesssim N^{a_1}$. Then, from (2.19), we see that the collection $\{F_m\}_{m \in \mathbb{Z}^d}$ consists of $O(N^{a_2})$ many elements for some $a_2 > a_1'$. Finally, denoting $\{F_m\}_{m \in \mathbb{Z}^d}$ by $\{f_{m_0}\}_{m_0 \in J}$ for some set $J \subset \mathbb{Z}^d$ with $|J| = O(N^{a_2})$ (where $f_{m_0}(n) = m_0 \cdot n$ with the domain D_{m_0} defined in (2.17)), we conclude that (2.18) holds. □

We now discuss a proof of Lemma 2.6 (ii) and (iii).

Proof of Lemma 2.6 (ii) and (iii). On the support of h, we have $|b_0 - c_0| \lesssim N$. We first restrict h to $|b_0'|, |c_0'| \lesssim N$, where $b_0' = b_0 - m$ and $c_0' = c_0 - m$ (for suitable $m \in \mathbb{Z}^d$). More precisely, we set

$$H^m_{(b_0',b_1),(c_0',c_1)} = \sum_{n_A} h^m_{(b_0',b_1),(c_0',c_1),n_A} \prod_{n \in \mathbb{Z}^d} \mathfrak{H}_{k_n}(g_n), \tag{2.22}$$

where h^m and k_n are as in (2.13) and (2.12), respectively. For fixed $m \in \mathbb{Z}^d$, Lemma 2.6 (i) yields

$$\begin{aligned} &\Big\| \| H^m_{(b_0',b_1),(c_0',c_1)} \|_{(b_0',b_1) \to (c_0',c_1)} \Big\|_{L^p(\Omega)} \\ &\quad \lesssim p^{\frac{k}{2}} N^{\frac{\theta}{2}} \max_{(B,C)} \| h^m_{(b_0',b_1),(c_0',c_1),n_A} \|_{(b_0',b_1),n_B \to (c_0',c_1),n_C}, \end{aligned} \tag{2.23}$$

where the implicit constant is independent of $m \in \mathbb{Z}^d$. By the hypothesis (ii.c), the tensor $h^m_{(b_0',b_1),(c_0',c_1),n_A}$ is determined by $(b_0', b_1, c_0', c_1, n_A)$ and

$$|b_0' + m|^2 - |c_0' + m|^2 = 2m \cdot (b_0' - c_0') + (|b_0'|^2 - |c_0'|^2).$$

[9] This follows from the almost orthogonality of the LLL reduced basis $\{\ell_i\}_{i=1}^r$; see [7, p. 56].

From Lemma 2.7, we see that there exists $\{m_j\}_{j=1}^J \subset \mathbb{Z}^d$ with $J = O(N^{a_2})$ such that for any $m \in \mathbb{Z}^d$, there exists $j \in \{1, \dots, J\}$ such that

$$|b_0' + m|^2 - |c_0' + m|^2 = |b_0' + m_j|^2 - |c_0' + m_j|^2 \tag{2.24}$$

for any $b_0', c_0' \in \mathbb{Z}^d$ such that $|b_0'|, |c_0'| \le C_1 N$ and $\big||b_0' + m|^2 - |c_0' + m|^2\big| \lesssim N^{a_1}$, (which in particular implies $|m \cdot (b_0' - c_0')| \lesssim N^{a_1}$).

Fix small $\theta > 0$. Then, by almost orthogonality, the observation above with $J = O(N^{a_2})$, and (2.23), we have

$$\begin{aligned}
\big\| \|H_{bc}\|_{b\to c} \big\|_{L^p(\Omega)} &\lesssim \Big\| \sup_{m\in\mathbb{Z}^d} \|H^m_{(b_0',b_1),(c_0',c_1)}\|_{(b_0',b_1)\to(c_0',c_1)} \Big\|_{L^p(\Omega)} \\
&= \Big\| \sup_{j=1,\dots,J} \|H^{m_j}_{(b_0',b_1),(c_0',c_1)}\|_{(b_0',b_1)\to(c_0',c_1)} \Big\|_{L^p(\Omega)} \\
&\le \Big\| \Big(\sum_{j=1,\dots,J} \|H^{m_j}_{(b_0',b_1),(c_0',c_1)}\|^p_{(b_0',b_1)\to(c_0',c_1)} \Big)^{\frac{1}{p}} \Big\|_{L^p(\Omega)} \\
&\lesssim N^{\frac{a_2}{p}} \sup_{j=1,\dots,J} \big\| \|H^{m_j}_{(b_0',b_1),(c_0',c_1)}\|_{(b_0',b_1)\to(c_0',c_1)} \big\|_{L^p(\Omega)} \\
&\lesssim p^{\frac{k}{2}} N^{\theta} \sup_{m\in\mathbb{Z}^d} \max_{(B,C)} \|h^m_{(b_0',b_1),(c_0',c_1),n_A}\|_{(b_0',b_1),n_B\to(c_0',c_1),n_C}
\end{aligned} \tag{2.25}$$

for any $p > \frac{2a_2}{\theta} \gg 1$. This proves (2.14).

(iii) As in Part (ii), with $b_0' = b_0 - m$ and $c_0' = c_0 - m$, we define

$$\widetilde{H}^m_{(b_0',b_1),(c_0',c)} = \sum_{n_A} \widetilde{h}^m_{(b_0',b_1),(c_0',c_1),n_A} \prod_{n\in\mathbb{Z}^d} \mathfrak{H}_{k_n}(g_n), \tag{2.26}$$

where $\widetilde{h}^m$ and k_n are as in (2.15) and (2.12), respectively. Then, it follows from Lemma 2.6 (i) that

$$\begin{aligned}
&\big\| \|\widetilde{H}^m_{(b_0',b_1),(c_0',c_1)}\|_{(b_0',b_1)\to(c_0',c_1)} \big\|_{L^p(\Omega)} \\
&\quad\lesssim p^{\frac{k}{2}} N^{\frac{\theta}{2}} \max_{(B,C)} \|\widetilde{h}^m_{(b_0',b_1),(c_0',c_1),n_A}\|_{(b_0',b_1),n_B\to(c_0',c_1),n_C},
\end{aligned} \tag{2.27}$$

where the implicit constant is independent of $m \in \mathbb{Z}^d$; see (2.23).

We first consider the case $|m| \ge 20C_1N$. In this case, we have $|b_0' + m|, |c_0' + m| \ge 19C_1N$ and thus the condition $|b_0' + m|, |c_0' + m| \ge 10C_1N$ on the right-hand side of (2.15) is vacuous. In particular, by comparing (2.15) with (2.13), we note that the tensor $\widetilde{h}^m_{(b_0',b_1),(c_0',c_1),n_A}$ depends only on $(b_0', b_1, c_0', c_1, n_A)$ and $|b_0' + m|^2 - |c_0' + m|^2$. Then, proceeding as in Part (ii) with Lemma 2.7, we see that there exists $\{m_j\}_{j=1}^J \subset \mathbb{Z}^d$ with $J = O(N^{a_2})$ such that for any $m \in \mathbb{Z}^d$ with $|m| \ge 20C_1N$, there exists $j \in \{1, \dots, J\}$ such that (2.24) holds for any $b_0', c_0' \in \mathbb{Z}^d$ such that $|b_0'|, |c_0'| \le C_1N$ and $\big||b_0' + m|^2 - |c_0' + m|^2\big| \lesssim N^{a_1}$. Without loss of generality, we assume that $|m_j| \ge 20C_1N$ for each $j = 1, \dots, J$, since, by definition, for each $j = 1, \dots, J$, (2.24) holds for some $|m| \ge 20C_1N$ and thus, if $|m_j| < 20C_1N$, we can replace m_j by such m.

Fix small $\theta > 0$. Proceeding as in (2.25) with (2.27), we then obtain

$$\begin{aligned}
\big\| \|\widetilde{H}_{bc}\|_{b\to c} \big\|_{L^p(\Omega)} &\lesssim \Big\| \sup_{m\in\mathbb{Z}^d} \|\widetilde{H}^m_{(b_0',b_1),(c_0',c_1)}\|_{(b_0',b_1)\to(c_0',c_1)} \Big\|_{L^p(\Omega)} \\
&\le \Big\| \sup_{|m|<20C_1N} \|\widetilde{H}^m_{(b_0',b_1),(c_0',c_1)}\|_{(b_0',b_1)\to(c_0',c_1)} \Big\|_{L^p(\Omega)} \\
&\quad + \Big\| \sup_{|m|\ge 20C_1N} \|\widetilde{H}^m_{(b_0',b_1),(c_0',c_1)}\|_{(b_0',b_1)\to(c_0',c_1)} \Big\|_{L^p(\Omega)} \\
&\lesssim N^{\frac{d}{p}} \sup_{|m|<20C_1N} \big\| \|\widetilde{H}^m_{(b_0',b_1),(c_0',c_1)}\|_{(b_0',b_1)\to(c_0',c_1)} \big\|_{L^p(\Omega)} \\
&\quad + N^{\frac{a_2}{p}} \sup_{j=1,\dots,J} \big\| \|\widetilde{H}^{m_j}_{(b_0',b_1),(c_0',c_1)}\|_{(b_0',b_1)\to(c_0',c_1)} \big\|_{L^p(\Omega)} \\
&\lesssim p^{\frac{k}{2}} N^{\theta} \sup_{m\in\mathbb{Z}^d} \max_{(B,C)} \|\widetilde{h}^m_{(b_0',b_1),(c_0',c_1),n_A}\|_{(b_0',b_1),n_B\to(c_0',c_1),n_C}
\end{aligned} \tag{2.28}$$

for any $p > \max\big(\frac{d}{\theta}, \frac{2a_2}{\theta}\big) \gg 1$, where we applied Part (i) of this lemma in estimating the contribution from $|m| < 20C_1N$ (since we have $|b_0|, |c_0| < 21C_1N$ under $|m| < 20C_1N$). This proves (2.16). □

Remark 2.8. (i) Instead of the hypothesis (ii.c) Lemma 2.6 (ii), suppose that

(ii.c') except for $O(N)$ many values of $m \in \mathbb{Z}^d$, h^m defined in (2.13) is determined by $(b_0', b_1, c_0', c_1, n_A)$ and $|b_0' + m|^2 - |c_0' + m|^2$.

Then, proceeding as in the proof of Lemma 2.6 (iii), we see that the conclusion of Lemma 2.6 (ii) holds.

(ii) From (2.14) with (2.9), we have

$$\big\| \|H_{bc}\|_{b\to c} \big\|_{L^p(\Omega)} \lesssim p^{\frac{k}{2}} N^{\theta} \max_{(B,C)} \|h\|_{bn_B\to cn_C},$$

which is closer to the format of [24, Proposition 4.15]. A similar comment applies to Lemma 2.6 (iii).

Remark 2.9. As it is stated, the random tensor estimate (Lemma 2.6) is for linear operators. However, given a random multilinear operator of the form:

$$H_{b_1,\dots,b_k,c} = \sum_{n_A} h_{b_1,\dots,b_k,c,n_A} \prod_{n\in\mathbb{Z}^d} \mathfrak{H}_{k_n}(g_n), \quad b_1,\dots,b_k \in \mathbb{Z}^d,\ c \in (\mathbb{Z}^d)^{q-k},$$

acting on $u_{b_1}, \dots, u_{b_k}$ (where k_n is as in (2.12)), we can view it as a linear operator acting on the tensor product $u_{b_1} \otimes \cdots \otimes u_{b_k}$ to which we can apply Lemma 2.6. By noting[10]

$$\|u_{b_1} \otimes \cdots \otimes u_{b_k}\|_{\bigotimes_{j=1}^k \ell^2_{b_j}} = \prod_{j=1}^k \|u_{k_j}\|_{\ell^2_{b_j}},$$

we then obtain a multilinear bound for $H_{b_1,\dots,b_k,c}$. We refer readers to [10, Lemma 8.1] on an improvement of this (trivial) argument in the bilinear case.

[10]Note that a bound $\|u_1 \otimes \cdots \otimes u_k\|_{\bigotimes_{j=1}^k B_j} \le \prod_{j=1}^k \|u_j\|_{B_j}$ holds true for any tensor norm (such as the projective norm; see [57, Section 2.1 and Chapter 6]).

2.4. **Lattice point counting & tensor norm estimates.** Given dyadic $N, N_1, N_2, N_3 \geq 1$ and $m \in \mathbb{Z}$, define the base tensor $h^{\mathbf{N},(m)}$ by

$$\begin{aligned} h^{\mathbf{N},(m)} &= h^{N,N_1,N_2,N_3,(m)}(n, n_1, n_2, n_3) \\ &= \mathbf{1}_{|n|\sim N} \cdot \Big(\prod_{j=1}^{3} \mathbf{1}_{|n_j|\sim N_j} \Big) \cdot \mathbf{1}_{n\neq n_1,n_3} \cdot \mathbf{1}_{\{n=n_1-n_2+n_3\}} \cdot \mathbf{1}_{\{\phi(\bar{n})=m\}}, \end{aligned} \tag{2.29}$$

where $\phi(\bar{n})$ is given by

$$\phi(\bar{n}) = |n|^2 - |n_1|^2 + |n_2|^2 - |n_3|^2. \tag{2.30}$$

Our main goal in this subsection is to establish basic bounds on the base tensor $h^{\mathbf{N},(m)}$; see Lemma 2.12. In Section 5 - 8, we estimate the random tensor terms I - IV by applying the random tensor estimate (Lemma 2.6) and the tensor norm estimates (Lemma 2.12) on the base tensor $h^{\mathbf{N},(m)}$ (with weights).

For this purpose, we first establish some counting estimates.

Lemma 2.10. (i) *Given $m \in \mathbb{Z} \setminus \{0\}$ and $a_0, b_0 \in \mathbb{Z}$, we have*

$$\big|\big\{(a,b) \in \mathbb{Z}^2 : m = ab,\ |a - a_0| \leq M,\ |b - b_0| \leq N\big\}\big| = O(M^\varepsilon N^\varepsilon). \tag{2.31}$$

for any $\varepsilon > 0$.

