Lec 2 : Nov 21, 2024 (THU)

Aside: **Riesz-Thorin interpolation**: $1 \le p_j, q_j \le \infty$, $j = 0, 1$

T, bdd from $L^{p_j}(X) \to L^{q_j}(Y)$

with $\|T\|_{L^{p_j} \to L^{q_j}} = M_j$

$\Rightarrow$ T is bdd from $L^p(X) \to L^q(Y)$

$0 < \theta < 1$,

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

with $\|T\|_{L^p \to L^q} \le M_0^{1-\theta} M_1^{\theta}$

true for unbdd ϕ satisfying some growth condition

$\Leftarrow$ **3 line lemma** from complex analysis: $\phi \in C_b$ on $0 \le \mathrm{Re}\, z \le 1$, holomorphic in the interior

$$\left.\begin{aligned} |\phi(z)| &\le M_0, \ \mathrm{Re}\, z = 0 \\ |\phi(z)| &\le M_1, \ \mathrm{Re}\, z = 1 \end{aligned}\right\} \Rightarrow |\phi(z)| \le M_0^{1-\theta} M_1^{\theta}, \quad \mathrm{Re}\, z = \theta.$$

Pf: $f \in L^p(X)$, $g \in L^{p'}(Y)$

$f = F e^{i\Phi}$

$g = G e^{i\Psi}$ $\quad F, G \geq 0$, Φ, Ψ, real

Define $f_z = F^{\frac{p}{az+b}} e^{i\Phi}$, $g_z = G^{\frac{p'}{cz+d}} e^{i\Psi}$.

Set $\Phi(z) = \int_Y (Tf_z)\, g_z$ $\Leftarrow$ analytic in z

Choose a, b, c, d s.t.

(i) $|f_z|^{p_0} = |f|^p$, $\quad |g_z|^{p_0'} = |g|^{p'}$ for $\mathrm{Re}\, z = 0$

(ii) $|f_z|^{p_1} = |f|^p$, $\quad |g_z|^{p_1'} = |g|^{p'}$ for $\mathrm{Re}\, z = 1$

(iii) $f_z = f$, $\quad g_z = g$ for $z = \theta$.

(EX: Show $\exists\, a, b, c, d$ satisfying (i) – (iii) if $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$

- $|\Phi(z)| = \left|\int_Y (Tf_z)\, g_z\right| \le M_0 \|f_z\|_{L^{p_0}} \|g_z\|_{L^{q_0'}}$

$$= M_0 \|f\|_{L^p}^{p/p_0} \|g\|_{L^{q'}}^{q'/q_0'} \quad \text{for } \mathrm{Re}\, z = 0.$$

- $|\Phi(z)| \le M_1 \|f\|_{L^p}^{p/p_1} \|g\|_{L^{q'}}^{q'/q_1'}$ for $\mathrm{Re}\, z = 1$

3 line lemma.

$\Rightarrow$ When $z = \theta$, we have

$$\underline{\left|\int (Tf)\, g\right|} \le M_0^{1-\theta} \left(\|f\|_{L^p}^{p/p_0} \|g\|_{L^{q'}}^{q'/q_0'} \right)^{1-\theta}$$

$$\times\, M_1^{\theta} \left(\|f\|_{L^p}^{p/p_1} \|g\|_{L^{q'}}^{q'/q_1'} \right)^{\theta}$$

$$= \underline{M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p} \|g\|_{L^{q'}}}$$

$\left(\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \right.$ and $\left. \frac{1}{q'} = \frac{1-\theta}{q_0'} + \frac{\theta}{q_1'} \right.$

Application of Riesz - Thorin interpolation

① Hausdorff - Young inequality: $2 \le p \le \infty$.

$$\|f\|_{L^p} \le \|\hat{f}\|_{L^{p'}}.$$

- $p = 2$: $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$ (Plancherel)
- $p = \infty$: $\|f\|_{L^\infty} \le \sup_x \left| \int \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi \right| \le \|\hat{f}\|_{L^1_\xi}$

$\Rightarrow$ R-Th interpolation

② Schur's test: $Tf(x) = \int K(x,y) f(y) dy$

Suppose $\sup_x \int |K(x,y)| dy \le M_0$

$$\sup_y \int |K(x,y)| dx \le M_1$$

$\Rightarrow$ T is bounded on L^p, $1 \le p \le \infty$.

