Lec 18 : May 27, 2025 (TUE)

It remains to prove convergence of $\Phi_{N_k}(\cdot)(u_{0,k}) \to \Phi(\cdot)(u_0)$

Prop 3.4.3: (PDE approximation lemma)

$u_0 \in H^s$ with $\|u_0\|_{H^s} \le A$ for some $A \ge 1$

- $u_{0,N} \to u_0$ in H^s s.t.

$$\|u_{0,N}\|_{H^s} \le A$$

$\forall N \in \mathbb{N}$

$$\|\underbrace{\Phi_N(t)(u_{0,N})}_{=: u_N(t) = \text{soln to the truncated dynamics.}}\|_{H^s} \le A \qquad \forall |t| \le T$$

$s_1 < s < \frac{1}{2}$

$s = \frac{1}{2} - \varepsilon$

$s_1 = \frac{1}{2} - 2\varepsilon$

$u(t) = \Phi(t)(u_0)$

Then,

$$\|u - u_N\|_{C_T H^{s_1}_x} \lesssim e^{C_1 A^{C_2} T} \left(\|u_0 - u_{0,N}\|_{H^{s_1}} + A \, N^{\overbrace{s_1 - s}^{<0}} \right)$$

for any $N \gg_T 1$

· Review of LWP in H^{s_1} :

(A1) · Suppose $\|u(t)\|_{H^{s_1}} \le 2A$ for $0 \le t \le \tau$ $(\le T)$.

· Let δ = local existence time with H^{s_1}-data $\le 2A$
$\sim A^{-\gamma}$ for some $\gamma > 0$. (see below)

· Let $I_j = [j\delta, (j+1)\delta]$, $j = 0, 1, \cdots, [T/\delta]$

Then, the LWP argument yields

(A2) $\|u\|_{X^{s_1,b}(I_j)} \le C_1 \|u(j\delta)\|_{H^{s_1}} + \boxed{C_2 \delta^\theta \|u\|^4_{X^{s_1,b}}} \|u\|_{X^{s_1,b}}$

$b = \frac{1}{2}+$

$\le C_3 A$ by LWP.

$\le C_2 C_3^4 \delta^\theta A^4 \le \frac{1}{2}$

by choosing $\delta \sim A^{-\gamma}$.

Persistence of regularity (see pp. 10-12 of Lec. 2).

Let $s > s_1$. By $\langle n \rangle^{\sigma} \lesssim \max_{j=1,\cdots,b} \langle n_j \rangle^{\sigma}$ for $\sigma \geq 0$,

a slight modification of (A2) yields

(A2)' $\| u \|_{X^{\underline{s},b}(I_j)} \leq C_1 \| u(j\delta) \|_{H^{\underline{s}}} + \underbrace{C_2' \delta^{\theta} \| u \|^{4}_{X^{\underline{s_1},b}} \| u \|_{X^{\underline{s},b}}}_{\leq \frac{1}{2} \text{ (by taking } \delta \text{ smaller by a multip. const)}}$

$\Rightarrow$ (A3) $\| u \|_{X^{\underline{s},b}(I_j)} \leq 2 C_1 \| u(j\delta) \|_{H^{\underline{s}}}$

- Recall : $b > \frac{1}{2} \Rightarrow$ (A4) $\| u \|_{C(I_j; H^{\underline{s}})} \leq C_4 \| u \|_{X^{\underline{s},b}(I_j)}$

- (A3) & (A4) $\Rightarrow$ (A5) $\| u \|_{C(I_j; H^{s})} \leq 2 C_1 C_4 \| u(j\delta) \|_{H^{s}}$.

↑ really on $I_j \cap [0, \tau]$.

We derived (A5) by assuming (A1) on p.2.

↑ (A5) holds on $I_j \cap [0,\tau]$.