(ii) *Given dyadic $N, N_1, N_2, N_3 \geq 1$ and $m \in \mathbb{Z}$, define*

$$\begin{aligned} S^{\mathbf{N},(m)} = \big\{(n, n_1, n_2, n_3) \in (\mathbb{Z}^2)^4 : n = n_1 - n_2 + n_3,\ n \neq n_1, n_3,\\ \phi(\bar{n}) = m,\ |n| \sim N, |n_j| \sim N_j,\ j = 1, 2, 3\big\}, \end{aligned} \tag{2.32}$$

where $\phi(\bar{n})$ is as in (2.30). *Given a subset A of $\{n, n_1, n_2, n_3\}$, we then let $S_A^{\mathbf{N},(m)}$ denote the collection of $(n, n_1, n_2, n_3) \in S^{\mathbf{N},(m)}$ when the frequencies in A are fixed. For example, $S_n^{\mathbf{N},(m)}$ denotes the collection of $(n, n_1, n_2, n_3) \in S^{\mathbf{N},(m)}$ when the frequency n is fixed, while*[11] *$S_{n_1 n_2}^{\mathbf{N},(m)}$ denotes the collection of $(n, n_1, n_2, n_3) \in S^{\mathbf{N},(m)}$ when the frequencies n_1 and n_2 are fixed. Then, the following bounds hold*;

$$\begin{aligned}
&\text{(ii.a)} \quad |S^{\mathbf{N},(m)}| \lesssim \min\big(N_1^2 N_3^2 (N \wedge N_2)^\varepsilon, N^2 N_2^2 (N_1 \wedge N_3)^\varepsilon,\\
&\qquad\qquad (N \wedge N_2)^2 (N_1 N_3)^{1+\varepsilon}, (N_1 \wedge N_3)^2 (N N_2)^{1+\varepsilon}\big),\\
&\text{(ii.b)} \quad |S_n^{\mathbf{N},(m)}| \lesssim \min\big(N_2^2 (N_1 \wedge N_3)^\varepsilon, (N_1 N_2)^{1+\varepsilon}, (N_1 N_3)^{1+\varepsilon}, (N_2 N_3)^{1+\varepsilon}\big),\\
&\text{(ii.c)} \quad |S_{n_1}^{\mathbf{N},(m)}| \lesssim \min\big(N_3^2 (N \wedge N_2)^\varepsilon, (N N_2)^{1+\varepsilon}, (N N_3)^{1+\varepsilon}, (N_2 N_3)^{1+\varepsilon}\big),\\
&\text{(ii.d)} \quad |S_{n_2}^{\mathbf{N},(m)}| \lesssim \min\big(N^2 (N_1 \wedge N_3)^\varepsilon, (N N_1)^{1+\varepsilon}, (N N_3)^{1+\varepsilon}, (N_1 N_3)^{1+\varepsilon}\big),\\
&\text{(ii.e)} \quad |S_{n_3}^{\mathbf{N},(m)}| \lesssim \min\big(N_1^2 (N \wedge N_2)^\varepsilon, (N N_1)^{1+\varepsilon}, (N N_2)^{1+\varepsilon}, (N_1 N_2)^{1+\varepsilon}\big),\\
&\text{(ii.f)} \quad |S_{nn_1}^{\mathbf{N},(m)}| \lesssim \min(N_2, N_3),\\
&\text{(ii.g)} \quad |S_{nn_2}^{\mathbf{N},(m)}| \lesssim \min(N_1, N_3)^\varepsilon,\\
&\text{(ii.h)} \quad |S_{nn_3}^{\mathbf{N},(m)}| \lesssim \min(N_1, N_2),\\
&\text{(ii.i)} \quad |S_{n_1 n_2}^{\mathbf{N},(m)}| \lesssim \min(N, N_3),\\
&\text{(ii.j)} \quad |S_{n_1 n_3}^{\mathbf{N},(m)}| \lesssim \min(N, N_2)^\varepsilon,
\end{aligned}$$

[11] For simplicity of notation, we used $n_1 n_2$ to denote the set $\{n_1, n_2\}$.

(ii.k) $|S^{\mathbf{N},(m)}_{n_2 n_3}| \lesssim \min(N, N_1)$,

where $a \wedge b = \min(a, b)$.

Proof. (i) The bound (2.31) follows from a divisor counting argument; see [23, Lemma 4.4] for details.

(ii) Some of these counting estimates and similar forms have been previously studied; see, for example, [6, Lemmas 1 and 2], [26, Section 4], [23, Lemma 4.4], and [25, Lemma 2.5].[12]

We first note that the bound (ii.f) is trivial. Given n and n_1, we fix n_2 which has $O(N_2)$ many choices. This in turn determines n_3, yielding $|S^{\mathbf{N},(m)}_{nn_1}| \lesssim N_2$. By symmetry, we obtain $|S^{\mathbf{N},(m)}_{nn_1}| \lesssim N_3$. The bounds (ii.h), (ii.i), and (ii.k) follow from a similar consideration.

Next, we prove (ii.a). We will discuss four different ways to count the lattices in $S^{\mathbf{N},(m)}$. First, note that there are $O(N_1^2 N_3^2)$ many choices for n_1 and n_3. With fixed n_1 and n_3, we count the number of choices for n and n_2. From (2.32) with (2.30), it suffices to count n satisfying

$$|n| \sim N \quad \text{and} \quad |n|^2 + |n - n_1 - n_3|^2 = c_1(n_1, n_3, m), \tag{2.33}$$

or n_2 satisfying

$$|n_2| \sim N_2 \quad \text{and} \quad |n_2|^2 + |n_2 - n_1 - n_3|^2 = c_1(n_1, n_3, m), \tag{2.34}$$

where c_1 is a constant depending on n_1, n_3, and m. By rewriting the second condition in (2.33) as

$$|2n - n_1 - n_3|^2 = c_2(n_1, n_3, m),$$

it suffices to count lattice points n with $|n| \sim N$ on a given circle (of radius $r > 0$), which is bounded by $O(N^\varepsilon)$; see the proof of [6, Lemma 1]. A similar argument shows that the number of choices of n_2 satisfying (2.34) is $O(N_2^\varepsilon)$. Hence, we obtain $|S^{\mathbf{N},(m)}_{n_1 n_3}| \lesssim (N \wedge N_2)^\varepsilon$, which implies $|S^{\mathbf{N},(m)}| \lesssim N_1^2 N_3^2 (N \wedge N_2)^\varepsilon$. By symmetry, we have $|S^{\mathbf{N},(m)}_{nn_2}| \lesssim (N_1 \wedge N_3)^\varepsilon$, which implies $|S^{\mathbf{N},(m)}| \lesssim N^2 N_2^2 (N_1 \wedge N_3)^\varepsilon$.

Let us proceed differently by fixing n_2 first. Then, it suffices to count (n_1, n_3) satisfying

$$2(n_2 - n_1) \cdot (n_2 - n_3) = m.$$

We denote the first (and second) coordinates of n_j by $n_j^{(1)}$ (and $n_j^{(2)}$, respectively). Let

$$(n_2^{(i)} - n_1^{(i)})(n_2^{(i)} - n_3^{(i)}) = c_i$$

for $i = 1, 2$. We first consider the case $c_1 = c_2 = 0$. Due to the condition $n_2 \neq n_1, n_3$, without loss of generality, we may assume $n_2^{(1)} = n_1^{(1)}$ and $n_2^{(2)} = n_3^{(2)}$. Namely, $n_1^{(1)}$ and $n_3^{(2)}$ are fixed since n_2 has been fixed. Then, there are $O(N_1 N_3)$ many choices for $n_1^{(2)}$ and $n_3^{(1)}$, (and thus for n_1 and n_3), which implies $|S^{\mathbf{N},(m)}| \lesssim N_2^2 (N_1 N_3)$ in this case. In the following, we assume that $c_1 \neq 0$ or $c_2 \neq 0$. By symmetry, we only consider the case $c_2 \neq 0$. By fixing the first coordinates $n_1^{(1)}$ and $n_3^{(1)}$ (which have $O(N_1 N_3)$ many choices), we count the number of choices for the second coordinates $n_1^{(2)}$ and $n_3^{(2)}$, satisfying

$$(n_2^{(2)} - n_1^{(2)})(n_2^{(2)} - n_3^{(2)}) = c_2$$

[12] Lemma 2.4 in the arXiv version.

for some fixed $c_2 \in \mathbb{R} \setminus \{0\}$. It follows from (2.31) that $|S_{n_2}^{\mathbf{N},(m)}| \lesssim (N_1N_3)^{1+\varepsilon}$, which implies $|S^{\mathbf{N},(m)}| \lesssim N_2^2(N_1N_3)^{1+\varepsilon}$. By symmetry, we obtain $|S_n^{\mathbf{N},(m)}| \lesssim (N_1N_3)^{1+\varepsilon}$, which implies $|S^{\mathbf{N},(m)}| \lesssim N^2(N_1N_3)^{1+\varepsilon}$. Also, $|S_{n_j}^{\mathbf{N},(m)}| \lesssim (NN_2)^{1+\varepsilon}$, $j = 1, 3$, and $|S^{\mathbf{N},(m)}| \lesssim (N_1 \wedge N_3)^2(NN_2)^{1+\varepsilon}$.

Lastly, we prove (ii.b). The bounds (ii.c), (ii.d), and (ii.e) follow from symmetry. We have already shown $|S_n^{\mathbf{N},(m)}| \lesssim (N_1N_3)^{1+\varepsilon}$. Given n, we fix n_2 which has $O(N_2^2)$ many choices. Then, by repeating the first part of the proof of (ii.a), we see that there are $O((N_1 \wedge N_3)^{\varepsilon})$ many choices for n_1 and n_3, which yields $|S_n^{\mathbf{N},(m)}| \lesssim N_2^2(N_1 \wedge N_3)^{\varepsilon}$. It remains to prove $|S_n^{\mathbf{N},(m)}| \lesssim (N_1N_2)^{1+\varepsilon}$, since $|S_n^{\mathbf{N},(m)}| \lesssim (N_2N_3)^{1+\varepsilon}$ follows from symmetry. Given n, it suffices to count (n_1, n_2) satisfying

$$2(n_2 - n_1) \cdot (n_1 - n) = m. \tag{2.35}$$

By writing (2.35) as

$$(n_2^{(i)} - n_1^{(i)})(n_1^{(i)} - n^{(i)}) = c_i, \quad i = 1, 2,$$

we first consider the case $c_1 = c_2 = 0$. Due to the condition $n_1 \neq n, n_2$, without loss of generality, we may assume $n_1^{(1)} = n^{(1)}$ and $n_1^{(2)} = n_2^{(2)}$. Given n, there are $O(N_1)$ many choices for $n_1^{(2)}$ (which fixes $n_2^{(2)}$) and $O(N_2)$ many choices for $n_2^{(1)}$.

Next, we consider the case $c_1 \neq 0$ or $c_2 \neq 0$. By symmetry, we only consider the case $c_2 \neq 0$. By fixing the first coordinates $n_1^{(1)}$ and $n_2^{(1)}$ (which have $O(N_1N_2)$ many choices), we count the number of choices for the second coordinates $n_1^{(2)}$ and $n_2^{(2)}$, satisfying

$$(n_2^{(2)} - n_1^{(2)})(n_1^{(2)} - n^{(2)}) = c_2$$

for some fixed $c_2 \in \mathbb{R} \setminus \{0\}$. It follows from (2.31) (with $a = n_2^{(2)} - n_1^{(2)}$, $a_0 = 0$, $b = n_1^{(2)} - n^{(2)}$, and $b_0 = -n^{(2)}$) that $|S_n^{\mathbf{N},(m)}| \lesssim (N_1N_2)^{1+\varepsilon}$. □

Remark 2.11. Note that the condition $n \neq n_1, n_3$ in (2.32) is crucial in the cases where we used the divisor counting argument. Define a set $\widetilde{S}^{\mathbf{N},(m)}$ by (2.32) after dropping the condition $n \neq n_1, n_3$. Let $m = 0$ and consider

$$\widetilde{S}^{\mathbf{N},(0)} \cap \{n = n_1\} = \{(n_1, n_2) \in (\mathbb{Z}^2)^2 : |n_j| \sim N_j,\ j = 1, 2\}.$$

and thus $|\widetilde{S}^{\mathbf{N},(0)}| \gtrsim N_1^2N_2^2$. When $N \sim N_1 \gg N_2 \sim N_3$, the bound above is much worse than (ii.a) in Lemma 2.10.

The next lemma follows from Schur's test and the counting lemma (Lemma 2.10).

Lemma 2.12. *Given $\varepsilon > 0$, the following estimates hold*;

$$\begin{aligned}
\|h^{\mathbf{N},(m)}\|_{n_1n_2n_3\to n} &\lesssim \min\big(N_2(N_1\wedge N_3)^{\varepsilon}, (N_{\mathrm{med}}N_{\mathrm{min}})^{\frac{1}{2}+\varepsilon}\big),\\
\|h^{\mathbf{N},(m)}\|_{n_1\to nn_2n_3} &\lesssim \min\big(N_3(N\wedge N_2)^{\varepsilon}, (NN_2)^{\frac{1}{2}+\varepsilon}, (NN_3)^{\frac{1}{2}+\varepsilon}, (N_2N_3)^{\frac{1}{2}+\varepsilon}\big),\\
\|h^{\mathbf{N},(m)}\|_{n_2\to nn_1n_3} &\lesssim \min\big(N(N_1\wedge N_3)^{\varepsilon}, (NN_1)^{\frac{1}{2}+\varepsilon}, (NN_3)^{\frac{1}{2}+\varepsilon}, (N_1N_3)^{\frac{1}{2}+\varepsilon}\big),\\
\|h^{\mathbf{N},(m)}\|_{n_3\to nn_1n_2} &\lesssim \min\big(N_1(N\wedge N_2)^{\varepsilon}, (NN_1)^{\frac{1}{2}+\varepsilon}, (NN_2)^{\frac{1}{2}+\varepsilon}, (N_1N_2)^{\frac{1}{2}+\varepsilon}\big),\\
\|h^{\mathbf{N},(m)}\|_{n_2n_3\to nn_1} &\lesssim \min(N,N_1)^{\frac{1}{2}}\cdot\min(N_2,N_3)^{\frac{1}{2}},\\
\|h^{\mathbf{N},(m)}\|_{n_1n_3\to nn_2} &\lesssim \min(N,N_2)^{\varepsilon}\cdot\min(N_1,N_3)^{\varepsilon},\\
\|h^{\mathbf{N},(m)}\|_{n_1n_2\to nn_3} &\lesssim \min\big(N,N_3\big)^{\frac{1}{2}}\cdot\min(N_1,N_2)^{\frac{1}{2}},
\end{aligned} \tag{2.36}$$

uniformly in dyadic $N, N_1, N_2, N_3 \geq 1$ and $m \in \mathbb{Z}$, where N_{med} and N_{min} are as in (2.2).

Proof. By Schur's test with (2.29),

$$\begin{aligned}
\|h^{\mathbf{N},(m)}\|_{n_1n_2n_3\to n} &\leq \bigg(\sup_{n\in\mathbb{Z}^2}\sum_{n_1,n_2,n_3\in\mathbb{Z}^2} h^{\mathbf{N},(m)}\bigg)^{\frac{1}{2}}\bigg(\sup_{n_1,n_2,n_3\in\mathbb{Z}^2}\sum_{n\in\mathbb{Z}^2} h^{\mathbf{N},(m)}\bigg)^{\frac{1}{2}}\\
&= |S_n^{\mathbf{N},(m)}|^{\frac{1}{2}}|S_{n_1,n_2,n_3}^{\mathbf{N},(m)}|^{\frac{1}{2}}\\
&\leq |S_n^{\mathbf{N},(m)}|^{\frac{1}{2}},
\end{aligned}$$

where the third step follows from the fact that n is uniquely determined by n_1, n_2, and n_3 under $n = n_1 - n_2 + n_3$. Then, the first bound in (2.36) follows from Lemma 2.10.