Pf: · Suppose $\sup_x \left| \int |K(x,y)|\, dy \right. \le M_0$

$$\Rightarrow \quad |Tf(x)| = \left| \int K(x,y) f(y)\, dy \right| \quad \Rightarrow \text{bdd on } L^\infty$$

$\uparrow$ $\sup_x$

$$\le \underbrace{\sup_x \int |K(x,y)|\, dy}_{\le M_0} \cdot \|f\|_{L^\infty}$$

· Suppose $\sup_y \int |K(x,y)|\, dx \le M_1$

$$\Rightarrow \quad \|Tf\|_{L^1_x} \le M_1 \|f\|_{L^1_y} \quad \Rightarrow \text{bdd on } L^1.$$

$\Rightarrow$ By R-Th, T is bounded on L^p.

with $\|T\|_{L^p \to L^p} \le M_0^{1-\frac{1}{p}} M_1^{1/p}$.

$$\left(\frac{1}{p} = \frac{1-\theta}{\infty} + \frac{\theta}{1} \Rightarrow \theta = \frac{1}{p}. \right.$$

Sec 1.2: conservation laws, global well-posedness, and persistence of regularity

(NLS) $i\partial_t u + \Delta u \mp |u|^{p-1}u = 0$

$\Rightarrow \partial_t u = i\Delta u \mp i\,|u|^{p-1}u.$

· Conservation laws

Mass $M(u) = \int |u|^2 dx$ ($= u\bar{u}$)

If u is a (smooth) soln to (NLS), then $M(u(t)) = M(u(0))$.

$$\partial_t M(u) = 2\,\mathrm{Re}\int \partial_t u\,\bar{u}\,dx$$

$$= 2\,\mathrm{Re}\, i\int \Delta u\,\bar{u}\,dx \mp \underbrace{2\,\mathrm{Re}\, i\int |u|^{p+1}dx}_{=0}$$

$$\overset{\text{IBP}}{=} -2\,\mathrm{Re}\, i\int |\nabla u|^2 dx$$

$$= 0.$$

Given $u_0 \in H^s(\mathbb{R}^d)$, take $\{u_{0,n}\}_{n=1}^{\infty} \subset \mathcal{S}(\mathbb{R}^d)$ s.t. $u_{0,n} \to u_0$ in $H^s(\mathbb{R}^d)$.

- u_n = soln to (NLS) with $u_n|_{t=0} = u_{0,n}$.
  u = " : $u|_{t=0} = u_0$

Lip dep $\Rightarrow$ $\|u_n - u\|_{C_T H^s_x} \lesssim \|u_{0,n} - u_0\|_{H^s} \to 0$

$\Rightarrow$ From the conservation of mass for u_n, we conclude conservation of mass for u.

- **Hamiltonian (= energy)**

$$H(u) = \underbrace{\frac{1}{2}\int |\nabla u|^2 \, dx}_{\text{kinetic energy}} \pm \underbrace{\frac{1}{p+1}\int |u|^{p+1} \, dx}_{\text{potential energy}}.$$

$+$ = defocusing case

$-$ = focusing case

- **Momentum**: $P(u) = \mathrm{Im} \int u \, \overline{\nabla u} \, dx = -i \int u \, \overline{\nabla u} \in \mathbb{C}^d$

Noether's thm: conservation laws $\Leftrightarrow$ symmetries of NLS.