- (A5) (under (A1): $\|u(t)\|_{H^{s_1}} \le 2A$ for $0 \le t \le \tau (\le T)$ implies

$$\|u(j\delta)\|_{H^s} \le \underbrace{(2C_1C_4)^{j+1}}_{\lesssim e^{cT/\delta} \sim e^{cA^\gamma T}} \|u_0\|_{H^s}, \quad j = 0, 1, \cdots \text{ s.t. } j\delta \le \tau$$

$\Rightarrow$ (A6) $\boxed{\|u\|_{X^{s,b}(I_j)} \lesssim e^{cA^\gamma T} \|u_0\|_{H^s}}$, $j = 0, 1, \cdots$ s.t. $j\delta \le \tau$.

Pf: $u(t) - u_N(t) = S(t)(u_0 - u_{0,N})$

$$- i \int_0^t S(t-t') \big[N(u)(t') - \pi_{\le N} N(\pi_{\le N} u_N)(t') \big] dt'$$

$$\pi_{\le N} \big(N(u)(t') - N(\pi_{\le N} u_N)(t') \big)$$

$$+ \pi_{>N} N(u)(t').$$

- $\delta \sim A^{-\sigma}$ as on p.2 , $I_j = [j\delta, (j+1)\delta]$.

$$\Rightarrow \|u - u_N\|_{X^{s_1,b}(I_0)} \le C_1 \|u_0 - u_{0,N}\|_{H^{s_1}}$$

$$+ C_2'' \delta^\theta \Big(1 + \|u\|^4_{X^{s_1,b}(I_0)} + \|u_N\|^4_{X^{s_1,b}(I_0)} \Big) \|u - u_N\|_{X^{s_1,b}(I_0)}$$

$\lesssim A^4$

(A7)

$$+ C \delta^\theta N^{s_1 - s} \|N(u)\|_{X^{s,b-1+\theta}(I_0)}$$

$\|\pi_{>N} f\|_{H^{s_1}} \lesssim N^{s_1 - s} \|f\|_{H^s}$, $s_1 < s$

· We have

$$\|\mathcal{N}(u)\|_{X^{s,\,b-1+\theta}(I_0)} \lesssim \|u\|^5_{X^{s,b}(I_0)}$$

(A8)

$$\overset{\text{(Ab) on p.4}}{\lesssim} e^{CA^\gamma T} A^5$$

In fact, proceeding as on p.3, we can have a better bd by

$$\|u\|^4_{X^{s_1,b}} \|u\|_{X^{s,b}}$$

· (A7) & (A8) (and by making δ smaller by a multip. const.

$$\Rightarrow \|u - u_N\|_{X^{s_1,b}(I_0)} \le 2C_1 \|u_0 - u_{0,N}\|_{H^{s_1}} + C e^{CA^\gamma T} A N^{s_1 - s}$$

↑ used $\delta^\theta A^4 \ll 1$.

$$\Rightarrow \|u - u_N\|_{C(I_0; H^{s_1})} \lesssim \|u - u_N\|_{X^{s_1, b}(I_0)}$$

(A9) $$\lesssim \|u_0 - u_{0,N}\|_{H^{s_1}} + e^{cA^\delta T} A N^{s_1 - s}.$$

$\Rightarrow$ By choosing $N = N_0 \gg 1$, ← Recall $\|u_N\|_{C(I_0; H^{s_1})} \le A$

$$\|u\|_{C(I_0; H^{s_1})} \le 2A$$

$\Rightarrow$ (A1) on p. 2 is satisfied with $T = \delta$ (i.e. on $[0, \delta]$.)

Then, by (Ab) on p. 4, we have

(A10) $$\|u\|_{X^{s, b}(I_1)} \lesssim e^{cA^\delta T} \|u_0\|_{H^s}.$$

$I_1 = [\delta, 2\delta]$

· Using (A9) and repeating the argument on $I_1 = [\delta, 2\delta]$, we obtain

(A11)
$$\|u - u_N\|_{X^{s_1,b}(I_1)} \le 2C_1 \|u(\delta) - u_N(\delta)\|_{H^{s_1}} + C e^{cA^r T} A N^{s_1 - s}.$$

$$\Rightarrow \|u - u_N\|_{C(I_1; H^{s_1})} \longrightarrow 0 \quad \text{as} \quad N \to \infty.$$