The other bounds in (2.36) follow from Lemma 2.10, once we note

$$\begin{aligned}
\|h^{\mathbf{N},(m)}\|_{n_1\to nn_2n_3} &\leq |S_{n_1}^{\mathbf{N},(m)}|^{\frac{1}{2}},\\
\|h^{\mathbf{N},(m)}\|_{n_2\to nn_1n_3} &\leq |S_{n_2}^{\mathbf{N},(m)}|^{\frac{1}{2}},\\
\|h^{\mathbf{N},(m)}\|_{n_3\to nn_1n_2} &\leq |S_{n_3}^{\mathbf{N},(m)}|^{\frac{1}{2}},\\
\|h^{\mathbf{N},(m)}\|_{n_2n_3\to nn_1} &\leq |S_{n_2n_3}^{\mathbf{N},(m)}|^{\frac{1}{2}}|S_{nn_1}^{\mathbf{N},(m)}|^{\frac{1}{2}},\\
\|h^{\mathbf{N},(m)}\|_{n_1n_3\to nn_2} &\leq |S_{n_1n_3}^{\mathbf{N},(m)}|^{\frac{1}{2}}|S_{nn_2}^{\mathbf{N},(m)}|^{\frac{1}{2}},\\
\|h^{\mathbf{N},(m)}\|_{n_1n_2\to nn_3} &\leq |S_{n_1n_2}^{\mathbf{N},(m)}|^{\frac{1}{2}}|S_{nn_3}^{\mathbf{N},(m)}|^{\frac{1}{2}},
\end{aligned}$$

each of which follows from Schur's test as above. □

3. Resonant contribution

In this section, we estimate the resonant term $\mathcal{R}(w+z)$ by following the presentation in [19, Subsection 4.2]. Due to the multilinearity and the symmetry, it suffices to treat

$$\mathcal{R}(w,w,w), \quad \mathcal{R}(z,z,z), \quad \mathcal{R}(w,z,z), \quad \text{and} \quad \mathcal{R}(w,w,z).$$

Since the complex conjugate on the second argument of $\mathcal{R}(v_1,v_2,v_3)$ in (1.19) does not play any role, we drop the complex conjugate sign in the following.

• **Case 1:** $\mathcal{R}(w,w,w)$.
This case is already included in (1.17), giving

$$\|\mathcal{R}(w,w,w)\|_{X_T^{s,-\frac{1}{2}+2\varepsilon}} \lesssim 1$$

for any w with $\|w\|_{X_T^{s,\frac{1}{2}+\varepsilon}} \leq 1$ and $0 < T \leq 1$.

• **Case 2:** $\mathcal{R}(z,z,z)$.
By Hölder's inequality ($\frac{1}{2} = \frac{1}{p} + \frac{1}{q}$ with $p \gg 1$ and $q > 2$ close to 2 such that $(\frac{1}{2} - 2\varepsilon)q > 1$), (1.21), Young's inequality, (2.3), and Lemma 2.3, we have

$$\begin{aligned}
&\|\mathcal{R}(z,z,z)\|_{X_T^{s,-\frac{1}{2}+2\varepsilon}}\\
&\lesssim \sup_{n\in\mathbb{Z}^2} \|\langle \tau + |n|^2\rangle^{-\frac{1}{2}+2\varepsilon}\|_{L^q_\tau} \bigg\| \langle n\rangle^{s-3}|g_n|^3 \int_{\tau=\tau_1-\tau_2+\tau_3} \prod_{j=1}^{3} \widehat{\eta}_T(\tau_j + |n|^2) d\tau_1 d\tau_2 \bigg\|_{\ell^2_n L^p_\tau}\\
&\lesssim T^{1-\frac{1}{p}} \big\| \langle n\rangle^{s-3}|g_n|^3 \big\|_{\ell^2_n} \lesssim T^{1-\frac{1}{p}-3\delta_1} \|\langle n\rangle^{s-3+3\delta_2}\|_{\ell^2_n} \lesssim 1,
\end{aligned}$$

yielding (1.23), outside an exceptional set of probability $< e^{-\frac{c_1}{T^{c_2}}}$, provided that $\delta_1, \delta_2 > 0$ are sufficiently small and $s < 2 - 3\delta_2$.

• **Case 3:** $\mathcal{R}(w,z,z)$.
By Hölder's inequality (with p and q as in Case 2), (1.21), Young's inequality, (2.3), and Lemma 2.3, we have

$$\begin{aligned}
&\|\mathcal{R}(w,z,z)\|_{X_T^{s,-\frac{1}{2}+2\varepsilon}}\\
&\lesssim \bigg\| \langle n\rangle^{s-2}|g_n|^2 \int_{\tau=\tau_1-\tau_2+\tau_3} \widehat{\widetilde{w}}(n,\tau_1) \prod_{j=2}^{3} \widehat{\eta}_T(\tau_j + |n|^2) d\tau_1 d\tau_2 \bigg\|_{\ell^2_n L^p_\tau} \qquad (3.1)\\
&\lesssim T^{\frac{1}{2}-\frac{1}{p}} \Big(\sup_{n\in\mathbb{Z}^2} \langle n\rangle^{-2}|g_n|^2 \Big) \|\langle n\rangle^s \widehat{\widetilde{w}}(n,\tau)\|_{\ell^2_n L^2_\tau} \lesssim T^{\frac{1}{2}-\frac{1}{p}-2\delta_1} \|\widetilde{w}\|_{X^{s,0}}
\end{aligned}$$

for any extension $\widetilde{w}$ of w with $\|w\|_{X_T^{s,\frac{1}{2}+\varepsilon}} \leq 1$, outside an exceptional set of probability $< e^{-\frac{c_1}{T^{c_2}}}$, provided that $\delta_1, \delta_2 > 0$ are sufficiently small. Hence, by taking an infimum over all extensions $\widetilde{w}$ of w, it follows from (3.1) and (2.4) that

$$\|\mathcal{R}(w,z,z)\|_{X_T^{s,-\frac{1}{2}+2\varepsilon}} \lesssim \|w\|_{X_T^{s,\frac{1}{2}+\varepsilon}} \leq 1$$

outside an exceptional set of probability $< e^{-\frac{c_1}{T^{c_2}}}$, yielding (1.23) in this case.

• **Case 4:** $\mathcal{R}(w,w,z)$.
By Hölder's inequality (with p and q as in Case 2), (1.21), Young's inequality with (2.3),

Lemma 2.3, Hölder's inequality (in τ), and $\ell^2_n \subset \ell^4_n$, we have

$$\begin{aligned}
&\|\mathcal{R}(w,w,z)\|_{X_T^{s,-\frac{1}{2}+2\varepsilon}}\\
&\lesssim \Big\| \langle n\rangle^{s-1}|g_n| \int_{\tau=\tau_1-\tau_2+\tau_3} \prod_{j=1}^{2} \widehat{\widetilde{w}}(n,\tau_j)\cdot \widehat{\eta}_T(\tau_3+|n|^2)d\tau_1 d\tau_2 \Big\|_{\ell^2_n L^p_\tau}\\
&\lesssim T^{\frac{1}{2}-\frac{1}{p}} \Big(\sup_{n\in\mathbb{Z}^2} \langle n\rangle^{-\delta_2}|g_n| \Big) \|\langle n\rangle^s \widehat{\widetilde{w}}(n,\tau)\|^2_{\ell^4_n L^{\frac{4}{3}}_\tau}\\
&\lesssim T^{\frac{1}{2}-\frac{1}{p}-\delta_1} \|\widetilde{w}\|^2_{X^{s,\frac{1}{4}+\varepsilon}}
\end{aligned} \tag{3.2}$$

for any extension $\widetilde{w}$ of w with $\|w\|_{X_T^{s,\frac{1}{2}+\varepsilon}} \leq 1$, outside an exceptional set of probability $< e^{-\frac{c_1}{T^{c_2}}}$, provided that $\delta_1, \delta_2 > 0$ are sufficiently small and $s \geq -1+\delta_2$. Hence, by taking an infimum over all extensions $\widetilde{w}$ of w, it follows from (3.2) and (2.4) that

$$\|\mathcal{R}(w,w,z)\|_{X_T^{s,-\frac{1}{2}+2\varepsilon}} \lesssim \|w\|^2_{X_T^{s,\frac{1}{2}+\varepsilon}} \leq 1$$

outside an exceptional set of probability $< e^{-\frac{c_1}{T^{c_2}}}$, yielding (1.23) in this case.

Therefore, the conclusion of Proposition 1.4 holds for the resonant term $\mathcal{R}(w+z)$.

4. Purely stochastic term

In this section, we study the purely stochastic term $\mathcal{N}(z,z,z)$, where $\mathcal{N}$ and z are as in (1.19) and (1.21), respectively.

Proposition 4.1. *Let $s < \frac{1}{2}$. Given small $\varepsilon > 0$, there exists $\theta > 0$ such that*

$$\Big\| \|\mathcal{N}(z,z,z)\|_{X_T^{s,-b'}} \Big\|_{L^p(\Omega)} \lesssim p^{\frac{3}{2}} T^\theta \tag{4.1}$$

for any finite $p \geq 1$ and $0 < T \ll 1$, where $b' = \frac{1}{2} - 2\varepsilon$ is as in (2.6). In particular, the conclusion of Proposition 1.4 holds for the purely stochastic term $\mathcal{N}(z,z,z)$.

Proof. We first prove

$$\Big\| \|\mathbf{Q}_N \mathcal{N}(\mathbf{Q}_{N_1} z, \mathbf{Q}_{N_2} z, \mathbf{Q}_{N_3} z)\|_{X_T^{s,-b'}} \Big\|_{L^p(\Omega)} \lesssim p^{\frac{3}{2}} T^\theta \frac{N_{\max}^\delta N^s}{N_1 N_2 N_3} \sup_{m\in\mathbb{Z}} |S^{\mathbf{N},(m)}|^{\frac{1}{2}} \tag{4.2}$$

for small $\theta > 0$ and any small $\delta > 0$, uniformly in dyadic $N, N_1, N_2, N_3 \geq 1$, where $\mathbf{Q}_N$ is the Littlewood-Paley projector and $S^{\mathbf{N},(m)}$ is as in (2.32).

From (1.19) and (1.21), we have

$$\begin{aligned}
&\mathcal{F}_{x,t}\big(e^{-it\Delta}\eta_T \mathbf{Q}_N \mathcal{N}(\mathbf{Q}_{N_1} z, \mathbf{Q}_{N_2} z, \mathbf{Q}_{N_3} z)\big)(n,\tau)\\
&\quad = \mathbf{1}_{|n|\sim N} \sum_{\substack{n=n_1-n_2+n_3\\ n\neq n_1,n_3}} \widehat{\eta}_T(\tau-\phi(\bar{n})) \Big(\prod_{j=1}^{3} \frac{g^*_{n_j}}{\langle n_j\rangle} \mathbf{1}_{|n_j|\sim N_j} \Big),
\end{aligned}$$

where $\phi(\bar{n})$ is as in (2.30) and $g^*_{n_j}$ is defined by

$$g^*_{n_j} = \begin{cases} g_{n_j}, & \text{if } j = 1,3,\\ \overline{g_{n_2}}, & \text{if } j = 2. \end{cases} \tag{4.3}$$

Then, from Minkowski's integral inequality, the Wiener chaos estimate (Lemma 2.4), and the orthogonality of $\big\{\mathbf{1}_{n_2\neq n_1,n_3}\prod_{j=1}^3 g_{n_j}^*\big\}_{n_1,n_2,n_3\in\mathbb{Z}^2}$ in $L^2(\Omega)$ (as elements in the homogeneous Wiener chaos of order 3), we have

$$\begin{aligned}
&\text{LHS of (4.2)}\\
&\lesssim p^{\frac{3}{2}}\bigg\|\langle\tau\rangle^{-b'}\langle n\rangle^{s}\bigg\|_{\substack{n=n_1-n_2+n_3\\ n\neq n_1,n_3}}\widehat{\eta}_T(\tau-\phi(\bar{n}))\bigg(\prod_{j=1}^3\frac{g_{n_j}^*}{\langle n_j\rangle}\mathbf{1}_{|n_j|\sim N_j}\bigg)\bigg\|_{L^2(\Omega)}\bigg\|_{L^2_\tau\ell^2_{|n|\sim N}} \qquad (4.4)\\
&\lesssim p^{\frac{3}{2}}\bigg\|\langle\tau\rangle^{-b'}\langle n\rangle^{s}\mathbf{1}_{A(n)}\widehat{\eta}_T(\tau-\phi(\bar{n}))\prod_{j=1}^3\frac{1}{\langle n_j\rangle}\bigg\|_{L^2_\tau\ell^2_{|n|\sim N}\ell^2_{n_1,n_2}},
\end{aligned}$$

where $A(n)$ is given by

$$A(n)=\big\{(n_1,n_2,n_3)\in(\mathbb{Z}^2)^3 : n=n_1-n_2+n_3,\ n\neq n_1,n_3,\ |n_j|\sim N_j\big\}. \qquad (4.5)$$

From Lemma 2.2 with (2.6), we have

$$\begin{aligned}
\int_{\mathbb{R}}\langle\tau\rangle^{-2b'}|\widehat{\eta}_T(\tau-m)|^2d\tau &\lesssim T^2\sum_{\mu\in\mathbb{Z}}\int_{\mu-\frac{1}{2}}^{\mu+\frac{1}{2}}\frac{1}{\langle\tau\rangle^{2b'}\langle T(\tau-m)\rangle^{1+\varepsilon}}d\tau\\
&\lesssim T^{1-\varepsilon}\sum_{\mu\in\mathbb{Z}}\frac{1}{\langle\mu\rangle^{2b'}\langle\mu-m\rangle^{1+\varepsilon}} \qquad (4.6)\\
&\lesssim T^{1-\varepsilon}\langle m\rangle^{-2b'}
\end{aligned}$$

for any $0<T\leq 1$, uniformly in $m\in\mathbb{Z}$. Thus, from (4.4) and (4.6), we obtain

$$\text{LHS of (4.2)}\lesssim p^{\frac{3}{2}}T^{\theta}\bigg\|\langle m\rangle^{-b'}\langle n\rangle^{s}\mathbf{1}_{A(n)}\mathbf{1}_{\phi(\bar{n})=m}\prod_{j=1}^3\frac{1}{\langle n_j\rangle}\bigg\|_{\ell^2_m(\mathbb{Z})\ell^2_{|n|\sim N}\ell^2_{n_1,n_2}} \qquad (4.7)$$

for some $\theta>0$. Recall from (2.6) that $-2b'=-1+4\varepsilon$. Then, by noting from (2.30) that the summand in (4.7) is non-trivial only for $|m|\lesssim N_{\max}^2$ (where $N_{\max}$ is as in (2.2)), we have

$$\begin{aligned}
\text{LHS of (4.2)}&\lesssim p^{\frac{3}{2}}T^{\theta}\frac{N_{\max}^{4\varepsilon}N^s}{N_1N_2N_3}\sup_{m\in\mathbb{Z}}\bigg(\sum_{\substack{n,n_1,n_2,n_3\in\mathbb{Z}^2\\ n=n_1-n_2+n_3\\ n\neq n_1,n_3\\ \phi(\bar{n})=m}}\mathbf{1}_{|n|\sim N}\prod_{j=1}^3\mathbf{1}_{|n_j|\sim N_j}\bigg)^{\frac{1}{2}}\\
&= p^{\frac{3}{2}}T^{\theta}\frac{N_{\max}^{4\varepsilon}N^s}{N_1N_2N_3}\sup_{m\in\mathbb{Z}}|S^{\mathbf{N},(m)}|^{\frac{1}{2}}.
\end{aligned}$$

This proves (4.2).