$M(u) \Leftarrow u \mapsto e^{i\theta} u$

· Consider the 1-d defocusing NLS, $s > 1/2$

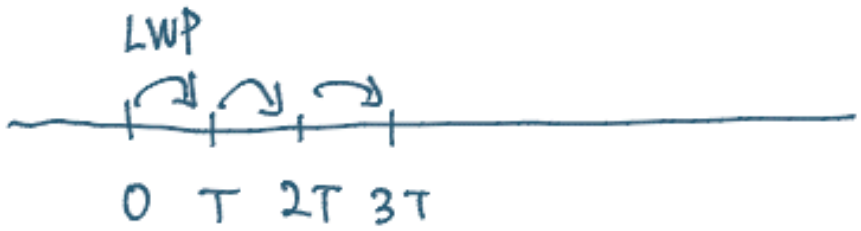
Let $u_0 \in H^1(\mathbb{R})$. $\Longrightarrow$ $u \in C_T H^1_x$, $T = T(\|u_0\|_{H^1}) > 0$.

$$\Rightarrow \|u(t)\|_{H^1}^2 = \|u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2$$

$$\leq M(u(t)) + 2H(u(t))$$

$$\overset{\text{cons}}{=} M(u_0) + 2H(u_0) = C^2(u_0) < \infty$$

$\Rightarrow$ Set $R = 2C(u_0)$ $\left(\geq 2\|u(t)\|_{H^1}\right) \Rightarrow T = T(R)$



$\Rightarrow$ We can iterate the LWP argument.

$\Rightarrow$ **Thm 2:** The 1-d defocusing NLS is globally well-posed in $H^1(\mathbb{R})$.

- **Persistence of regularity:** $u_0 \in H^s(\mathbb{R})$, $s > 1$.

$\Rightarrow u_0 \in H^1(\mathbb{R}) \to u \in C(\mathbb{R}; H^1(\mathbb{R}))$.

Q: $u \in C(\mathbb{R}; H^s(\mathbb{R}))$?

Yes. Need the following product estimate:

$$\|fg\|_{H^s} \lesssim \|f\|_{L^\infty} \|g\|_{H^s} + \|f\|_{H^s} \|g\|_{L^\infty}$$

($\Leftarrow$ follows from the paraproduct formula

. Fractional Leibniz rule: $s>0$, $1<p_j, q_j \le \infty$, $j=1,2$

$1 \le r \le \infty$

$$\frac{1}{r} = \frac{1}{p_j} + \frac{1}{q_j}$$

$$\|\langle\nabla\rangle^s (fg)\|_{L^r} \lesssim \|f\|_{L^{p_1}} \|\langle\nabla\rangle^s g\|_{L^{q_1}} + \|\langle\nabla\rangle^s f\|_{L^{p_2}} \|g\|_{L^{q_2}}$$

· Similar result for $|\nabla|^s$

($\Leftarrow$ Coifman-Meyer thm (non-endpt case)

$\Leftarrow$ paraproduct formula

See Bényi-Oh-Zhao PAMS '24 for further references.

Back to NLS, $d=1$, $p \in 2\mathbb{N}+1$

$$\| |u|^{p-1}u \|_{H^s} \lesssim \|u\|_{L^\infty}^{p-1} \|u\|_{H^s}$$

$$\overset{\text{Sob}}{\lesssim} \|u\|_{H^1}^{p-1} \|u\|_{H^s}.$$

$$\Rightarrow \|u(t)\|_{H^s} \le \|u_0\|_{H^s} + C R^{p-1} \int_0^t \|u(t')\|_{H^s} dt'.$$

Gronwall

$$\Rightarrow \|u(t)\|_{H^s} \le e^{C(R)t} \|u_0\|_{H^s} \Leftarrow \text{finite for each fixed } t$$

LWP in H^s

$$\Rightarrow u \in C(\mathbb{R}; H^s(\mathbb{R})).$$

Rmk: ① This argument gives an exp growth bd.

· It's possible to show a poly(in t) bd.

Bourgain, Staffilani, 90's.

Q: subpolynomial growth bd?

Q: Can we construct a soln s.t. the H^s-norm diverges as $t \to \pm\infty$?

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^s} = \infty \ ?$$

$s > 1$. · Colliander-Keel-Staffilani-Takaoka-Tao '10.

(CKSTT, or I-team)

· $d=2$: defocusing cubic NLS

LWP in $H^s(\mathbb{R}^2)$, $s > 1$

Thm 3: The 2-d def. cubic NLS is globally well-posed in $H^2(\mathbb{R}^d)$.