$\Rightarrow$ By choosing $N = N_1 \ (\ge N_0)$,

$$\|u\|_{C([0,2\delta]; H^{s_1})} \le 2A$$

$\Rightarrow$ we can repeat the argument on I_j, $j = 0, 1, \cdots, \left[\frac{T}{\delta}\right]$.

$$\Rightarrow \quad \| u - u_N \|_{C([0,T];H^{s_1})} \lesssim e^{CA^{\gamma}T} \left(\| u_0 - u_{0,N} \|_{H^{s_1}} + A N^{s_1 - s} \right).$$

$C^{\frac{I}{\delta}} \sim e^{CA^{\gamma}T}$

Let $X_j = \| u - u_N \|_{C(I_j;H^{s_1})}$. Then, we have

$$X_{j+1} \leq C_0 X_j + \square \leq C_0 \left(C_0 X_{j-1} + \square \right) + \square$$

$$= C_0^j X_0 + \sum_{k=0}^{j} C_0^k \square$$

$$\leq C_0^j X_0 + C_0^{j+1} \square$$

$\square$

By Prop 3.4.3, we have the following:

Given $u_0 \in \Sigma = \bigcup_{i=1}^{\infty} \Sigma^i$, $\exists\, i$ s.t. $u_0 \in \Sigma^i$.

$\Rightarrow \exists\, N_k \nearrow \infty$ & $u_{0,k} \in \Sigma^i_{N_k}$ s.t.

$u_{0,k} \to u_0$ in H^s

↖ uniform (in k) growth bd.

Cor 3.4.2 $\Longrightarrow$ Prop 3.4.3

$$\Phi_{N_k}(\cdot)(u_{0,k}) \longrightarrow \Phi(\cdot)(u_0) \text{ in } C(\mathbb{R}; H^{s_1})$$

Cor: $\forall\, u_0 \in \Sigma$ $\exists\, N_k \nearrow \infty$ s.t. ($N_k = N_k(u_0)$)

$$\Phi_{N_k}(\cdot)(u_0) \longrightarrow \Phi(\cdot)(u_0)$$

(by a diagonal argument)

Also, $\Phi_N(\cdot)(u_0) \longrightarrow \Phi(\cdot)(u_0)$

· Invariance of the Gibbs meas ρ:

WTS $\rho(A) = \rho(\Phi(t)A)$

By time reversibility of $\Phi(t)$, it suffices to show

$$\text{(*)} \quad \boxed{\rho(A) \le \rho(\Phi(t)A)}$$

A, meas in H^s

$\forall\, t \in \mathbb{R}$.

By inner regularity, $\rho(A) = \sup_{\substack{F \subset A \\ F, \text{ closed}}} \rho(F)$

$\Rightarrow \exists\, \{F_n\}$, closed sets in H^s s.t.

$$F_n \subset A \quad \text{and} \quad \rho(A) = \lim_{n\to\infty} \rho(F_n)$$

Claim: Suffices to prove (*) for closed sets.

$$\rho(A) = \lim_{n\to\infty} \rho(F_n) \le \overline{\lim_{n\to\infty}}\, \rho(\Phi(t)F_n) \le \rho(\Phi(t)A).$$

uniqueness

$F_n \subset A \Rightarrow \Phi(t)F_n \subset \Phi(t)A$

· Given a closed set F in H^s,

let $K_n = \{u \in F : \|u\|_{H^\sigma} \le n\}$, $\quad s < \sigma < \frac{1}{2}$

Rellich's lemma $\Rightarrow$ K_n is compact in H^s.

$\Rightarrow$ $\rho(F) = \lim_{n\to\infty} \rho(K_n)$

i.e. suffices to prove $\circledast$ for compact sets in H^s.

Let K be a cpt set in H^s.