From (ii.a) in Lemma 2.10, we have

$$|S^{\mathbf{N},(m)}|^{\frac{1}{2}}\lesssim N_{\max}^{\frac{1}{2}+\varepsilon}N_{\text{med}}^{\frac{1}{2}+\varepsilon}N_{\min}, \qquad (4.8)$$

where $N_{\max}$, $N_{\med}$, and $N_{\min}$ are as in (2.2). Then, by summing (4.2) over dyadic blocks $N, N_1, N_2, N_3 \geq 1$ and applying (4.8), we have

$$\begin{aligned}
\Big\| \|\mathcal{N}(z,z,z)\|_{X_T^{s,-b'}} \Big\|_{L^p(\Omega)} &\lesssim p^{\frac{3}{2}} T^\theta \sup_{\substack{N,N_1,N_2,N_3\geq 1\\ \text{dyadic}}} \frac{N_{\max}^{5\varepsilon} N^s}{N_1 N_2 N_3} \sup_{m\in\mathbb{Z}} |S^{\mathbf{N},(m)}|^{\frac{1}{2}} \\
&\lesssim p^{\frac{3}{2}} T^\theta \sup_{\substack{N,N_1,N_2,N_3\geq 1\\ \text{dyadic}}} N_{\max}^{-\frac{1}{2}+s+6\varepsilon} N_{\med}^{-\frac{1}{2}+\varepsilon} \\
&\lesssim p^{\frac{3}{2}} T^\theta,
\end{aligned}$$

provided that $s < \frac{1}{2}$ and $\varepsilon > 0$ is sufficiently small. This proves (4.1). The second claim in Proposition 4.1 follows from (4.1) and Chebyshev's inequality (see also Lemma 4.5 in [61]). □

5. Random tensor term I

In this section, we estimate the random tensor term $\mathcal{N}(z,w,z)$. Define the random operator $\mathcal{T}_1$ by setting

$$\mathcal{T}_1(w) = \mathcal{N}(z,w,z), \tag{5.1}$$

where z denotes the random linear solution in (1.21). Then, our main goal is to prove the following proposition.

Proposition 5.1. *Given small $s > 0$ and $\varepsilon > 0$, there exists $\theta > 0$ such that*

$$\Big\| \|\mathcal{T}_1\|_{X_T^{s,b}\to X_T^{s,-b'}} \Big\|_{L^p(\Omega)} \lesssim pT^\theta \tag{5.2}$$

for any finite $p \geq 1$ and $0 < T \ll 1$, where $b = \frac{1}{2} + \varepsilon$ and $b' = \frac{1}{2} - 2\varepsilon$ are as in (2.6). In particular, the conclusion of Proposition 1.4 holds for the random tensor term $\mathcal{N}(z,w,z)$.

Proof. We write

$$\mathcal{T}_1(w) = \sum_{\substack{N_1,N_3\geq 1\\ \text{dyadic}}} \mathcal{T}_1^{N_1,N_3}(w),$$

where $\mathcal{T}_1^{N_1,N_3}$ is given by

$$\mathcal{T}_1^{N_1,N_3}(w) = \mathcal{N}(\mathbf{Q}_{N_1} z, w, \mathbf{Q}_{N_3} z). \tag{5.3}$$

Here, $\mathbf{Q}_{N_j}$ denotes the Littlewood-Paley projector. In the following, we prove that there exist small $\theta, \delta > 0$ such that

$$\Big\| \|\mathcal{T}_1^{N_1,N_3}\|_{X_T^{s,b}\to X_T^{s,-b'}} \Big\|_{L^p(\Omega)} \lesssim pT^\theta (N_1 N_3)^{-\delta} \tag{5.4}$$

for any finite $p \geq 1$ and $0 < T \ll 1$. Then, (5.2) follows from summing over dyadic $N_1, N_3 \geq 1$, and the second claim in Proposition 5.1 follows from Chebyshev's inequality.

From (5.3) with (5.1) and (1.19), we have

$$\begin{aligned}
&\mathcal{F}_{x,t}\big(\eta_T \mathcal{T}_1^{N_1,N_3}(w)\big)(n,\tau)\\
&\quad = \sum_{\substack{n=n_1-n_2+n_3\\ n\neq n_1,n_3}} \Big(\prod_{j\in\{1,3\}} \frac{g_{n_j}}{\langle n_j\rangle} \mathbf{1}_{|n_j|\sim N_j}\Big)\\
&\qquad \times \int_{\mathbb{R}} \widehat{\eta}_T(\tau+\tau_2+|n_1|^2-|n_2|^2+|n_3|^2)\overline{\widehat{w}(n_2,\tau_2-|n_2|^2)}d\tau_2.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\mathcal{F}_{x,t}\big(\eta_T \mathcal{T}_1^{N_1,N_3}(w)\big)(n,\tau-|n|^2)\\
&\quad = \langle n\rangle^{-s} \sum_{n_2\in\mathbb{Z}^2} \langle n_2\rangle^s \int_{\mathbb{R}} \mathbf{H}_1^{N_1,N_3}(n,n_2,\tau,\tau_2)\overline{\widehat{w}(n_2,\tau_2-|n_2|^2)}d\tau_2,
\end{aligned} \tag{5.5}$$

where

$$\begin{aligned}
&\mathbf{H}_1^{N_1,N_3}(n,n_2,\tau,\tau_2)\\
&\quad = \frac{\langle n\rangle^s}{\langle n_2\rangle^s} \sum_{\substack{n+n_2=n_1+n_3\\ n\neq n_1,n_3}} \Big(\prod_{j\in\{1,3\}} \frac{g_{n_j}}{\langle n_j\rangle} \mathbf{1}_{|n_j|\sim N_j}\Big) \widehat{\eta}_T(\tau+\tau_2-\phi(\bar{n}))
\end{aligned} \tag{5.6}$$

with $\phi(\bar{n})$ as in (2.30). Then, from (2.4), (5.5), and Cauchy-Schwarz's inequality, we have

$$\begin{aligned}
\|\mathcal{T}_1^{N_1,N_3}(w)\|_{X_T^{s,-b'}} &\le \|\eta_T \mathcal{T}_1^{N_1,N_3}(\widetilde{w})\|_{X^{s,-b'}}\\
&= \big\|\langle n\rangle^s\langle\tau\rangle^{-b'}\mathcal{F}_{x,t}\big(\eta_T \mathcal{T}_1^{N_1,N_3}(\widetilde{w})\big)(n,\tau-|n|^2)\big\|_{\ell^2_n L^2_\tau}\\
&= \Big\|\langle\tau\rangle^{-b'} \sum_{n_2\in\mathbb{Z}^2} \langle n_2\rangle^s \int_{\mathbb{R}} \mathbf{H}_1^{N_1,N_3}(n,n_2,\tau,\tau_2)\overline{\widehat{\widetilde{w}}(n_2,\tau_2-|n_2|^2)}d\tau_2\Big\|_{\ell^2_n L^2_\tau}\\
&\le \Big\|\langle\tau\rangle^{-b'} \int_{\mathbb{R}} \|\mathbf{H}_1^{N_1,N_3}(n,n_2,\tau,\tau_2)\|_{\ell^2_{n_2}\to\ell^2_n} \|\langle n_2\rangle^s \widehat{\widetilde{w}}(n_2,\tau_2-|n_2|^2)\|_{\ell^2_{n_2}} d\tau_2\Big\|_{L^2_\tau}\\
&\le \big\|\langle\tau\rangle^{-b'}\langle\tau_2\rangle^{-b}\|\mathbf{H}_1^{N_1,N_3}(n,n_2,\tau,\tau_2)\|_{\ell^2_{n_2}\to\ell^2_n}\big\|_{L^2_{\tau,\tau_2}} \cdot \|\widetilde{w}\|_{X^{s,b}}
\end{aligned} \tag{5.7}$$

for any extension $\widetilde{w}$ of w. By taking an infimum over all extensions $\widetilde{w}$ of w and then a supremum over w with $\|w\|_{X_T^{s,b}}\le 1$, we obtain

$$\|\mathcal{T}_1^{N_1,N_3}\|_{X_T^{s,b}\to X_T^{s,-b'}} \le \big\|\langle\tau\rangle^{-b'}\langle\tau_2\rangle^{-b}\|\mathbf{H}_1^{N_1,N_3}(n,n_2,\tau,\tau_2)\|_{\ell^2_{n_2}\to\ell^2_n}\big\|_{L^2_{\tau,\tau_2}}. \tag{5.8}$$

Hence, (5.4) follows once we prove

$$\Big\|\big\|\langle\tau\rangle^{-b'}\langle\tau_2\rangle^{-b}\|\mathbf{H}_1^{N_1,N_3}(n,n_2,\tau,\tau_2)\|_{\ell^2_{n_2}\to\ell^2_n}\big\|_{L^2_{\tau,\tau_2}}\Big\|_{L^p(\Omega)} \lesssim pT^\theta(N_1N_3)^{-\delta}. \tag{5.9}$$

Given dyadic $N, N_1, N_2, N_3 \ge 1$, we set

$$\mathbf{H}_1^{N,N_1,N_2,N_3}(n,n_2,\tau,\tau_2) = \mathbf{1}_{|n|\sim N}\mathbf{1}_{|n_2|\sim N_2}\mathbf{H}_1^{N_1,N_3}(n,n_2,\tau,\tau_2). \tag{5.10}$$

By introducing the following short-hand notation:

$$\mathbf{H}_1^{\mathbf{N}}(n,n_2,\tau,\tau_2) = \mathbf{H}_1^{N,N_1,N_2,N_3}(n,n_2,\tau,\tau_2), \tag{5.11}$$

we write

$$\mathbf{H}_1^{N_1,N_3} = \sum_{\substack{N,N_2\geq 1,\text{ dyadic}\\ \max(N_1,N_3)\gtrsim N_2}} \mathbf{H}_1^{\mathbf{N}} + \sum_{\substack{N,N_2\geq 1,\text{ dyadic}\\ N_2\gg \max(N_1,N_3)}} \mathbf{H}_1^{\mathbf{N}}. \tag{5.12}$$

In the following, we separately estimate the first and second terms on the right-hand side of (5.12). Before proceeding further, we state a lemma on the operator norm of $\mathbf{H}_1^{\mathbf{N}}$ (possibly with further frequency localizations).

Lemma 5.2. *Let Q_0 and Q_2 be cubes of side length $\lesssim \max(N_1, N_3)$. Fix small $s > 0$. Then, we have*

$$\sup_{\tau,\tau_2\in\mathbb{R}} \Big\| \|\mathbf{1}_{Q_0}(n)\mathbf{1}_{Q_2}(n_2)\mathbf{H}_1^{\mathbf{N}}(n,n_2,\tau,\tau_2)\|_{\ell^2_{n_2}\to\ell^2_n}\Big\|_{L^p(\Omega)} \lesssim p(N_1N_3)^{-\frac{1}{2}+s+\varepsilon} \tag{5.13}$$

for any finite $p \geq 1$ and dyadic $N, N_1, N_2, N_3 \geq 1$, uniformly in $0 < T \ll 1$ and the cubes Q_0 and Q_2.

We note that the supremum over τ and τ_2 is outside the $L^p(\Omega)$-norm. In the following, we first prove (5.9) by assuming Lemma 5.2. We present a proof of Lemma 5.2 at the end of this section.

- **Case 1:** $\max(N_1, N_3) \gtrsim N_2$.

  In this case, we have

$$\max(N, N_1, N_2, N_3) \sim \max(N_1, N_3), \tag{5.14}$$

since $n = n_1 - n_2 + n_3$. Then, (5.9) follows once we prove

$$\Big\| \big\| \langle\tau\rangle^{-b'}\langle\tau_2\rangle^{-b} \|\mathbf{H}_1^{\mathbf{N}}(n,n_2,\tau,\tau_2)\|_{\ell^2_{n_2}\to\ell^2_n}\big\|_{L^2_{\tau,\tau_2}}\Big\|_{L^p(\Omega)} \lesssim pT^\theta(N_1N_3)^{-2\delta}, \tag{5.15}$$

since the extra factor $(N_1N_3)^{-\delta}$ allows us to sum over dyadic $N, N_2 \geq 1$. Furthermore, we claim that (5.15) follows once we prove

$$\Big\| \big\| \langle\tau\rangle^{-b}\langle\tau_2\rangle^{-b} \|\mathbf{H}_1^{\mathbf{N}}(n,n_2,\tau,\tau_2)\|_{\ell^2_{n_2}\to\ell^2_n}\big\|_{L^2_{\tau,\tau_2}}\Big\|_{L^p(\Omega)} \lesssim p(N_1N_3)^{-3\delta}. \tag{5.16}$$

(Note from (2.6) that $-b = -\frac{1}{2} - \varepsilon < -\frac{1}{2} < -b' = -\frac{1}{2} + 2\varepsilon$.) Indeed, by applying (2.10) and crudely estimating the $\ell^2_{n_2,n}$-norm by the $\ell^1_{n_2,n}$-norm with (5.10) and (5.6), we have

$$\begin{aligned}
&\Big\| \big\| \langle\tau_2\rangle^{-b} \|\mathbf{H}_1^{\mathbf{N}}(n,n_2,\tau,\tau_2)\|_{\ell^2_{n_2}\to\ell^2_n}\big\|_{L^2_{\tau,\tau_2}}\Big\|_{L^p(\Omega)}\\
&\leq \Big\| \big\| \langle\tau_2\rangle^{-b} \|\mathbf{H}_1^{\mathbf{N}}(n,n_2,\tau,\tau_2)\|_{\ell^2_{n_2,n}}\big\|_{L^2_{\tau,\tau_2}}\Big\|_{L^p(\Omega)}\\
&\lesssim N^s \Big\| \Big\| \langle\tau_2\rangle^{-b} \sum_{\substack{n,n_1,n_2\in\mathbb{Z}^2\\ |n|\sim N}} \mathbf{1}_{A(n)} \frac{|g_{n_1}g_{n_3}|}{\langle n_1\rangle\langle n_3\rangle} |\widehat{\eta}_T(\tau+\tau_2-\phi(\bar{n}))|\Big\|_{L^2_{\tau,\tau_2}}\Big\|_{L^p(\Omega)}\\
&\lesssim N^s \Big\| \sum_{\substack{n,n_1,n_2\in\mathbb{Z}^2\\ |n|\sim N}} \mathbf{1}_{A(n)} \frac{|g_{n_1}g_{n_3}|}{\langle n_1\rangle\langle n_3\rangle} \|\langle\tau_2\rangle^{-b}\widehat{\eta}_T(\tau+\tau_2-\phi(\bar{n}))\|_{L^2_{\tau,\tau_2}}\Big\|_{L^p(\Omega)}\\
&\lesssim T^{\frac{1}{2}}N^s \sum_{\substack{n,n_1,n_2\in\mathbb{Z}^2\\ |n|\sim N}} \frac{\mathbf{1}_{A(n)}}{\langle n_1\rangle\langle n_3\rangle} \|g_{n_1}g_{n_3}\|_{L^p(\Omega)} \lesssim pT^{\frac{1}{2}}\max(N_1,N_3)^6
\end{aligned} \tag{5.17}$$

for small $s > 0$, where the set $A(n)$ is as in (4.5). Here, we used (2.3) and the fact that $b > \frac{1}{2}$ in the penultimate step and (5.14) in the last step. Then, (5.15) follows from interpolating (5.16) and (5.17).

We now prove (5.16). Without loss of generality, assume $p \geq 2$. Then, from Minkowski's integral inequality, the fact that $b > \frac{1}{2}$, and Lemma 5.2 with (5.14), we obtain

$$\begin{aligned} \text{LHS of (5.16)} &\leq \Big\| \langle \tau \rangle^{-b} \langle \tau_2 \rangle^{-b} \big\| \|\mathbf{H}_1^{\mathbf{N}}(n, n_2, \tau, \tau_2)\|_{\ell^2_{n_2} \to \ell^2_n} \big\|_{L^p(\Omega)} \Big\|_{L^2_{\tau,\tau_2}} \\ &\lesssim \sup_{\tau,\tau_2 \in \mathbb{R}} \Big\| \|\mathbf{H}_1^{\mathbf{N}}(n, n_2, \tau, \tau_2)\|_{\ell^2_{n_2} \to \ell^2_n} \Big\|_{L^p(\Omega)} \\ &\lesssim p(N_1 N_3)^{-\frac{1}{4}} \end{aligned} \tag{5.18}$$

for small $s > 0$. This proves (5.16).

