Lec 4 : Dec 03, 2024 (TUE)

Strichartz estimate : $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ Schrödinger admissible $(q, r, d) \neq (2, \infty, 2)$

(i) homog estimate :
$$\| S(t) f \|_{L_t^q L_x^r} \lesssim \| f \|_{L^2}$$

(ii) dual estimate :
$$\| \int_{\mathbb{R}} S(-t) F(t) \, dt \|_{L_x^2} \lesssim \| F \|_{L_t^{q'} L_x^r}$$

Last time: TT^* argument, $T = S(t)$

$$TT^* F(t) = \int_{\mathbb{R}} S(t - t') F(t') \, dt' .$$

$\Rightarrow$ Suffices to show
$$\| TT^* F \|_{L_t^q L_x^r} \lesssim \| F \|_{L_t^{q'} L_x^{r'}} \quad (*)$$

duality $\Longleftrightarrow$
$$| \langle TT^* F, G \rangle_{L_{t,x}^2} | \lesssim \| F \|_{L_t^{q'} L_x^{r'}} \| G \|_{L_t^{q'} L_x^{r'}} \quad (**)$$

$$| \langle T^* F, T^* G \rangle_{L_{t,x}^2} | = \left| \int_t \int_{t'} \langle S(-t') F(t'), S(-t) G(t) \rangle_{L_x^2} \, dt' dt \right|$$

can assume $t' < t$

① $(q,r)=(\infty,2)$ is trivial $\Leftarrow$ L^2-unitarity of $S(t)$ ②

② nonendpoint case: $2<q<\infty$

- We proved (*) via L^p-$L^{p'}$ estimate: $\boxed{\|S(t)f\|_{L^p} \lesssim \frac{1}{|t|^{d(\frac12-\frac1p)}}\|f\|_{L^{p'}}}$ $\quad 2\le p\le\infty$

and H-L-S ineq (in t).

③ Endpt case: $d\ge 3$, $q=2$, $r=\frac{2d}{d-2}$.

WTS (**)

$\Leftarrow$ By symmetry, we may assume $t'<t$ in (**)

· Suffices to show

(***) $$\sum_{j\in\mathbb{Z}} |R_j(F,G)| \lesssim \|F\|_{L_t^2 L_x^{r'}} \|G\|_{L_t^2 L_x^{r'}}$$

where $R_j(F,G)=\iint_{\substack{2^j\le t-t'<2^{j+1}\\ t'<t}} \langle S(t-t')F(t'), S(-t)G(t)\rangle_{L_x^2}\, dt'dt$

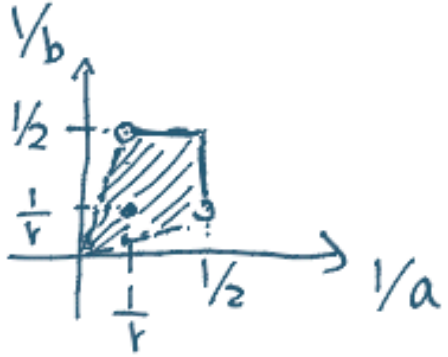
We proved

Lemma A: $|R_j(F,G)| \lesssim 2^{-j\beta(a,b)} \|F\|_{L^2_t L^{a'}_x} \|G\|_{L^2_t L^{b'}_x}$

$$\beta(a,b) = \frac{d}{2} - 1 - \frac{d}{2a} - \frac{d}{2b}$$

uniformly in $j \in \mathbb{Z}$

$\forall$ $(\frac{1}{a}, \frac{1}{b})$ in a nbhd of $(\frac{1}{r}, \frac{1}{r})$



Model case:

$$F(t) = 2^{-k/r'} f(t) \mathbb{1}_{E(t)}(x), \qquad G(t) = 2^{-\tilde{k}/r'} g(t) \mathbb{1}_{\tilde{E}(t)}(x)$$

$$|E(t)| = 2^k$$

$$|\tilde{E}(t)| = 2^{\tilde{k}}$$

$k, \tilde{k} \in \mathbb{Z}$.