From $\rho_N \to \rho$ & portmanteau thm, we have

(1) $\quad \rho(\Phi(t)K + \overline{B}_\varepsilon) \ge \overline{\lim}\, \rho_N(\Phi(t)K + \overline{B}_\varepsilon)$

local thery local unif continuity

· Fix $t \ll 1$.

$$\Phi_N(t)(K + B_\delta) \subset \Phi_N(t)K + B_{\varepsilon/2}$$

$$\subset \Phi(t)K + B_\varepsilon$$

↑ Approx lemma.

By the invariance of the truncated Gibbs meas ρ_N,

$$\rho_N(K + B_\delta) = \rho_N(\Phi_N(t)(K + B_\delta))$$

$$\leq \rho_N(\Phi(t)K + B_\varepsilon)$$

(2)

$$\rho(K) \leq \rho(K + B_\delta) \leq \underline{\lim}\, \rho_N(K + B_\delta)$$

$$\overset{(2)}{\leq} \underline{\lim}\, \rho_N(\Phi(t)K + B_\varepsilon) \leq \overline{\lim}\, \rho_N(\Phi(t)K + \overline{B}_\varepsilon).$$

$$\overset{(1)}{\leq} \rho(\Phi(t)K + \overline{B}_\varepsilon).$$

Take $\varepsilon \to 0$. $\Rightarrow \quad \rho(K) \le \rho(\Phi(t) K)$ for $t \ll 1$

↙ for any $t \ge 0$

(and $\forall\, t \in \mathbb{R}$

This proves invariance of ρ.

· Another approach to prove invariance of ρ.

(**) WTS: $\int_{H^s} F(\Phi(t) u)\, d\rho(u) = \int_{H^s} F(u)\, d\rho(u)$

$\forall\, F \in C_b(H^s; \mathbb{R})$.

① $\rho_N \to \rho$ implies

(B1) $\int_{H^s} F(u)\, d\rho(u) = \lim_{N\to\infty} \int_{H^s} F(u)\, d\rho_N(u)$

(B2) $\int_{H^s} F(\Phi(t) u)\, d\rho(u) = \lim_{N\to\infty} \int_{H^s} F(\Phi(t) u)\, d\rho_N(u)$

② invariance of ρ_N implies

(B3) $$\int_{H^s} F(\Phi_N(t)u)\, d\rho_N(u) = \int_{H^s} F(u)\, d\rho_N(u)$$

③
$$\left| \int_{H^s} F(\Phi(t)u) - F(\Phi_N(t)u)\, d\rho_N \right|$$

$$\leq Z_N^{-1} \| F(\Phi(t)u) - F(\Phi_N(t)u) \|_{L^2(\mu_1)} \underbrace{\| R_N(u) \|_{L^2(\mu_1)}}_{\lesssim 1,\ \text{unif in } N}$$

$$\begin{cases} d\rho_N = Z_N^{-1} R_N(u)\, d\mu_1 \\ Z_N = \| R_N \|_{L^1(\mu_1)} \to Z_\infty \quad \text{as } N \to \infty \end{cases}$$

$$\lesssim \| \mathbb{1}_{A_{N,\delta}} (F(\Phi(t)u) - F(\Phi_N(t)u) \|_{L^2(\mu_1)}$$
$$+ \| \mathbb{1}_{A_{N,\delta}^c} (F(\Phi(t)u) - F(\Phi_N(t)u) \|_{L^2(\mu_1)}$$

where $A_{N,\delta} = \{ u \in H^s : |F(\Phi(t)u) - F(\Phi_N(t)u)| \leq \delta \}$

$$\leq \delta + 2\|F\|_{L^\infty}\left(\mu_1\left(A_{N,\delta}^c\right)\right)^{1/2}$$

We have $\Phi_N(t)u \to \Phi(t)u$ for $u \in H^s$ a.s. w.r.t. μ_1

$\Rightarrow$ in probability. (Egoroff)

$\Rightarrow$ $F(\Phi_N(t)u) \to F(\Phi(t)u)$ in probability

(conti. mapping thm and F is conti.)