• **Case 2:** $N_2 \gg \max(N_1, N_3)$.

Let $\mathcal{T}_{12}^{N_1,N_3}$ denote the contribution from this case to $\mathcal{T}_1^{N_1,N_3}$ defined in (5.3). Let $\phi(\bar{n})$ be as in (2.30). Then, we have $\phi(\bar{n}) \sim N_2^2$ in this case. Then, by the triangle inequality, we have

$$\max(\langle \sigma \rangle, \langle \sigma_1 \rangle, \langle \sigma_2 \rangle, \langle \sigma_3 \rangle) \gtrsim N_2^2 \gg \max(N_1, N_3)^2 \tag{5.19}$$

under $\tau = \tau_1 - \tau_2 + \tau_3$, where

$$\sigma = \tau + |n|^2 \qquad \text{and} \qquad \sigma_j = \tau_j + |n_j|^2 \quad \text{for} \quad j = 1, 2, 3. \tag{5.20}$$

From (2.7), we have

$$\begin{aligned} &\big\| \mathcal{N}(\mathbf{Q}_{N_1} u_1, \boldsymbol{\mathcal{S}}^{\perp}_{10 \max(N_1,N_3)} u_2, \mathbf{Q}_{N_3} u_3) \big\|_{X_T^{s,-b'+2\varepsilon_1}} \\ &\quad \lesssim \|u_2\|_{X_T^{s,b-\varepsilon_1}} \prod_{j \in \{1,3\}} \|\mathbf{Q}_{N_j} u_j\|_{X_T^{\varepsilon_2, b-\varepsilon_1}} \end{aligned} \tag{5.21}$$

for any $0 < T \leq 1$, $s > 0$, and $\varepsilon_2 > 0$, provided that $\varepsilon_1 > 0$ is sufficiently small. Here, $\boldsymbol{\mathcal{S}}^{\perp}_N$ is as in (2.1) and b and b' are as in (2.6). Then, from Lemma 2.1, (5.19), and (5.21), we obtain

$$\|\mathcal{T}_{12}^{N_1,N_3}(w)\|_{X_T^{s,-b'}} \lesssim T^{\varepsilon_1} \max(N_1, N_3)^{-2\varepsilon_1} \|w\|_{X_T^{s,b}} \prod_{j \in \{1,3\}} \|\mathbf{Q}_{N_j} z\|_{X_T^{\varepsilon_2, b}}.$$

Hence, by taking a supermum over $\|w\|_{X_T^{s,b}} \leq 1$ and applying the homogeneous linear estimate (2.5) (with (1.21)) and the Wiener chaos estimate (Lemma 2.4), we have

$$\begin{aligned} \Big\| \|\mathcal{T}_{12}^{N_1,N_3}\|_{X_T^{s,b} \to X_T^{s,-b'}} \Big\|_{L^p(\Omega)} &\lesssim T^{\varepsilon_1} \max(N_1, N_3)^{-2\varepsilon_1 + 4\varepsilon_2} \big\| \|u_0^{\omega}\|_{H^{-\varepsilon_2}} \big\|^2_{L^{2p}(\Omega)} \\ &\lesssim p T^{\varepsilon_1} \max(N_1, N_3)^{-2\varepsilon_1 + 4\varepsilon_2}. \end{aligned}$$

Therefore, by choosing $0 < \varepsilon_2 \ll \varepsilon_1 < \varepsilon \ll 1$, we obtain (5.4) in this subcase.

This concludes the proof of Proposition 5.1. □

We conclude this section by presenting a proof of Lemma 5.2.

Proof of Lemma 5.2. Given dyadic $N, N_1, N_2, N_3 \geq 1$, write $\mathbf{H}_1^{\mathbf{N}}$ in (5.10) and (5.11) as

$$\mathbf{H}_1^{\mathbf{N}}(n, n_2, \tau, \tau_2) = \sum_{m \in \mathbb{Z}} \mathbf{H}_1^{\mathbf{N},(m)}(n, n_2) \widehat{\eta}_T(\tau + \tau_2 - m), \tag{5.22}$$

where $\mathbf{H}_1^{\mathbf{N},(m)}$ is defined by

$$\begin{aligned}\mathbf{H}_1^{\mathbf{N},(m)}(n,n_2) &= \frac{\langle n\rangle^s}{\langle n_2\rangle^s}\mathbf{1}_{\substack{|n|\sim N\\ |n_2|\sim N_2}} \sum_{\substack{n+n_2=n_1+n_3\\ n\neq n_1,n_3}} \Big(\prod_{j\in\{1,3\}} \frac{g_{n_j}}{\langle n_j\rangle}\mathbf{1}_{|n_j|\sim N_j}\Big)\mathbf{1}_{\phi(\bar{n})=m}\\ &= \frac{\langle n\rangle^s}{\langle n_2\rangle^s}\sum_{n_1,n_3\in\mathbb{Z}^2} h^{\mathbf{N},(m)}(n,n_1,n_2,n_3)\frac{g_{n_1}g_{n_3}}{\langle n_1\rangle\langle n_3\rangle}.\end{aligned} \tag{5.23}$$

Here, $h^{\mathbf{N},(m)}$ is the base tensor defined in (2.29). Hence, from (5.22), $\widehat{\eta}_T(\tau) = T\widehat{\eta}(T\tau)$, Lemma 2.6 (i) with (5.23), and Lemma 2.12, we have

$$\begin{aligned}&\sup_{\tau,\tau_2\in\mathbb{R}} \Big\| \|\mathbf{1}_{Q_0}(n)\mathbf{1}_{Q_2}(n_2)\mathbf{H}_1^{\mathbf{N}}(n,n_2,\tau,\tau_2)\|_{\ell^2_{n_2}\to\ell^2_n}\Big\|_{L^p(\Omega)}\\ &\quad\le \sup_{\tau,\tau_2\in\mathbb{R}}\sum_{m\in\mathbb{Z}} |\widehat{\eta}_T(\tau+\tau_2-m)|\Big\| \|\mathbf{1}_{Q_0}(n)\mathbf{1}_{Q_2}(n_2)\mathbf{H}_1^{\mathbf{N},(m)}(n,n_2)\|_{\ell^2_{n_2}\to\ell^2_n}\Big\|_{L^p(\Omega)}\\ &\quad\lesssim p\max(N_1,N_3)^{\varepsilon}N^sN_1^{-1}N_2^{-s}N_3^{-1}\sup_{m\in\mathbb{Z}}\max_{(B,C)}\|h^{\mathbf{N},(m)}\|_{n_2n_B\to nn_C},\\ &\quad\lesssim p(N_1N_3)^{-\frac{1}{2}+s+2\varepsilon},\end{aligned}$$

where (B,C) is a partition of $\{1,3\}$. Namely, the maximum is taken over

$$n_1n_2n_3\to n,\qquad n_2n_3\to nn_1,\qquad n_1n_2\to nn_3,\qquad \text{and}\qquad n_2\to nn_1n_3.$$

This proves (5.13). □

6. Random tensor term II

In this section, we estimate the random tensor term:

$$\mathcal{T}_2(w) = \mathcal{N}(w,z,z), \tag{6.1}$$

where z denotes the random linear solution in (1.21). Then, our main goal is to prove the following proposition.

Proposition 6.1. *Given small $s>0$ and $\varepsilon>0$, there exists $\theta>0$ such that*

$$\Big\| \|\mathcal{T}_2\|_{X^{s,b}_T\to X^{s,-b'}_T}\Big\|_{L^p(\Omega)} \lesssim pT^{\theta}$$

for any finite $p\ge 1$ and $0<T\ll 1$, where $b=\frac{1}{2}+\varepsilon$ and $b'=\frac{1}{2}-2\varepsilon$ are as in (2.6). In particular, the conclusion of Proposition 1.4 holds for the random tensor term $\mathcal{N}(w,z,z)$.

Proof. Given dyadic $N_2,N_3\ge 1$, set

$$\mathcal{T}_2^{N_2,N_3}(w) = \mathcal{N}(w,\mathbf{Q}_{N_2}z,\mathbf{Q}_{N_3}z), \tag{6.2}$$

where $\mathbf{Q}_{N_j}$ denotes the Littlewood-Paley projector. Then, as in the proof of Proposition 5.1, it suffices to prove that there exist small $\theta,\delta>0$ such that

$$\Big\| \|\mathcal{T}_2^{N_2,N_3}\|_{X^{s,b}_T\to X^{s,-b'}_T}\Big\|_{L^p(\Omega)} \lesssim pT^{\theta}(N_2N_3)^{-\delta} \tag{6.3}$$

for any finite $p\ge 1$ and $0<T\ll 1$.

Proceeding as in (5.5) with (6.2), (6.1), and (1.19), we have

$$\begin{aligned}\mathcal{F}_{x,t}\big(\eta_T \mathcal{T}_2^{N_2,N_3}(w)\big)(n,\tau-|n|^2)\\ = \langle n\rangle^{-s}\sum_{n_1\in\mathbb{Z}^2}\langle n_1\rangle^{s}\int_{\mathbb{R}}\mathbf{H}_2^{N_2,N_3}(n,n_1,\tau,\tau_1)\widehat{w}(n_1,\tau_1-|n_1|^2)d\tau_1,\end{aligned}$$

where

$$\begin{aligned}&\mathbf{H}_2^{N_2,N_3}(n,n_1,\tau,\tau_1)\\ &\quad=\frac{\langle n\rangle^s}{\langle n_1\rangle^s}\sum_{\substack{n-n_1=-n_2+n_3\\ n\neq n_1,n_3}}\bigg(\prod_{j=2}^{3}\frac{g_{n_j}^*}{\langle n_j\rangle}\mathbf{1}_{|n_j|\sim N_j}\bigg)\widehat{\eta}_T(\tau-\tau_1-\phi(\bar{n}))\end{aligned}\tag{6.4}$$

with $\phi(\bar{n})$ as in (2.30) and $g_{n_j}^*$ as in (4.3). Then, by repeating the computations in (5.7) and (5.8), we see that (6.3) follows once we prove

$$\Big\|\big\|\langle\tau\rangle^{-b'}\langle\tau_1\rangle^{-b}\|\mathbf{H}_2^{N_2,N_3}(n,n_1,\tau,\tau_1)\|_{\ell^2_{n_1}\to\ell^2_n}\big\|_{L^2_{\tau,\tau_1}}\Big\|_{L^p(\Omega)}\lesssim pT^{\theta}(N_2N_3)^{-\delta}.\tag{6.5}$$

Given dyadic $N,N_1,N_2,N_3\geq 1$, we set

$$\begin{aligned}\mathbf{H}_2^{\mathbf{N}}(n,n_1,\tau,\tau_1)&=\mathbf{H}_2^{N,N_1,N_2,N_3}(n,n_1,\tau,\tau_1)\\ &=\mathbf{1}_{|n|\sim N}\mathbf{1}_{|n_1|\sim N_1}\mathbf{H}_2^{N_2,N_3}(n,n_1,\tau,\tau_1).\end{aligned}$$

Then, by writing

$$\mathbf{H}_2^{N_2,N_3}=\sum_{\substack{N,N_1\geq 1,\text{ dyadic}\\ \max(N_2,N_3)\gtrsim N_1}}\mathbf{H}_2^{\mathbf{N}}+\sum_{\substack{N,N_1\geq 1,\text{ dyadic}\\ N_1\gg\max(N_2,N_3)}}\mathbf{H}_2^{\mathbf{N}},\tag{6.6}$$

we separately estimate the first and second terms on the right-hand side of (6.6).

• **Case 1:** $\max(N_2,N_3)\gtrsim N_1$.

In this case, a straightforward modification of the argument in Case 1 of the proof of Proposition 5.1 yields (6.5), and thus we omit details.

• **Case 2:** $N_1\gg\max(N_2,N_3)$.

In this case, we have $N\sim N_1\gg\max(N_2,N_3)$. Unlike Case 2 in the proof of Proposition 5.1, there is no effective lower bound on $|\phi(\bar{n})|$. We separately consider the following subcases:

(2.a) $\big||n|^2-|n_1|^2\big|\gg\max(N_2,N_3)^{a_1}$,

(2.b) $\big||n|^2-|n_1|^2\big|\lesssim\max(N_2,N_3)^{a_1}$

for some fixed $a_1>2$.

∘ **Subcase 2.a:** $\big||n|^2-|n_1|^2\big|\gg\max(N_2,N_3)^{a_1}$.

Let $\mathcal{T}_{2a}^{N_2,N_3}$ denote the contribution from this subcase to $\mathcal{T}_2^{N_2,N_3}$ defined in (6.2). We will directly prove (6.3) for $\mathcal{T}_{2a}^{N_2,N_3}$. In this subcase, we have

$$|\phi(\bar{n})|=\big||n|^2-|n_1|^2+|n_2|^2-|n_3|^2\big|\gg\max(N_2,N_3)^{a_1}.$$

Then, by the triangle inequality, we have

$$\max(\langle\sigma\rangle,\langle\sigma_1\rangle,\langle\sigma_2\rangle,\langle\sigma_3\rangle)\gg\max(N_2,N_3)^{a_1}\tag{6.7}$$

under $\tau = \tau_1 - \tau_2 + \tau_3$, where σ and σ_j are as in (5.20). Then, proceeding as in Case 2 of the proof of Proposition 5.1 with (6.7), we have

$$\begin{aligned}\Big\| \|\mathcal{T}_{2b}^{N_2,N_3}\|_{X_T^{s,b}\to X_T^{s,-b'}} \Big\|_{L^p(\Omega)} &\lesssim T^{\varepsilon_1} \max(N_2,N_3)^{-a_1\varepsilon_1+4\varepsilon_2} \big\| \|u_0^\omega\|_{H^{-\varepsilon_2}} \big\|_{L^{2p}(\Omega)}^2 \\ &\lesssim pT^{\varepsilon_1} \max(N_2,N_3)^{-a_1\varepsilon_1+4\varepsilon_2}.\end{aligned}$$

Therefore, by choosing $0 < \varepsilon_2 \ll \varepsilon_1 < \varepsilon \ll 1$, we obtain (6.3) in this subcase.

∘ **Subcase 2.b:** $\big||n|^2 - |n_1|^2\big| \lesssim \max(N_2,N_3)^{a_1}$.