- **Brezis – Gallouët inequality** '80, $s > d/2$

$$\|f\|_{L^\infty} \le C_s \|f\|_{H^{d/2}} \left[\log\left(2 + \frac{\|f\|_{H^s}}{\|f\|_{H^{d/2}}}\right)\right]^{1/2}$$

(*) $\|f\|_{L^p} \lesssim \sqrt{p}\, \|f\|_{H^{d/2}}$ for $2 \le p < \infty$.

pf of B-G ineq: $g = f / \|f\|_{H^{d/2}} \Rightarrow \|g\|_{H^{d/2}} = 1$

WTS: $\|g\|_{L^\infty} \le C_s \left[\log(2 + \|g\|_{H^s})\right]^{1/2}$

· $\|g\|_{L^\infty} \le \|\hat{g}\|_{L^1} = \int_{|\xi|\le R} |\hat{g}(\xi)|\,d\xi + \int_{|\xi|>R} |\hat{g}(\xi)|\,d\xi$

$= \int_{|\xi|\le R} \langle\xi\rangle^{d/2} |\hat{g}(\xi)| \cdot \frac{1}{\langle\xi\rangle^{d/2}}\,d\xi + \int_{|\xi|>R} \langle\xi\rangle^{s} |\hat{g}(\xi)| \frac{1}{\langle\xi\rangle^{s}}\,d\xi$

C-S

$\le \|g\|_{H^{d/2}} \left(\log(2+R) \right)^{1/2} + \|g\|_{H^s} \left(\int_{|\xi|>R} \frac{1}{\langle\xi\rangle^{2s}}\,d\xi \right)^{1/2}$

$\|g\|_{H^{d/2}} = 1$; $\int_{|\xi|>R} \frac{1}{\langle\xi\rangle^{2s}}\,d\xi \sim R^{\frac{d}{2}-2s}$

Set $R = \|g\|_{H^s}^{\theta}$, $\theta = \frac{1}{s-d/2} > 0$

$\lesssim \log\left(2 + \|g\|_{H^s}\right)$

□

$$\|u(t)\|_{H^2} \overset{\text{product}}{\leq} \|u_0\|_{H^2} + C\int_0^t \|u(t')\|_{L^\infty}^2 \|u(t')\|_{H^2}\, dt'.$$

$$\overset{\text{B-G}}{\leq} \|u_0\|_{H^2} + C\int_0^t \|u(t')\|_{H^2}\left(1+\log\left(2+\|u(t')\|_{H^2}\right)\right) dt'.$$

$$=: F(t).$$

Yudovich: unique of soln to 2-d Euler.

$$\Rightarrow F'(t) \leq F(t)\left(1+\log\left(2+F(t)\right)\right)$$

$$\Rightarrow \frac{d}{dt}\log\left(1+\log\left(2+F(t)\right)\right) \leq C$$

$$\Rightarrow \|u(t)\|_{H^2} \leq F(t) \leq e^{e^{ct}} C(u_0)$$

$\Rightarrow$ GWP. $\square$

Pf of ⊛ : on P.14. $\|f\|_{L^p} \lesssim \sqrt{p}\, \|f\|_{H^{d/2}}$, $2 \le p < \infty$.

Write $f = f_{\le \lambda} + f_{>\lambda}$. $\widehat{f_{\le\lambda}}(\xi) = \mathbf{1}_{|\xi|\le\lambda} \hat{f}(\xi)$

$$\cdot \|f_{\le\lambda}\|_{L^\infty} \le \|\hat{f}_{\le\lambda}\|_{L^1} = \int_{|\xi|\le\lambda} |\hat{f}(\xi)|\, d\xi$$

$$\uparrow \langle\xi\rangle^{d/2} \langle\xi\rangle^{-d/2}$$

$$\le C \|f\|_{H^{d/2}} \left(\log(2+\lambda)\right)^{1/2}$$

Given $\alpha > 0$, choose $\lambda = \lambda(\alpha)$ s.t.