$$\leq 2\delta \quad \text{for any } N \geq N_\delta \gg 1.$$

- Since the choice of $\delta > 0$ was arbitrary, we obtain

(B4) $$\lim_{N\to\infty}\left|\int_{H^s} F(\Phi(t))\,d\rho_N(u) - \int_{H^s} F(\Phi_N(t)u)\,d\rho_N(u)\right| = 0.$$

Finally, $\int_{H^s} F(\Phi(t)u)\, d\rho(u)$

$$\overset{(B2)}{=} \lim_{N\to\infty} \int_{H^s} F(\Phi(t)u)\, d\rho_N(u)$$

$$\overset{(B4)}{=} \lim_{N\to\infty} \int_{H^s} F(\Phi_N(t)u)\, d\rho_N(u)$$

$$\overset{(B3)}{=} \lim_{N\to\infty} \int_{H^s} F(u)\, d\rho_N(u)$$

$$\overset{(B1)}{=} \int_{H^s} F(u)\, d\rho(u) \,.$$

This proves (**) on p. 14.

- In the case of a stochastic PDE, the soln map is random due to a stochastic forcing which is assumed to be independent of the random initial data.
  - ρ, ρ_N = prob. measures on H^s
  - $\mathbb{P}_2$ = prob. meas for the stochastic forcing:
    $\omega_2 \in \Omega_2 \longmapsto \xi(\omega_2) \in \mathcal{D}'_{t,x}(\mathbb{R}_+ \times \mathbb{T}^d)$.

In this setting, we say that ρ is invariant if

$$\int_{H^s} \int_{\Omega_2} F(\Phi^{\omega_2}(t)u)\, d\mathbb{P}_2\, d\rho(u) = \int_{H^s} F(u)\, d\rho(u)$$

Then, Markov semigroup $P_t F(u) = \mathbb{E}_{\omega_2}[F(\Phi^{\omega_2}(t)u)]$

$\Leftrightarrow$ $$\int_{H^s} P_t F(u)\, d\rho(u) = \int_{H^s} F(u)\, d\rho(u)$$

- Compare these with (**) on p. 14.

  In particular, the argument on pp 14-17 can be easily adapted in the SPDE setting.

  - See Subsec 5.2 of Oh-Robert-Wang, CMP '21

    (See also Sec 5 of Oh-Robert-Tzvetkov, AHL '23.

- Define the adjoint P_t^* by $\langle F, P_t^* \rho \rangle = \langle P_t F, \rho \rangle$

  $\left(= \int P_t F(u)\, d\rho(u) \right.$

  invariance $\Longleftrightarrow P_t^* \rho = \rho$. $\forall\, t \geq 0$.

  See Chapter 5 of Da Prato's book "An intro to ∞-dim'l analysis."

Sec 3.5: Gibbs measures on $\mathbb{T}^2$ $= \Phi^k_2$ - measure

$\uparrow d=2$

$$d\rho = Z^{-1} e^{\mp \frac{1}{k}\int_{\mathbb{T}^2} |u|^k dx} d\mu_1$$

where $d\mu_1 = Z^{-1} e^{-\frac{1}{2}\|u\|_{H^1}} du$ on $\mathbb{T}^2$.

- $\operatorname{supp}\rho \subset H^s(\mathbb{T}^2) \setminus L^2(\mathbb{T}^2)$, $s < 0 = 1 - \frac{d}{2}$

$$\Rightarrow \int_{\mathbb{T}^2} |u|^k dx = \infty, \text{ a.s.}$$

We need to renormalize $\int_{\mathbb{T}^2} |u|^k dx$.

In the following, we only consider the defocusing case

(and real-valued.

Focusing case:

- $k=4$: non-normalizability. Brydges-Slade, CMP '96
  Oh-Seong-Tolomeo, FMσ '24
  Greco-Oh-Tao-Tolomeo, PAMS '25
- $k=3$: real-valued $e^{\int u^3 dx}$

  with a renormalized L^2-cutoff, ρ can be constructed
  Jaffe 90's, Oh-Seong-Tolomeo. '24.