Let $\mathbf{H}_{2b}^{N_2,N_3}$ denote the contribution from this subcase to $\mathbf{H}_2^{N_2,N_3}$ in (6.4), namely,

$$\begin{aligned}\mathbf{H}_{2b}^{N_2,N_3}(n,n_1,\tau,\tau_1) = \frac{\langle n\rangle^s}{\langle n_1\rangle^s} \sum_{\substack{N,N_1\geq 1,\text{ dyadic}\\ N_1\gg\max(N_2,N_3)\\ ||n|^2-|n_1|^2|\lesssim \max(N_2,N_3)^{a_1}}} \mathbf{1}_{|n|\sim N}\mathbf{1}_{|n_1|\sim N_1} \\ \sum_{\substack{n-n_1=-n_2+n_3\\ n\neq n_1,n_3}} \bigg(\prod_{j=2}^3 \frac{g_{n_j}^*}{\langle n_j\rangle}\mathbf{1}_{|n_j|\sim N_j}\bigg)\widehat{\eta}_T(\tau-\tau_1-\phi(\bar{n})).\end{aligned} \tag{6.8}$$

Note that, under the condition $N_1 \gg \max(N_2,N_3)$, the condition $n \neq n_3$ (i.e. $n_1 \neq n_2$) in (6.8) is vacuous. Since we have $|n| \sim |n_1|$ on the support of $\mathbf{H}_{2b}^{N_2,N_3}$, we have

$$\|\mathbf{H}_{2b}^{N_2,N_3}(n,n_1,\tau,\tau_1)\|_{\ell^2_{n_1}\to\ell^2_n} \lesssim \|\widetilde{\mathbf{H}}_{2b}^{N_2,N_3}(n,n_1,\tau,\tau_1)\|_{\ell^2_{n_1}\to\ell^2_n} \tag{6.9}$$

for each $\tau,\tau_2 \in \mathbb{R}$, where $\widetilde{\mathbf{H}}_{2b}^{N_2,N_3}$ is defined by

$$\widetilde{\mathbf{H}}_{2b}^{N_2,N_3}(n,n_1,\tau,\tau_1) = \frac{\langle n_1\rangle^s}{\langle n\rangle^s}\mathbf{H}_{2b}^{N_2,N_3}(n,n_1,\tau,\tau_1).$$

Then, by writing

$$\widetilde{\mathbf{H}}_{2b}^{N_2,N_3}(n,n_1,\tau,\tau_1) = \sum_{\substack{n_2,n_3\in\mathbb{Z}^2\\ |n_2|\sim N_2\\ |n_3|\sim N_3}} \widetilde{h}_{n,n_1,n_2,n_3}^{\tau,\tau_1}\overline{g_{n_2}}g_{n_3},$$

we see that, for fixed τ and τ_1, the tensor $\widetilde{h}_{n,n_1,n_2,n_3}^{\tau,\tau_1}$ depends only on $n-n_1$, $\big||n|^2-|n_1|^2\big|$, and (n_2,n_3) with the restriction $|n|,|n_1| \gg \max(N_2,N_3)$ on the input and output frequencies, to which Lemma 2.6 (iii) can be applied.

By an interpolation argument as in Case 1 of the proof of Proposition 5.1 and (6.9), we claim that (6.5) in this subcase follows once we prove

$$\Big\| \big\|\langle\tau\rangle^{-b}\langle\tau_1\rangle^{-b}\|\widetilde{\mathbf{H}}_{2b}^{N_2,N_3}(n,n_1,\tau,\tau_1)\|_{\ell^2_{n_1}\to\ell^2_n}\big\|_{L^2_{\tau,\tau_1}} \Big\|_{L^p(\Omega)} \lesssim p(N_2N_3)^{-\delta}. \tag{6.10}$$

where the power on $\langle\tau\rangle$ is replaced by $-b$.

As in (2.22), we define $\widetilde{\mathbf{H}}_2^{N_2,N_3,m}$ by setting

$$\widetilde{\mathbf{H}}_2^{N_2,N_3,m}(n',n_1',\tau,\tau_1) = \widetilde{\mathbf{H}}_2^{N_2,N_3}(n'+m,n_1'+m,\tau,\tau_1)\cdot \mathbf{1}_{\substack{|n'|\lesssim\max(N_2,N_3)\\ |n_1'|\lesssim\max(N_2,N_3)}}.$$

Then, arguing as in the proof of Lemma 2.6 (iii) with Lemma 2.7 (see (2.28)), we see that there exists $J = O(\max(N_2, N_3)^{a_2})$ and $\{m_j\}_{j=1}^J \subset \mathbb{Z}^2$ with $|m_j| \gg \max(N_2, N_3)$ such that

$$\begin{aligned}
&\Big\| \big\| \langle \tau_1 \rangle^{-b} \|\widetilde{\mathbf{H}}_2^{N_2,N_3}(n, n_1, \tau, \tau_1)\|_{\ell^2_{n_1} \to \ell^2_n} \big\|_{L^2_{\tau,\tau_1}} \Big\|_{L^p(\Omega)} \\
&\lesssim \Big\| \big\| \langle \tau_1 \rangle^{-b} \sup_{m \in \mathbb{Z}^2} \|\widetilde{\mathbf{H}}_2^{N_2,N_3,m}(n', n_1', \tau, \tau_1)\|_{\ell^2_{n_1'} \to \ell^2_{n'}} \big\|_{L^2_{\tau,\tau_1}} \Big\|_{L^p(\Omega)} \\
&\lesssim \Big\| \big\| \langle \tau_1 \rangle^{-b} \sup_{|m| \lesssim \max(N_2,N_3)} \|\widetilde{\mathbf{H}}_2^{N_2,N_3,m}(n', n_1', \tau, \tau_1)\|_{\ell^2_{n_1'} \to \ell^2_{n'}} \big\|_{L^2_{\tau,\tau_1}} \Big\|_{L^p(\Omega)} \\
&\quad + \Big\| \big\| \langle \tau_1 \rangle^{-b} \sup_{j=1,\dots,J} \|\widetilde{\mathbf{H}}_2^{N_2,N_3,m_j}(n', n_1', \tau, \tau_1)\|_{\ell^2_{n_1'} \to \ell^2_{n'}} \big\|_{L^2_{\tau,\tau_1}} \Big\|_{L^p(\Omega)} \qquad (6.11) \\
&\lesssim \max(N_2, N_3) \\
&\qquad \times \sup_{|m| \lesssim \max(N_2,N_3)} \Big\| \big\| \langle \tau_1 \rangle^{-b} \|\widetilde{\mathbf{H}}_2^{N_2,N_3,m}(n', n_1', \tau, \tau_1)\|_{\ell^2_{n_1',n'}} \big\|_{L^2_{\tau,\tau_1}} \Big\|_{L^p(\Omega)} \\
&\quad + \max(N_2, N_3)^{\frac{a_2}{2}} \sup_{j=1,\dots,J} \Big\| \big\| \langle \tau_1 \rangle^{-b} \|\widetilde{\mathbf{H}}_2^{N_2,N_3,m_j}(n', n_1', \tau, \tau_1)\|_{\ell^2_{n_1',n'}} \big\|_{L^2_{\tau,\tau_1}} \Big\|_{L^p(\Omega)}
\end{aligned}$$

for any $p \geq 2$. Then, proceeding as in (5.17), we obtain

$$\Big\| \big\| \langle \tau_1 \rangle^{-b} \|\widetilde{\mathbf{H}}_2^{N_2,N_3}(n, n_1, \tau, \tau_1)\|_{\ell^2_{n_1} \to \ell^2_n} \big\|_{L^2_{\tau,\tau_1}} \Big\|_{L^p(\Omega)} \lesssim p T^{\frac{1}{2}} \max(N_2, N_3)^{a_3} \qquad (6.12)$$

for some $a_3 \gg 1$. Then, (6.5) in this subcase follows from interpolating (6.10) and (6.12).

It remains to prove (6.10). Since $b > \frac{1}{2}$, we can proceed as in (5.18) by applying Lemma 2.6 (iii) (instead of Lemma 2.6 (i) used in the proof of Lemma 5.2 presented at the end of Section 5). This proves (6.10) (with $\delta = \frac{1}{2} - \varepsilon$).

This concludes the proof of Proposition 6.1. □

7. Random tensor term III

In this section, we estimate the random tensor term:

$$\mathcal{T}_3(w_2, w_3) = \mathcal{N}(z, w_2, w_3), \qquad (7.1)$$

where z denotes the random linear solution in (1.21). Then, our main goal is to prove the following proposition.

Proposition 7.1. *Given small $s > 0$ and $\varepsilon > 0$, there exists $\theta > 0$ such that*

$$\Big\| \|\mathcal{T}_3\|_{X^{s,b}_T \times X^{s,b}_T \to X^{s,-b'}_T} \Big\|_{L^p(\Omega)} \lesssim p^{\frac{1}{2}} T^{\theta}$$

for any finite $p \geq 1$ and $0 < T \ll 1$, where $b = \frac{1}{2} + \varepsilon$ and $b' = \frac{1}{2} - 2\varepsilon$ are as in (2.6). In particular, the conclusion of Proposition 1.4 holds for the random tensor term $\mathcal{N}(z, w, w)$.

Proof. Given dyadic $N_1 \geq 1$, define

$$\mathcal{T}_3^{N_1}(w_2, w_3) = \mathcal{N}(\mathbf{Q}_{N_1} z, w_2, w_3). \qquad (7.2)$$

Then, it suffices to prove that there exist small $\theta, \delta > 0$ such that

$$\Big\| \|\mathcal{T}_3^{N_1}\|_{X^{s,b}_T \times X^{s,b}_T \to X^{s,-b'}_T} \Big\|_{L^p(\Omega)} \lesssim p^{\frac{1}{2}} T^{\theta} N_1^{-\delta} \qquad (7.3)$$

for any finite $p \geq 1$ and $0 < T \ll 1$. In the following, we use n_1, n_2, and n_3 to denote the spatial frequencies of $\mathbf{Q}_{N_1} z$, w_2, and w_3, respectively.

• **Case 1:** $N_1 \lesssim \min(|n_2|, |n_3|)$.

Let $\mathcal{T}_{31}^{N_1}$ denote the contribution from this case to $\mathcal{T}_3^{N_1}$ defined in (7.2). Without loss of generality, assume that $N_1 \lesssim |n_2| \leq |n_3|$. Then, from Lemma 2.1 and (2.7) with small $\varepsilon_1, \varepsilon_2 > 0$, we have

$$\begin{aligned}
\big\|\mathcal{N}(\mathbf{Q}_{N_1} u_1, u_2, u_3)\big\|_{X_T^{s,-b'}} &\lesssim T^{\varepsilon_1} \sum_{\substack{N_2 \geq 1 \\ \text{dyadic}}} \mathbf{1}_{N_2 \gtrsim N_1} \big\|\mathcal{N}(\mathbf{Q}_{N_1} u_1, \mathbf{Q}_{N_2} u_2, u_3)\big\|_{X_T^{s,-b'+\varepsilon_1}} \\
&\lesssim T^{\varepsilon_1} \sum_{\substack{N_2 \geq 1 \\ \text{dyadic}}} \mathbf{1}_{N_2 \gtrsim N_1} \|\mathbf{Q}_{N_1} u_1\|_{X_T^{\varepsilon_2,b}} \|\mathbf{Q}_{N_2} u_2\|_{X_T^{\varepsilon_2,b}} \|u_3\|_{X_T^{s,b}} \\
&\lesssim T^{\varepsilon_1} N_1^{-\varepsilon_2} \|u_1\|_{X_T^{-\varepsilon_2,b}} \|u_2\|_{X_T^{s,b}} \|u_3\|_{X_T^{s,b}}
\end{aligned}$$

for any $0 < T \leq 1$, provided that $s \geq 4\varepsilon_2$. In particular, we have

$$\|\mathcal{T}_{31}^{N_1}(w_2, w_3)\|_{X_T^{s,-b'}} \lesssim T^{\varepsilon_1} N_1^{-\varepsilon_2} \|z\|_{X_T^{-\varepsilon_2,b}} \|w_2\|_{X_T^{s,b}} \|w_3\|_{X_T^{s,b}}$$

Hence, by taking a supermum over $\|w_j\|_{X_T^{s,b}} \leq 1$, $j = 2, 3$, and applying the homogeneous linear estimate (2.5) (with (1.21)) and the Wiener chaos estimate (Lemma 2.4), we have

$$\begin{aligned}
\Big\| \|\mathcal{T}_{31}^{N_1}\|_{X_T^{s,b} \times X_T^{s,b} \to X_T^{s,-b'}} \Big\|_{L^p(\Omega)} &\lesssim T^{\varepsilon_1} N_1^{-\varepsilon_2} \big\| \|u_0^\omega\|_{H^{-\varepsilon_2}} \big\|_{L^p(\Omega)} \\
&\lesssim p^{\frac{1}{2}} T^{\varepsilon_1} N_1^{-\varepsilon_2},
\end{aligned}$$

which yields (7.3) in this case.

• **Case 2:** $|\phi(\bar{n})| \gg N_1^2$.

Let $\mathcal{T}_{32}^{N_1}$ denote the contribution from this case to $\mathcal{T}_3^{N_1}$ defined in (7.2). Then, proceeding as in Case 2 of the proof of Proposition 5.1, we have

$$\|\mathcal{T}_{32}^{N_1}(w_2, w_3)\|_{X_T^{s,-b'}} \lesssim T^{\varepsilon_1} N_1^{-2\varepsilon_1} \|z\|_{X_T^{-\varepsilon_2,b}} \|w_2\|_{X_T^{s,b}} \|w_3\|_{X_T^{s,b}}.$$

Then, by taking a supermum over $\|w_j\|_{X_T^{s,b}} \leq 1$, $j = 2, 3$, and applying the homogeneous linear estimate (2.5) (with (1.21)) and the Wiener chaos estimate (Lemma 2.4), we obtain (7.3) in this case.

Before proceeding further, let us carry out a basic reduction. From (7.2) with (7.1) and (1.19), we have

$$\begin{aligned}
&\mathcal{F}\big(\eta_T \mathcal{T}_3^{N_1}(w_2, w_3)\big)(n, \tau - |n|^2) \\
&\quad = \langle n \rangle^{-s} \sum_{n_2, n_3 \in \mathbb{Z}^2} \langle n_2 \rangle^s \langle n_3 \rangle^s \int_{\mathbb{R}^2} \mathbf{H}_3^{N_1}(n, n_2, n_3, \tau, \tau_2, \tau_3) \\
&\qquad\qquad \times \overline{\widehat{w}_2(n_2, \tau_2 - |n_2|^2)} \widehat{w}_3(n_3, \tau_3 - |n_3|^2) d\tau_2 d\tau_3
\end{aligned}$$

where[13]

$$\begin{aligned}\mathbf{H}_3^{N_1}(n,n_2,n_3,\tau,\tau_2,\tau_3) &= \frac{\langle n\rangle^s}{\langle n_2\rangle^s\langle n_3\rangle^s}\\ &\quad\times \sum_{\substack{n+n_2-n_3=n_1\\ n_1\neq n,n_2}} \frac{g_{n_1}}{\langle n_1\rangle}\mathbf{1}_{|n_1|\sim N_1}\widehat{\eta}_T(\tau+\tau_2-\tau_3-\phi(\bar{n}))\end{aligned} \tag{7.4}$$

with $\phi(\bar{n})$ as in (2.30). Arguing as in (5.7), we see that (7.3) follows once we prove

$$\begin{aligned}&\Big\| \big\| \langle\tau\rangle^{-b'}\langle\tau_2\rangle^{-b}\langle\tau_3\rangle^{-b} \|\mathbf{H}_3^{N_1}(n,n_2,n_3,\tau,\tau_2,\tau_3)\|_{\ell^2_{n_2,n_3}\to\ell^2_n}\big\|_{L^2_{\tau,\tau_2,\tau_3}}\Big\|_{L^p(\Omega)}\\ &\quad\lesssim p^{\frac{1}{2}}T^\theta N_1^{-\delta},\end{aligned} \tag{7.5}$$

where b and b' are as in (2.6). See also Remark 2.9.