$$C \|f\|_{H^{d/2}} \left(\log(2+\lambda(\alpha))\right)^{1/2} = \frac{\alpha}{2}$$

$$\|f\|_{L^p}^p \overset{\text{Fubini}}{=} p\int_0^\infty \alpha^{p-1}\, d_f(\alpha)\, d\alpha$$

$$d_f(\alpha) = \text{distribution function of } f$$

$$= |\{x : |f(x)| > \alpha\}|$$

Grafakos, vol I, Chap 1.

$$\leq p\int_0^\infty \alpha^{p-1} \left|\left\{x : |f_{>\lambda(\alpha)}(x)| > \frac{\alpha}{2}\right\}\right| d\alpha$$

$$\overset{\text{Chebyshev}}{\lesssim} p\int_0^\infty \alpha^{p-3} \|f_{>\lambda(\alpha)}\|_{L^2_x}^2 \, d\alpha$$

$$= \int \mathbb{1}_{|\xi|>\lambda(\alpha)} |\hat{f}(\xi)|^2 d\xi.$$

$$\overset{\text{Fubini}}{=} p \int_{\mathbb{R}^3} \int_0^{2C\|f\|_{H^{d/2}}(\log(2+|\xi|))^{1/2}} \alpha^{p-3}\, d\alpha \; |\hat{f}(\xi)|^2\, d\xi.$$

$$\lesssim \underbrace{\frac{p}{p-2}}_{\gtrsim 1} \|f\|_{H^{d/2}}^{p-2} \int_{\mathbb{R}^3} \left[\log(2+|\xi|)\right]^{\frac{p-2}{2}} |\hat{f}(\xi)|^2\, d\xi$$

$$\lesssim p^{p/2} \|f\|_{H^{1/2}}^2$$

$$\left[\log(2+|\xi|)\right]^k \le k!\,(2+|\xi|) \lesssim k^k (1+|\xi|^2)^{1/2}$$

$$\Leftarrow \quad e^x = 2+|\xi|$$

$$\Leftarrow \quad x^k \le k!\, e^x$$

$$\lesssim p^{p/2} \|f\|_{H^{d/2}}^p$$

□

EX: Modify the proof of ⊛ to prove a special case of Sobolev's ineq:

$$\|f\|_{L^p} \lesssim \|f\|_{\dot{H}^s}, \quad \frac{s}{d} = \frac{1}{2} - \frac{1}{p}, \quad p > 2.$$

- A glimpse of the focusing case:

interpolation: $s = \theta s_1 + (1-\theta) s_2$

$$1 < p, p_1, p_2 < \infty, \quad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$$

$$\|f\|_{W^{s,p}} \lesssim \|f\|^{\theta}_{W^{s_1,p_1}} \|f\|^{1-\theta}_{W^{s_2,p_2}}, \quad 0 < \theta < 1$$

$\|f\|_{W^{s,p}} \overset{\text{Littlewood-Paley}}{\sim} \| \left(\sum_{N \geq 1 \text{ dyadic}} N^{2s} |P_N f|^2 \right)^{1/2} \|_{L^p}$

↑ LP projector onto $\{|\xi| \sim N\}$

$N = 2^j$, $j \in \mathbb{Z}_{\geq 0} = \mathbb{N} \cup \{0\}$.

and apply Hölder

Thm 1.3.6 in Grafakos, vol 2

· Gagliardo – Nirenberg's ineq:

$$\|f\|_{L^q} \lesssim \|f\|_{L^r}^{\theta} \|f\|_{\dot{W}^{s,r}}^{1-\theta}$$

$1 < r < q \le \infty$

$s > 0$

$$\frac{1}{q} = \frac{1}{r} - \frac{(1-\theta)s}{d}$$

Tao's book (Appendix)

· $d = 1$, $s = 1$, $r = 2$

$$\frac{1}{q} = \frac{1}{2} - \frac{1-\theta}{1} = -\frac{1}{2} + \theta \Rightarrow \theta = \frac{1}{2} + \frac{1}{q}.$$

G-N $\Rightarrow$

$$\|f\|_{L^q}^q \lesssim \|f\|_{L^2}^{\frac{q}{2}+1} \|\nabla f\|_{L^2}^{\frac{q}{2}-1}$$

Sharp constant for G-N ineq: $C_{GN} = \frac{q}{2} \|Q\|_{L^2(\mathbb{R}^d)}^{2-q}$

Nagy '41
Weinstein '83

$$(q-2)\, d\, \Delta Q + 4 Q^{q-1} - (4 + (q-2)(2-d)) Q = 0$$

Q = ground state (positive, radial, exp dec.)
= optimizer of G-N ineq.