Under μ_1, we have $u = \sum_{n\in\mathbb{Z}^2} \frac{g_n}{\langle n\rangle} e_n \in H^{s}(\mathbb{T}^2)\setminus L^2(\mathbb{T}^2)$ a.s.

$\{g_n\}_{n\in\mathbb{Z}^d}$ = indep. std. $\mathbb{C}$-valued Gaussian r.v.s

$\overline{g_{-n}} = g_n$, $\forall\, n\in\mathbb{Z}^2$. $\quad \mathrm{Var}(g_n) = 1$.

Given $N \in \mathbb{N}$, let $u_N = \mathbb{T}_{\le N} u$.

$\Rightarrow$ For each fixed $x \in \mathbb{T}^2$,

$u_N(x)$ = mean 0, $\mathbb{R}$-valued Gaussian r.v. with variance

$$\sigma_N := \mathbb{E}\left[u_N^2(x) \right] = \sum_{|n| \le N} \frac{1}{\langle n \rangle^2} \sim \log N \to \infty.$$

↑ indep of $x \in \mathbb{T}^2$.

$$\mathbb{E}\left[g_n^k \, \overline{g}_m^\ell \right] = \delta_{k\ell} \delta_{nm} \cdot k!$$

(On a manifold, σ_N depends on $x \in M$.

Similarly, $\mathbb{E}\left[\int_{\mathbb{T}^2} u_N^2(x) \, dx \right] \sim \sum_{|n| \le N} \frac{1}{\langle n \rangle^2} \sim \log N$

· Wick-renormalized power: $u_N^{\diamond 2}$ or $\mathbb{I}[u_N^2]\mathbb{I}$

$$:u_N^2(x): = u_N^2(x) - \mathbb{E}[u_N^2(x)]$$
$$= u_N^2(x) - \sigma_N$$
$$= H_2(u_N(x); \sigma_N)$$

renormalization is a ptwise operation.

↑ Hermite polynomial of degree 2.

$H_2(x;\sigma) = x^2 - \sigma$

$$\Rightarrow \int_{\mathbb{T}^2} :u_N^2(x): dx = \sum_{|n|\le N} \frac{|g_n|^2 - 1}{\langle n\rangle^2}$$

$\mathbb{E}[(|g_n|^2-1)(|g_m|^2-1)]$
$= 0$ unless $n=m$

· $\mathbb{E}\left[\int_{\mathbb{T}^2} :u_N^2: dx\right] = 0.$

· $\mathbb{E}\left[\left(\int_{\mathbb{T}^2} :u_N^2: dx\right)^2\right] \lesssim \sum_{|n|\le N} \frac{\overbrace{\mathbb{E}[(|g_n|^2-1)^2]}^{\sim 1}}{\langle n\rangle^4} \lesssim 1$, unif in N.

$$\Rightarrow \quad \int_{\mathbb{T}^2} :u^2: \, dx = \lim_{N\to\infty} \int_{\mathbb{T}^2} :u_N^2: \, dx$$

exists in $L^p(\Omega)$, $\forall\, p < \infty$.

· Hermite polynomials, white noise functional, Wick renormalization

· Hermite poly $H_k(x;\sigma)$ is defined through the generating function.

$$G(t,x;\sigma) = e^{tx - \frac{1}{2}\sigma t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x;\sigma)$$

$$H_0(x;\sigma) = 1, \quad H_1(x;\sigma) = x$$

$$H_2(x;\sigma) = x^2 - \sigma, \quad H_3(x;\sigma) = x^3 - 3\sigma x$$

$$H_4(x;\sigma) = x^4 - 6\sigma x^2 + 3\sigma^2.$$

When $\sigma = 1$, we write $H_k(x) = H_k(x;\sigma=1)$.

· $1, x, x^2, x^3, \cdots$

Apply the Gram-Schmidt orgonalization in $L^2(\mathbb{R}; \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}})$

$$\Rightarrow \quad H_k(x;\sigma) = \sigma^{k/2} H_k\left(\frac{x}{\sigma^{1/2}}\right)$$

· White noise functional: $d\mu_0 = Z^{-1} e^{-\frac{1}{2}\|u\|_{L^2}^2} du$ on $\mathbb{T}^d$.