Given dyadic $N, N_1, N_2, N_3 \geq 1$, we set

$$\begin{aligned}\mathbf{H}_3^{\mathbf{N}}(n,n_2,n_3,\tau,\tau_2,\tau_3) &= \mathbf{H}_3^{N,N_1,N_2,N_3}(n,n_2,n_3,\tau,\tau_2,\tau_3)\\ &= \mathbf{1}_{|n|\sim N}\Big(\prod_{j=2}^{3}\mathbf{1}_{|n_j|\sim N_j}\Big)\mathbf{H}_3^{N_1}(n,n_2,n_3,\tau,\tau_2,\tau_3).\end{aligned} \tag{7.6}$$

Then, by writing

$$\mathbf{H}_3^{N_1} = \sum_{\substack{N,N_2,N_3\geq 1,\text{ dyadic}\\ N_1\gtrsim\max(N_2,N_3)}} \mathbf{H}_3^{\mathbf{N}} + \sum_{\substack{N,N_2,N_3\geq 1,\text{ dyadic}\\ N_1\ll\max(N_2,N_3)}} \mathbf{H}_3^{\mathbf{N}}, \tag{7.7}$$

we separately estimate the first and second terms on the right-hand side of (7.7) (after removing the contributions from Cases 1 and 2).

We now state a lemma whose proof is presented at the end of this section.

Lemma 7.2. *Given dyadic $N, N_1, N_2, N_3 \geq 1$, suppose that $N_1 \gtrsim N_{\text{med}}$, where N_{med} is as in (2.2). Let Q_0, Q_2, and Q_3 be cubes of side length $\lesssim N_1$. Fix small $s > 0$. Then, we have*

$$\begin{aligned}\sup_{\tau,\tau_2,\tau_3\in\mathbb{R}} \Big\| &\|\mathbf{1}_{Q_0}(n)\mathbf{1}_{Q_2}(n_2)\mathbf{1}_{Q_3}(n_3)\\ &\times \mathbf{H}_3^{\mathbf{N}}(n,n_2,n_3,\tau,\tau_2,\tau_3)\|_{\ell^2_{n_2,n_3}\to\ell^2_n}\Big\|_{L^p(\Omega)} \lesssim p^{\frac{1}{2}}N_1^{-s+\varepsilon}\end{aligned} \tag{7.8}$$

for any finite $p \geq 1$ and dyadic $N, N_1, N_2, N_3 \geq 1$, uniformly in $0 < T \ll 1$ and the cubes Q_0, Q_2, and Q_3.

• **Case 3:** $N_1 \gtrsim \max(N_2, N_3)$.

We proceed by an interpolation argument as in Case 1 of the proof of Proposition 5.1. Proceeding as in (5.17) with (2.10), (7.6), (7.4), and the Wiener chaos estimate (Lemma 2.4),

[13]Note that there is at most one term in the sum over n_1.

we have

$$\begin{aligned}
&\Big\| \big\| \langle \tau_2\rangle^{-b}\langle \tau_3\rangle^{-b} \|\mathbf{H}_3^{\mathbf{N}}(n,n_2,n_3,\tau,\tau_2,\tau_3)\|_{\ell^2_{n_2,n_3}\to\ell^2_n}\big\|_{L^2_{\tau,\tau_2,\tau_3}}\Big\|_{L^p(\Omega)}\\
&\quad\le \Big\| \big\| \langle \tau_2\rangle^{-b}\langle \tau_3\rangle^{-b} \|\mathbf{H}_3^{\mathbf{N}}(n,n_2,n_3,\tau,\tau_2,\tau_3)\|_{\ell^2_{n_2,n_3,n}}\big\|_{L^2_{\tau,\tau_2,\tau_3}}\Big\|_{L^p(\Omega)}\\
&\quad\lesssim p^{\frac12} \sum_{\substack{n,n_1,n_2,n_3\in\mathbb{Z}^2\\ |n|\sim N}} \mathbf{1}_{A(n)} \frac{N^s}{N_1N_2^sN_3^s} \|\langle \tau_2\rangle^{-b}\langle \tau_3\rangle^{-b} \widehat{\eta}_T(\tau+\tau_2-\tau_3-\phi(\bar n))\|_{L^2_{\tau,\tau_2,\tau_3}}\\
&\quad\lesssim p^{\frac12}T^{\frac12}N_1^6
\end{aligned} \tag{7.9}$$

for small $s>0$, where $A(n)$ is as in (4.5). Hence, by interpolating with (7.9), we see that (7.5) follows once we prove

$$\begin{aligned}
&\Big\| \big\| \langle \tau\rangle^{-b}\langle \tau_2\rangle^{-b}\langle \tau_3\rangle^{-b} \|\mathbf{H}_3^{N_1}(n,n_2,n_3,\tau,\tau_2,\tau_3)\|_{\ell^2_{n_2,n_3}\to\ell^2_n}\big\|_{L^2_{\tau,\tau_2,\tau_3}}\Big\|_{L^p(\Omega)}\\
&\qquad\lesssim p^{\frac12}N_1^{-2\delta},
\end{aligned} \tag{7.10}$$

since the extra factor $N_1^{-\delta}$ allows us to sum over dyadic $N,N_2,N_3\ge 1$.

Recalling $b>\frac12$, the desired bound (7.10) in this case follows from Minkowski's integral inequality and Lemma 7.2; see (5.18).

• **Case 4:** $N_1 \ll \max(N_2,N_3)$.

In view of Case 1, we have $N_1 \gg \min(N_2,N_3)$. We separately consider the following subcases:

$$\text{(4.a) } N\sim N_2 \gg N_1 \gg N_3 \qquad \text{and} \qquad \text{(4.b) } N\sim N_3 \gg N_1 \gg N_2.$$

∘ **Subcase 4.a:** $N\sim N_2\gg N_1\gg N_3$.

In this case, we have $|\phi(\bar n)|\sim N^2 \gg N_1^2$, which is already treated in Case 2.

∘ **Subcase 4.b:** $N\sim N_3\gg N_1\gg N_2$.

In view of Case 2, we may assume that $|\phi(\bar n)|\lesssim N_1^2$. Since $\big||n_1|^2-|n_2|^2\big|\lesssim N_1^2$, we obtain $\big||n|^2-|n_3|^2\big|\lesssim N_1^2$ in this case, and thus we can proceed as in Subcase 2.b of the proof of Proposition 6.1.

Let $\mathbf{H}_{3b}^{N_1}$ denote the contribution from this subcase to $\mathbf{H}_3^{N_1}$ in (7.4), namely,

$$\begin{aligned}
\mathbf{H}_{3b}^{N_1}(n,n_2,n_3,\tau,\tau_2,\tau_3) = \frac{\langle n\rangle^s}{\langle n_2\rangle^s\langle n_3\rangle^s} &\sum_{\substack{N,N_2,N_3\ge1,\text{ dyadic}\\ N\sim N_3\gg N_1\gg N_2\\ ||n|^2-|n_3|^2|\lesssim N_1^2}} \mathbf{1}_{|n|\sim N}\Big(\prod_{j=2}^3 \mathbf{1}_{|n_j|\sim N_j}\Big)\\
&\times \sum_{\substack{n+n_2-n_3=n_1\\ n_1\ne n,n_2}} \frac{g_{n_1}}{\langle n_1\rangle}\mathbf{1}_{|n_1|\sim N_1}\widehat{\eta}_T(\tau+\tau_2-\tau_3-\phi(\bar n)).
\end{aligned} \tag{7.11}$$

Note that the condition $n_1\ne n,n_2$ is vacuous since $N\gg N_1\gg N_2$. Since we have $|n|\sim|n_3|$ on the support of $\mathbf{H}_{3b}^{N_1}$, we have

$$\|\mathbf{H}_{3b}^{N_1}(n,n_2,n_3,\tau,\tau_2,\tau_3)\|_{\ell^2_{n_2,n_3}\to\ell^2_n} \lesssim \|\widetilde{\mathbf{H}}_{3b}^{N_1}(n,n_2,n_3,\tau,\tau_2,\tau_3)\|_{\ell^2_{n_2,n_3}\to\ell^2_n} \tag{7.12}$$

for each $\tau, \tau_2, \tau_3 \in \mathbb{R}$, where $\widetilde{\mathbf{H}}_{3b}^{N_1}(n, n_2, n_3, \tau, \tau_2, \tau_3)$ is defined by

$$\widetilde{\mathbf{H}}_{3b}^{N_1}(n, n_2, n_3, \tau, \tau_2, \tau_3) = \frac{\langle n_3\rangle^s}{\langle n\rangle^s}\mathbf{H}_{3b}^{N_1}(n, n_2, n_3, \tau, \tau_2, \tau_3).$$

Then, by writing

$$\widetilde{\mathbf{H}}_{3b}^{N_1}(n, n_2, n_3, \tau, \tau_2, \tau_3) = \sum_{\substack{n_1\in\mathbb{Z}^2\\ |n_1|\sim N_1}} \widetilde{h}_{n,n_1,n_2,n_3}^{\tau,\tau_2,\tau_3} g_{n_1},$$

we see that, for fixed τ, τ_2, and τ_3, the tensor $\widetilde{h}_{n,n_1,n_2,n_3}^{\tau,\tau_2,\tau_3}$ depends only on $n-n_3$, $\big||n|^2-|n_3|^2\big|$, and (n_1, n_2) with the restriction $|n|, |n_3| \gg N_1$ on the input and output frequencies, to which Lemma 2.6 (iii) can be applied.

By an interpolation argument as before with (7.12), we claim that (7.5) in this subcase follows once we prove

$$\begin{aligned}\Big\| \big\| \langle\tau\rangle^{-b}\langle\tau_2\rangle^{-b}\langle\tau_3\rangle^{-b} \|\widetilde{\mathbf{H}}_{3b}^{N_1}(n, n_2, n_3, \tau, \tau_2, \tau_3)\|_{\ell^2_{n_2,n_3}\to\ell^2_n}\big\|_{L^2_{\tau,\tau_2,\tau_3}}\Big\|_{L^p(\Omega)}\\ \lesssim p^{\frac12} N_1^{-\delta},\end{aligned} \tag{7.13}$$

where the power on $\langle\tau\rangle$ is replaced by $-b$.

As in (2.22), we define $\widetilde{\mathbf{H}}_{3b}^{N_1,m}$ by setting

$$\widetilde{\mathbf{H}}_{3b}^{N_1,m}(n', n_2, n_3', \tau, \tau_2, \tau_3) = \widetilde{\mathbf{H}}_{3b}^{N_1}(n'+m, n_2, n_3'+m, \tau, \tau_2, \tau_3)\cdot \mathbf{1}_{\substack{|n'|\lesssim N_1\\ |n_3'|\lesssim N_1}}.$$

Then, proceeding as in (6.11) with Lemma 2.7 (see also (2.28)), we see that there exists $J = O(N_1^{a_2})$ (with some $a_2 \gg 1$) and $\{m_j\}_{j=1}^J \subset \mathbb{Z}^2$ with $|m_j| \gg N_1$ such that

$$\begin{aligned}&\Big\| \big\| \langle\tau_2\rangle^{-b}\langle\tau_3\rangle^{-b} \|\widetilde{\mathbf{H}}_{3b}^{N_1}(n, n_2, n_3, \tau, \tau_2, \tau_3)\|_{\ell^2_{n_2n_3}\to\ell^2_n}\big\|_{L^2_{\tau,\tau_2,\tau_3}}\Big\|_{L^p(\Omega)}\\
&\lesssim N_1 \sup_{|m|\lesssim N_1} \Big\| \big\| \langle\tau_2\rangle^{-b}\langle\tau_3\rangle^{-b} \|\widetilde{\mathbf{H}}_{3b}^{N_1,m}(n', n_2, n_3', \tau, \tau_2, \tau_3)\|_{\ell^2_{n_2,n_3',n'}}\big\|_{L^2_{\tau,\tau_2,\tau_3}}\Big\|_{L^p(\Omega)}\\
&\quad + N_1^{\frac{a_2}{2}} \sup_{j=1,\dots,J} \Big\| \big\| \langle\tau_2\rangle^{-b}\langle\tau_3\rangle^{-b} \|\widetilde{\mathbf{H}}_{3b}^{N_1,m_j}(n', n_2, n_3', \tau, \tau_2, \tau_3)\|_{\ell^2_{n_2,n_3',n'}}\big\|_{L^2_{\tau,\tau_2,\tau_3}}\Big\|_{L^p(\Omega)}\\
&\lesssim p^{\frac12} T^{\frac12} N_1^{\frac{a_2}{2}+6}\end{aligned} \tag{7.14}$$

for any $p \geq 2$, where the second step follows from (a slight modification of) (7.9), Then, (7.5) in this subcase follows from interpolating (7.13) and (7.14).

It remains to prove (7.13). Since $b > \frac12$, we can proceed as in (5.18) by applying Lemma 2.6 (iii) (instead of Lemma 2.6 (i) used in the proof of Lemma 7.2 presented at the end of this section). This proves (7.13) (with $\delta = s-\varepsilon$).

This concludes the proof of Proposition 7.1. □

We conclude this section by presenting a proof of Lemma 7.2.

Proof of Lemma 7.2. Given dyadic $N, N_1, N_2, N_3 \geq 1$, write $\mathbf{H}_3^{\mathbf{N}}$ in (7.6) (see also (7.4)) as

$$\mathbf{H}_3^{\mathbf{N}}(n, n_2, n_3, \tau, \tau_2, \tau_3) = \sum_{m\in\mathbb{Z}} \mathbf{H}_3^{\mathbf{N},(m)}(n, n_2, n_3)\widehat{\eta}_T(\tau+\tau_2-\tau_3-m), \tag{7.15}$$

where $\mathbf{H}_3^{\mathbf{N},(m)}$ is defined by

$$\begin{aligned}\mathbf{H}_3^{\mathbf{N},(m)}(n,n_2,n_3) &= \frac{\langle n\rangle^s}{\langle n_2\rangle^s\langle n_3\rangle^s}\mathbf{1}_{\substack{|n|\sim N\\ |n_2|\sim N_2\\ |n_3|\sim N_3}}\sum_{\substack{n+n_2-n_3=n_1\\ n_1\neq n,n_2}}\frac{g_{n_1}}{\langle n_1\rangle}\mathbf{1}_{|n_1|\sim N_1}\mathbf{1}_{\phi(\bar{n})=m}\\ &= \frac{\langle n\rangle^s}{\langle n_2\rangle^s\langle n_3\rangle^s}\sum_{n_1\in\mathbb{Z}^2} h^{\mathbf{N},(m)}(n,n_1,n_2,n_3)\frac{g_{n_1}}{\langle n_1\rangle}.\end{aligned} \tag{7.16}$$

Here, $h^{\mathbf{N},(m)}$ is the base tensor defined in (2.29). Hence, from (7.15), $\widehat{\eta}_T(\tau) = T\widehat{\eta}(T\tau)$, Lemma 2.6 (i) with (7.16), and Lemma 2.12, we have

$$\begin{aligned}&\sup_{\tau,\tau_2,\tau_3\in\mathbb{R}}\Big\|\|\mathbf{1}_{Q_0}(n)\mathbf{1}_{Q_2}(n_2)\mathbf{1}_{Q_3}(n_3)\mathbf{H}_3^{\mathbf{N}}(n,n_2,n_3,\tau,\tau_2,\tau_3)\|_{\ell^2_{n_2n_3}\to\ell^2_n}\Big\|_{L^p(\Omega)}\\ &\le \sup_{\tau,\tau_2,\tau_3\in\mathbb{R}}\sum_{m\in\mathbb{Z}}|\widehat{\eta}_T(\tau+\tau_2-\tau_3-m)|\\ &\qquad\times\Big\|\|\mathbf{1}_{Q_0}(n)\mathbf{1}_{Q_2}(n_2)\mathbf{1}_{Q_3}(n_3)\mathbf{H}_3^{\mathbf{N},(m)}(n,n_2,n_3)\|_{\ell^2_{n_2n_3}\to\ell^2_n}\Big\|_{L^p(\Omega)}\\ &\lesssim pN_1^{\varepsilon}N^sN_1^{-1}N_2^{-s}N_3^{-s}\sup_{m\in\mathbb{Z}}\max\Big(\|h^{\mathbf{N},(m)}\|_{n_1n_2n_3\to n},\|h^{\mathbf{N},(m)}\|_{n_2n_3\to nn_1}\Big)\\ &\lesssim pN_1^{-s+3\varepsilon},\end{aligned}$$

where, in the last step, we used the fact that $N_1\gtrsim N_{\text{med}}$. This proves (7.8). □

8. Random tensor term IV

In this section, we briefly discuss how to treat the random tensor term:

$$\mathcal{T}_4(w_1,w_3) = \mathcal{N}(w_1,z,w_3),$$

where z denotes the random linear solution in (1.21).