LemA $\Rightarrow$

$$|R_j(F,G)| \lesssim 2^{-j\beta(a,b)}\, 2^{-k/r'} \|f\|_{L^2_t}\, 2^{k/a'} \times 2^{-\tilde{k}/r'} \|g\|_{L^2_t}\, 2^{\tilde{k}/b'}.$$

$$\begin{cases} -k/r' + k/a' = k\left(\frac{1}{r}-\frac{1}{a}\right) \\ -\tilde{k}/r' + \tilde{k}/b' = \tilde{k}\left(\frac{1}{r}-\frac{1}{b}\right) \end{cases}$$

$$-\beta = -\frac{d}{2} + 1 + \frac{d}{2a} + \frac{d}{2b}$$

$$= \left(-\frac{d}{4}+\frac{1}{2}+\frac{d}{2a}\right) + \left(-\frac{d}{4}+\frac{1}{2}+\frac{d}{2b}\right)$$

$$= -\frac{d}{2}\left(\frac{1}{r}-\frac{1}{a}\right) - \frac{d}{2}\left(\frac{1}{r}-\frac{1}{b}\right)$$

using $\frac{1}{r} = \frac{d-2}{2d}$

$$\lesssim 2^{\left(k-\frac{d}{2}j\right)\left(\frac{1}{r}-\frac{1}{a}\right)} 2^{\left(\tilde{k}-\frac{d}{2}j\right)\left(\frac{1}{r}-\frac{1}{b}\right)} \times \|f\|_{L^2_t} \|g\|_{L^2_t}$$

$k - \frac{d}{2}j \geq 0$: $\frac{1}{a} = \frac{1}{r} + \varepsilon$

$k - \frac{d}{2}j < 0$: $\frac{1}{a} = \frac{1}{r} - \varepsilon$

$$\Rightarrow |R_j(F,G)| \lesssim \underbrace{2^{-\varepsilon|k-\frac{d}{2}j| - \varepsilon|\tilde{k}-\frac{d}{2}j|}}_{\leq\, 2^{-\frac{\varepsilon}{2}|k-\tilde{k}|}\, 2^{-\frac{\varepsilon}{2}|k-\frac{d}{2}j|}\, 2^{-\frac{\varepsilon}{2}|\tilde{k}-\frac{d}{2}j|}} \|f\|_{L^2_t} \|g\|_{L^2_t}.$$

⊕

Lemma B (atomic decomposition of $L^p(\mathbb{R}^d)$, $1 \leq p < \infty$)

Given $f \in L^p_x$, we can write f as

$$f = \sum_{k\in\mathbb{Z}} c_k \chi_k$$

- $\|\chi_k\|_{L^\infty} \lesssim 2^{-k/p}$
- $|\operatorname{supp} \chi_k| \simeq O(2^k)$
- $c_k \geq 0$ s.t. $\|c_k\|_{l^p} \lesssim \|f\|_{L^p_x}$

For now, assume Lem B.

Given F, G on $\mathbb{R}_t \times \mathbb{R}^d_x$,

$\overset{\text{Lem B}}{\Longrightarrow}$ $F(t') = \sum_{k \in \mathbb{Z}} f_k(t') \chi_k(t', x)$, $G(t) = \sum_{k \in \mathbb{Z}} g_k(t) \widetilde{\chi}_k(t)$

$\| \chi_k(t) \|_{L^\infty} \lesssim 2^{-k/r'}$, unif in t.

$|\operatorname{supp} \chi_k| = O(2^k)$.

and $\left\| \| f_k \|_{\ell^{r'}_k} \right\|_{L^2_t} \lesssim \| F \|_{L^2_t L^{r'}_x}$.

Similar bounds for G.

⊕ on p. 5.

$$\Longrightarrow \sum_{j\in\mathbb{Z}} |R_j(F,G_T)|$$

$$\lesssim \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \sum_{\tilde{k}\in\mathbb{Z}} \underbrace{|R_j(f_k\chi_k, g_{\tilde{k}}\widetilde{\chi}_{\tilde{k}})|}_{\text{w.r.t. } \ell^2_{\tilde{k}}}$$

$$\lesssim 2^{-\frac{\varepsilon}{2}|k-\tilde{k}|}\, 2^{-\frac{\varepsilon}{2}|k-\frac{d}{2}j|}\, 2^{-\frac{\varepsilon}{2}|\tilde{k}-\frac{d}{2}j|} \times \|f_k\|_{L^2_t} \|g_{\tilde{k}}\|_{L^2_t}$$

↑ put in $\ell^2_{\tilde{k}}$

Schur's test

$$\lesssim \big\| \|f_k\|_{L^2_t} \big\|_{\ell^2_k} \big\| \|g_{\tilde{k}}\|_{L^2_t} \big\|_{\ell^2_{\tilde{k}}}$$