Under μ_0, we have $u^\omega = \sum_{n\in\mathbb{Z}^d} g_n e_n \in H^s(\mathbb{T}^d) \setminus H^{-\frac{d}{2}}(\mathbb{T}^d)$, $s < -\frac{d}{2}$

↑ same "intensity" (of randomness

$\Rightarrow$ white noise.

· $W_\cdot : L^2(\mathbb{T}^d) \to L^2(\Omega)$

$$W_f(\omega) = \langle f, u^\omega \rangle_{L^2} = \sum_{n\in\mathbb{Z}^d} \hat{f}(n)\, \overline{g_n(\omega)}$$

i.e. Wiener integral "$\int f\, dB$."

- Given $f \in L^2(\mathbb{T}^d)$, W_f is a mean 0 Gaussian r.v.

$$\mathrm{Var}(W_f) = \mathbb{E}\Big[\sum_{n \in \mathbb{Z}^d} \hat{f}(n)\, \overline{g_n} \cdot \sum_{m \in \mathbb{Z}^d} \overline{\hat{f}(m)}\, g_m \Big]$$

$$= \|f\|_{L^2}^2 \qquad (n = m)$$

Lem 3.5.1:

$f, h \in L^2(\mathbb{T}^2)$ with $\|f\|_{L^2} = \|h\|_{L^2} = 1$

Then,

$$\mathbb{E}\left[H_k(W_f)\, H_\ell(W_h) \right] = \delta_{k\ell}\, k! \left[\langle f, h \rangle_{L^2} \right]^k.$$

Another equiv def of a white noise Ξ

$\mathbb{E}[\Xi(f)] = 0$

$\mathbb{E}[\Xi(f)\,\Xi(g)] = \langle f, g \rangle_{L^2(\mathbb{T}^d)}$

Pf: Recall $\int_\Omega e^{Wf}\,dP = e^{\frac{1}{2}\|f\|_{L^2}^2}$

$G(t,x;1) = e^{tx-\frac{1}{2}t^2}$

- $\int_\Omega G(t, W_f; 1)\, G(s, W_h; 1)\, dP$

$= e^{-\frac{1}{2}(t^2+s^2)} \underbrace{\int_\Omega e^{W_{tf+sh}}\,dP}_{= e^{\frac{1}{2}\|tf+sh\|_{L^2}^2}}$

$tW_f + sW_h = W_{tf+sh}$ ↑ Gaussian

$\|f\|_{L^2} = \|h\|_{L^2} = 1$

$= e^{ts\langle f,h\rangle_{L^2}}$

$= \sum_{m=0}^{\infty} \frac{(ts)^m}{m!} (\langle f,h\rangle_{L^2})^m$

- On the other hand, $(LHS) = \sum_{k,l=0}^{\infty} \frac{t^k s^l}{k!\,l!} \int_\Omega H_k(W_f)\, H_l(W_h)\, dP$

Compare the coefficients

□

· Wick-ordered monomial:

$$:u_N^k(x): \;=\; H_k(u_N(x);\sigma_N)$$

Prop 3.5.2: $k \in \mathbb{N}$.

$\{:u_N^k:\}_{N\in\mathbb{N}}$ is Cauchy in $W^{s,\infty}(\mathbb{T}^2)$, a.s., $s<0$. (also $\mathcal{C}^s(\mathbb{T}^2)$)

also in $L^p(\mu_1; W^{s,\infty}(\mathbb{T}^2))$

$\forall\, p<\infty$

· Define

$$:u^k: \;=\; \lim_{N\to\infty} :u_N^k: \quad \text{in } W^{s,\infty}(\mathbb{T}^2), \text{ a.s. } s<0$$

$$\text{in } L^p(\mu_1; W^{s,\infty}(\mathbb{T}^2)).$$

No lecture next week!
(June 03, 2025)