Proposition 8.1. *Given small $s>0$ and $\varepsilon>0$, there exists $\theta>0$ such that*

$$\Big\|\|\mathcal{T}_4\|_{X_T^{s,b}\times X_T^{s,b}\to X_T^{s,-b'}}\Big\|_{L^p(\Omega)}\lesssim p^{\frac12}T^{\theta}$$

for any finite $p\ge1$ and $0<T\ll1$, where $b=\frac12+\varepsilon$ and $b'=\frac12-2\varepsilon$ are as in (2.6). In particular, the conclusion of Proposition 1.4 holds for the random tensor term $\mathcal{N}(w,z,w)$.

Proof. Given dyadic $N_2\ge1$, define

$$\mathcal{T}_4^{N_2}(w_1,w_3) = \mathcal{N}(w_1,\mathbf{Q}_{N_2}z,w_3).$$

Then, it suffices to prove that there exist small $\theta,\delta>0$ such that

$$\Big\|\|\mathcal{T}_4^{N_2}\|_{X_T^{s,b}\times X_T^{s,b}\to X_T^{s,-b'}}\Big\|_{L^p(\Omega)}\lesssim p^{\frac12}T^{\theta}N_2^{-\delta} \tag{8.1}$$

for any finite $p\ge1$ and $0<T\ll1$.

• **Case 1:** $N_2\lesssim\min(|n_1|,|n_3|)$ or $|\phi(\bar{n})|\gg N_2^2$.

A slight modification of the argument in Case 1 or Case 2 of the proof of Proposition 7.1 yields (8.1).

As seen in the proof of Proposition 7.1, (8.1) follows once we prove

$$\begin{aligned}&\Big\| \big\| \langle \tau \rangle^{-b'} \langle \tau_1 \rangle^{-b} \langle \tau_3 \rangle^{-b} \| \mathbf{H}_4^{N_2}(n, n_1, n_3, \tau, \tau_1, \tau_3) \|_{\ell^2_{n_1,n_3} \to \ell^2_n} \big\|_{L^2_{\tau,\tau_1,\tau_3}} \Big\|_{L^p(\Omega)} \\ &\quad \lesssim p^{\frac{1}{2}} T^\theta N_2^{-\delta},\end{aligned} \tag{8.2}$$

where $\mathbf{H}_4^{N_2}$ is given by

$$\begin{aligned}\mathbf{H}_4^{N_2}(n, n_1, n_3, \tau, \tau_1, \tau_3) = \frac{\langle n \rangle^s}{\langle n_1 \rangle^s \langle n_3 \rangle^s} \\ \times \sum_{\substack{n - n_1 - n_3 = n_2 \\ n_2 \neq n_1, n_3}} \frac{\overline{g_{n_2}}}{\langle n_2 \rangle} \mathbf{1}_{|n_2| \sim N_2} \widehat{\eta}_T(\tau - \tau_1 - \tau_3 - \phi(\bar{n})).\end{aligned}$$

Given dyadic $N, N_1, N_2, N_3 \geq 1$, we set

$$\mathbf{H}_4^{\mathbf{N}}(n, n_1, n_3, \tau, \tau_1, \tau_3) = \mathbf{1}_{|n| \sim N} \bigg(\prod_{j \in \{1,3\}} \mathbf{1}_{|n_j| \sim N_j} \bigg) \mathbf{H}_4^{N_2}(n, n_1, n_3, \tau, \tau_1, \tau_3).$$

Then, by writing

$$\mathbf{H}_4^{N_2} = \sum_{\substack{N, N_1, N_3 \geq 1, \text{ dyadic} \\ N_2 \gtrsim \max(N_1, N_3)}} \mathbf{H}_4^{\mathbf{N}} + \sum_{\substack{N, N_1, N_3 \geq 1, \text{ dyadic} \\ N_2 \ll \max(N_1, N_3)}} \mathbf{H}_4^{\mathbf{N}}, \tag{8.3}$$

we separately estimate the first and second terms on the right-hand side of (8.3) (after removing the contribution from Case 1).

• **Case 2:** $N_2 \gtrsim \max(N_1, N_3)$.

We can conclude (8.2) by an interpolation argument as in Case 3 of the proof of Proposition 7.1, once we have following lemma.

Lemma 8.2. *Given dyadic $N, N_1, N_2, N_3 \geq 1$, suppose that $N_2 \gtrsim N_{\text{med}}$, where N_{med} is as in (2.2). Let Q_0, Q_1, and Q_3 be cubes of side length $\lesssim N_2$. Fix small $s > 0$. Then, we have*

$$\begin{aligned}\sup_{\tau, \tau_1, \tau_3 \in \mathbb{R}} \Big\| \| \mathbf{1}_{Q_0}(n) \mathbf{1}_{Q_1}(n_1) \mathbf{1}_{Q_3}(n_3) \\ \times \mathbf{H}_4^{\mathbf{N}}(n, n_1, n_3, \tau, \tau_1, \tau_3) \|_{\ell^2_{n_1,n_3} \to \ell^2_n} \Big\|_{L^p(\Omega)} \lesssim p^{\frac{1}{2}} N_2^{-s+\varepsilon}\end{aligned}$$

for any finite $p \geq 1$ and dyadic $N, N_1, N_2, N_3 \geq 1$, uniformly in $0 < T \ll 1$ and the cubes Q_0, Q_1, and Q_3.

Since Lemma 8.2 follows from a minor modification of the proof of Lemma 7.2, we omit details.

• **Case 3:** $N_2 \ll \max(N_1, N_3)$.

By assuming $N_1 \geq N_3$ without loss of generality, we have $N \sim N_1 \gg N_2 \gg N_3$. In view of Case 1, we may assume that $|\phi(\bar{n})| \lesssim N_2^2$. Since $\big||n_2|^2 - |n_3|^2\big| \lesssim N_2^2$, we obtain $\big||n|^2 - |n_1|^2\big| \lesssim N_2^2$ in this case, and thus we can proceed as in Subcase 4.b of the proof of Proposition 7.1, using an interpolation argument and Lemma 2.6 (iii) (instead of Lemma 2.6 (i) which is needed for a proof of Lemma 8.2). □

Appendix A. Auxiliary lemma

In this appendix, we prove an auxiliary lemma used in the proof of Lemma 2.7.

Lemma A.1. *Given $r \in \mathbb{N}$, fix $y \in \mathbb{R}^r$. Given $N \gg 1$, let $\{\alpha_i\}_{i=1}^r \subset \mathbb{R}^r$ be a collection of linearly independent vectors, satisfying $|\alpha_i| \lesssim N$, $i = 1, \dots, r$, such that*

$$\mathrm{Vol}_r\big(\Delta(\{\alpha_i\}_{i=1}^r)\big) \geq \frac{1}{r!}, \tag{A.1}$$

where $\Delta(\{\alpha_i\}_{i=1}^r)$ denotes the r-dimensional simplex formed by the vectors $\alpha_1, \dots, \alpha_r$, and Vol_r denotes the r-dimensional volume. Suppose that

$$|y \cdot \alpha_i| \lesssim N^{a_1}, \quad i = 1, \dots, r \tag{A.2}$$

for some $a_1 \geq 1$. Then, there exists $C_r \gg 1$, independent of $N \gg 1$, $\{\alpha_i\}_{i=1}^r \subset \mathbb{R}^r$, and $a_1 \geq 1$, such that $|y| \lesssim N^{C_r a_1}$.

Proof. We argue by induction. When $r = 1$, we have $\alpha_1 \in \mathbb{R} \setminus \{0\}$. Then, given $y \in \mathbb{R}$ satisfying (A.2), the conclusion follows from $|y\alpha_1| \lesssim N^{a_1}$ and $|\alpha_1| \geq 1$.

Now, suppose that the conclusion holds for some $r \geq 1$. Let $\{\alpha_i\}_{i=1}^{r+1} \subset \mathbb{R}^{r+1}$ be a collection of linearly independent vectors, satisfying $|\alpha_i| \leq c_0 N$, $i = 1, \dots, r+1$, for some $c_0 > 0$ such that

$$\mathrm{Vol}_{r+1}\big(\Delta(\{\alpha_i\}_{i=1}^{r+1})\big) \geq \frac{1}{(r+1)!}, \tag{A.3}$$

Define the hyperplane $\mathcal{P}$ by

$$\mathcal{P} = \mathrm{span}\{\alpha_i\}_{i=1}^r \subset \mathbb{R}^{r+1}.$$

Given $y \in \mathbb{R}^{r+1}$ satisfying

$$|y \cdot \alpha_i| \leq c_1 N^{a_1}, \quad i = 1, \dots, r+1, \tag{A.4}$$

for some $c_1 > 0$, write it as

$$y = y^{\parallel} + y^{\perp},$$

where $y^{\parallel} \in \mathcal{P}$ and $y^{\perp} \perp \mathcal{P}$. From (A.4), we have

$$|y^{\parallel} \cdot \alpha_i| = |y \cdot \alpha_i| \leq c_1 N^{a_1} \tag{A.5}$$

for $i = 1, \dots, r$. Now, recall that the volume of a $(r+1)$-dimensional simplex A is given by

$$\mathrm{Vol}_{r+1}(A) = \frac{1}{r+1} \cdot (\text{base volume}) \cdot (\text{height}). \tag{A.6}$$

Then, from (A.6), (A.3), and $|\alpha_{r+1}| \leq c_0 N$, we see that

$$\mathrm{Vol}_r\big(\Delta(\{\alpha_i\}_{i=1}^r)\big) \geq \frac{r+1}{|\alpha_{r+1}|}\mathrm{Vol}_{r+1}\big(\Delta(\{\alpha_i\}_{i=1}^{r+1})\big) \geq (c_0 N)^{-1} \cdot \frac{1}{r!}.$$

By setting $\beta_i = c_0^{\frac{1}{r}} N^{\frac{1}{r}} \alpha_i$, $i = 1, \dots, r$, we see that β_i's are linearly independent, $|\beta_i| \leq c_0^{1+\frac{1}{r}} N^{1+\frac{1}{r}} =: c_0 M$ for any $i = 1, \dots, r$, and $\mathrm{Vol}_r\big(\Delta(\{\beta_i\}_{i=1}^r)\big) \geq \frac{1}{r!}$. Moreover, from (A.5) with $a_1 \geq 1$, we have

$$|y^{\parallel} \cdot \beta_i| \leq c_1 c_0^{\frac{1}{r}} N^{\frac{1}{r}} N^{a_1} \leq c_1 M^{a_1}$$

for any $i = 1, \dots, r$. Hence, by the inductive hypothesis, we conclude that

$$|y^{\|}| \lesssim M^{C_r a_1} \sim N^{C_r(1+\frac{1}{r})a_1} \tag{A.7}$$

for some $C_r \gg 1$.

It remains to show that $|y^{\perp}| \lesssim N^{C_r' a_1}$ for some $C_r' > 0$. We prove this by contradiction. Suppose that for any $k \in \mathbb{N}$, there exists $N = N(k) \gg 1$ and $y_k \in \mathbb{R}^{r+1}$, satisfying (A.2), such that

$$|y_k^{\perp}| \gg N^{ka_1}, \tag{A.8}$$

where $y_k^{\perp}$ is the component of y_k orthogonal to the hyperplane $\mathcal{P}$.

Recall that α_{r+1} is not on the hyperplane $\mathcal{P}$. Write $\alpha_{r+1} = \alpha_{r+1}^{\|} + \alpha_{r+1}^{\perp}$, where $\alpha_{r+1}^{\|} \in \mathcal{P}$ and $\alpha_{r+1}^{\perp} \perp \mathcal{P}$. Then, we claim that

$$|\alpha_{r+1}^{\perp}| = \operatorname{dist}(\alpha_{r+1}, \mathcal{P}) \gtrsim N^{-r}. \tag{A.9}$$

Since $|\alpha_i| \lesssim N$, $i = 1, \dots, r$, we have

$$\operatorname{Vol}_r\big(\Delta(\{\alpha_i\}_{i=1}^r)\big) \lesssim N^r. \tag{A.10}$$

Then, (A.9) follows from (A.6) and (A.10).

From (A.8) and (A.9), we have

$$|y_k^{\perp} \cdot \alpha_{r+1}| = |y_k^{\perp}||\alpha_{r+1}^{\perp}| \gtrsim N^{ka_1} N^{-r} \gg N^{\frac{1}{2}ka_1} \tag{A.11}$$

for any $N \gg 1$, provided that $k > 2r$. Hence, from (A.7) and (A.11) with $|\alpha_{r+1}| \lesssim N$, we see that there exist $c_2 \gg c_3 > 0$ such that

$$|y_k \cdot \alpha_{r+1}| \geq |y_k^{\perp} \cdot \alpha_{r+1}| - |y_k^{\|} \cdot \alpha_{r+1}| \geq c_2 N^{\frac{1}{2}ka_1} - c_3 N^{C_r(1+\frac{1}{r})a_1+1} \gg N^{a_1}$$

for any $N \gg 1$, provided that $k > 2C_r(1+\frac{1}{r})+2$, which is a contradiction to (A.4). Therefore, there exists $C_{r+1}' > 0$ such that, given any $y \in \mathbb{R}^{r+1}$ satisfying (A.4), we have $|y^{\perp}| \lesssim N^{C_{r+1}' a_1}$. Together with (A.7), we conclude that there exists $C_{r+1} > 0$ such that $|y| \lesssim N^{C_{r+1} a_1}$ for any $y \in \mathbb{R}^{r+1}$ satisfying (A.4). We note that, from the discussion above, it suffices to choose $C_{r+1}' > \max(2r, 2C_r(1+\frac{1}{r})+2)$, which shows that the choice of C_{r+1}' (and hence of C_{r+1}) is independent of $N \gg 1$, $\{\alpha_i\}_{i=1}^{r+1}$, and $a_1 \geq 1$.

Finally, by induction, we concludes the proof of Lemma A.1. □

Remark A.2. Let $\{\alpha_i\}_{i=1}^r \subset \mathbb{Z}^r$ be a collection of linearly independent *integer* vectors, satisfying $|\alpha_i| \lesssim N$, $i = 1, \dots, r$. Then, we have

$$\operatorname{Vol}_r\big(\Delta(\{\alpha_i\}_{i=1}^r)\big) = \frac{1}{r!}|\det(\alpha_1, \alpha_2, \cdots, \alpha_r)| \geq \frac{1}{r!},$$

and thus the hypothesis (A.1) is automatically satisfied.

Appendix B. Declarations

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