· $d=1$, $s=1$, $r=2$, $q=b$.

$$\Rightarrow \quad \frac{1}{b}\|f\|_{L^b}^b \le \frac{1}{2}\left(\frac{\|f\|_{L^2}}{\|Q\|_{L^2}}\right)^4 \|\nabla f\|_{L^2}^2 .$$

· 1-d focusing quintic NLS ($p=5$)

$$H(u) = \frac{1}{2}\int |\nabla u|^2 dx - \frac{1}{b}\int |u|^b dx$$

$$\ge \frac{1}{2}\int |\nabla u|^2 \left(1 - \left(\frac{\|u\|_{L^2}}{\|Q\|_{L^2}}\right)^4\right) dx$$

$$\gtrsim_u \|\nabla u\|_{L^2} \quad \text{if} \quad \|u\|_{L^2} < \|Q\|_{L^2}.$$

$\Rightarrow$ GWP if $\|u_0\|_{L^2} < \|Q\|_{L^2}$ ($\&$ scattering).

in L^2: Dodson, Adv Math '15

· 1-d focusing sub-quintic NLS ($p<5$)

G-N ineq with $d=1,\ s=1,\ q<6,\ r=2$

$$\Rightarrow \frac{1}{q}\|f\|_{L^q}^q \le \frac{1}{2}\|Q\|_{L^2}^{2-q}\|f\|_{L^2}^{\frac{q}{2}+1}\|\nabla f\|_{L^2}^{\frac{q}{2}-1}$$

($\frac{q}{2}-1 < 2$.)

$$\overset{\text{Young}}{\le} C_\varepsilon \|Q\|_{L^2}^{\alpha_1}\|f\|_{L^2}^{\alpha_2} + \varepsilon\|\nabla f\|_{L^2}^2$$

· $H(u) = \frac{1}{2}\int|\nabla u|^2dx - \frac{1}{p+1}\int|u|^{p+1}dx \quad (q=p+1)$

$$\Rightarrow H(u) + C'_\varepsilon M(u)^{\alpha_2/2} \gtrsim \|\nabla u\|_{L^2}^2.$$

$\Rightarrow$ GWP in $H^1(\mathbb{R})$ for $p<5$.

GWP for $\|u\|_{L^2} < \|Q\|_{L^2}$

GWP

Rmk: same holds for $p = 1+\frac{4}{d}$ and $p < 1+\frac{4}{d}$

Fact: ∃ finite time blowup soln to 1-d focusing quintic NLS on $\mathbb{R}$ with $\|u_0\|_{L^2} = \|Q\|_{L^2}$

(and $\|u_0\|_{L^2} \geq \|Q\|_{L^2}$)

- $\|u\|_{L^2} = \|Q\|_{L^2}$.

  - Soliton: $u(t) = e^{it}Q$ is a global non-scattering soln.

  - pseudo-conformal symmetry ($s_{crit} = 0$)

    ↑ scaling critical Sobolev index.

$$u(t,x) \longmapsto v(t,x)$$

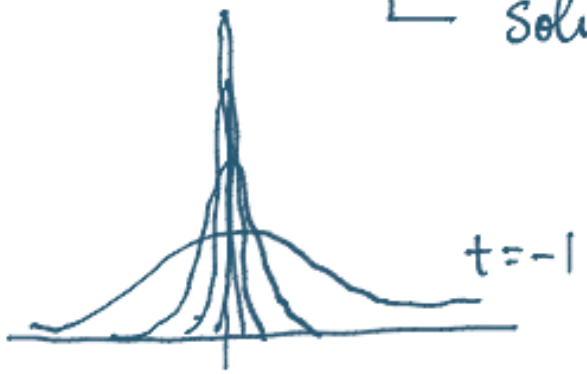
$$= \frac{1}{|t|^{d/2}} u\left(-\frac{1}{t}, \frac{x}{t}\right) e^{i\frac{|x|^2}{4t}}, \quad t \neq 0$$

$\Rightarrow$ v satisfies the same NLS.

Apply pseudoconformal symmetry to $e^{it}Q$.

$$\Rightarrow \quad Q^*(t,x) = \frac{1}{|t|^{d/2}} Q\left(\frac{x}{t}\right) e^{i|x|^2/4t + \frac{i}{t}}$$

↑ soln to NLS for $t<0$.



Q^* blows up at time $t=0$
(starting at $t=0$)

- $|Q^*(t)|^2 \longrightarrow \|Q\|_{L^2}^2 \, \delta_{x=0}$ as $t \nearrow 0$.

$\|\nabla Q^*(t)\|_{L^2} \sim \frac{1}{|t|}$ ⇐ blowup rate.

- Q^* is the minimal mass finite time blowup soln. (unstable)

(HW: scaling symmetry → rate $\gtrsim \frac{1}{\sqrt{T^*-t}}$ GWP Q^* $e^{it}Q$

· Other finite time blowup solns?

Merle-Raphaël
DD's. ~

$$M(Q) < M(u_0) < M(Q) + \varepsilon$$

loglog blowup solns (stable) $\sim \sqrt{\dfrac{\log\log(T^*-t)}{T^*-t}}$

· Virial Identity: $i\partial_t u + \Delta u = \lambda |u|^{p-1}u$, $\lambda = 1$, def.
$\lambda = -1$, foc.

Virial potential $V_a(t) = \int a(x)|u(x)|^2 dx$

a = "nice" func
$a(x) = |x|^2$, $|x|$, etc

Compute $\partial_t^2 V_a$ (see the course note from '16).

$$\partial_t^2 V_a = 4\int \mathrm{Re}(\partial_k u\, \partial_j \bar{u})\, \partial_k \partial_j a + 2\lambda\frac{p-1}{p+1}\int |u|^{p+1}\Delta a - \int |u|^2 \Delta^2 a.$$

Einstein's summation notation
(sum over repeated indices)

ex: Virial identity (Viriel) $\quad a(x) = |x|^2 = \sum_{j=1}^{d} x_j^2$

$\Delta a = 2d$

$\Delta^2 a = 0$

$\partial_k \partial_j a = 2\delta_{jk}$ ↑ Kronecker delta

- $V(t) = \int |x|^2 |u(x)|^2 dx \geq 0$

(= variance if we view $|u(x)|^2 dx$ as a prob. measure.

$$\Rightarrow \partial_t^2 V(t) = 8 \int |\nabla u|^2 + 4d\lambda \frac{p-1}{p+1} \int |u|^{p+1}$$

$$= 16\, H(u) + \frac{4\lambda d}{p+1}\left(p - \left(1 + \frac{4}{d}\right)\right) \int |u|^{p+1}$$

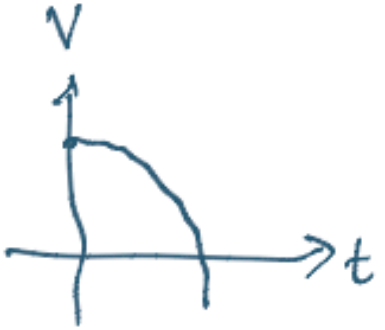
$\lambda = -1$

- Mass-critical case: $p = 1 + 4/d$

→ If $H(u_0) < 0$, then $\partial_t^2 V(t) = 16\, H(u_0) < 0$.

$\Rightarrow V(t^*) < 0$ for some $t^* > 0$.

but $V(t) \geq 0$, $\forall\, t \in \mathbb{R}$



$\Rightarrow$ u must blow up before time $t^* > 0$

obstruction-type argument (Glassey's argument
Zakharov's

Lec 3 on Nov 26 (TUE), 11 AM