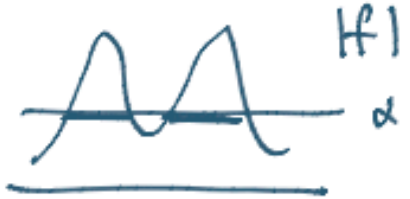
$$\Big(\Leftarrow\quad A\big(\|f_k\|_{L^2_t}\big)(\tilde{k}) = \sum_{k\in\mathbb{Z}} 2^{-\frac{\varepsilon}{2}|k-\tilde{k}|} \|f_k\|_{L^2_t}$$

$$\leq \|f_k\|_{L^2_t \ell^{r'}_k} \|g_{\tilde{k}}\|_{L^2_t \ell^{r'}_{\tilde{k}}} \underset{\text{LemB}}{\lesssim} \|F\|_{L^2_t L^{r'}_x} \|G\|_{L^2_t L^{r'}_x} \quad \square$$

$r'<2$

Pf of Lem B: d_f is right-cont.

$$d_f(\alpha) = |\{x \in \mathbb{R}^d : |f(x)| > \alpha\}| \quad \Leftarrow \text{non-inc}$$



Given $k \in \mathbb{Z}$, set

$$\alpha_k = \inf_{d_f(\alpha) < 2^k} \alpha \quad \Leftarrow \text{non-inc}$$

$$c_k = 2^{k/p}\,\alpha_k$$

$$\chi_k = \frac{1}{c_k}\,\mathbb{1}_{(\alpha_{k+1},\,\alpha_k]}(f)\,f.$$

$$\Rightarrow f = \sum_k c_k \chi_k$$

$$\Rightarrow |\chi_k| \le 2^{-k/p} \quad (\le \alpha_k)$$

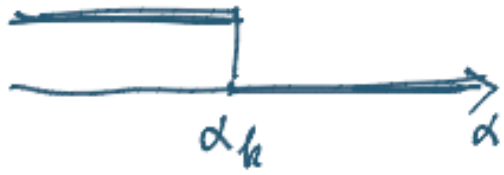
- $|\operatorname{supp}\chi_k| = |\{x \in \mathbb{R}^d : \alpha_{k+1} < |f(x)| \le \alpha_k\}|$

$$\le |\{|f(x)| > \alpha_{k+1}\}| \le 2^{k+1}.$$

$$\sum_{k\in\mathbb{Z}} c_k^p = \sum_k 2^k \alpha_k^p$$

$$= \int \alpha^p \underbrace{\left(\sum_k 2^k f_{\alpha_k}(\alpha)\right)}_{= -F'(\alpha)} d\alpha,$$

where $F(\alpha) = \sum_{k\in\mathbb{Z}} 2^k H(\alpha_k - \alpha)$ ← Heaviside func



$$= \sum_{\substack{k\in\mathbb{Z} \\ \alpha_k > \alpha}} 2^k = \sum_{k\le k_*} 2^k$$

k_* largest s.t. $\alpha_{k_*} > \alpha$.

$$= 2\cdot 2^{k_*} \le 2\, d_f(\alpha)$$

$$d_f(\alpha_{k_*}) \le 2^{k_*}$$

$$\overset{\text{LBP}}{=} p\int \alpha^{p-1} F(\alpha)\, d\alpha \lesssim p\int \alpha^{p-1} d_f(\alpha)\, d\alpha$$

$$\overset{\text{Fubini}}{=} \|f\|_{L^p}^p \qquad \square$$

· We still need to prove

(iii) non-homog. estimate

(++) $$\left\| \int_0^t S(t-t') F(t') \, dt' \right\|_{L_t^q L_x^r} \lesssim \| F \|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

(q,r), $(\tilde{q},\tilde{r})$, admissible.

2 proofs: ① direct

② Christ-Kiselev lemma.

① Direct approach: CASE 1: $(q,r) = (\tilde{q},\tilde{r})$

$$\mathbb{1}_{[0,t]}(t') = \mathbb{1}_{(-\infty,t]}(t') - \mathbb{1}_{(-\infty,0)}(t')$$

↑ enough to consider (++) with $\int_{t'<t}$

← put together with $F(t')$

TT^* bound ((**) on p.1)

$$|\langle TT^* F, G\rangle_{L^2_{t,x}}| \lesssim \|F\|_{L_t^{q'}L_x^{r'}} \|G\|_{L_t^{q'}L_x^{r'}}$$

$$\| \iint \langle S(-t')F(t'), S(-t)G(t)\rangle_{L^2_x} \, dt' dt |$$

↑ by symmetry, we can insert $t' < t$. ⇒ (iii) in this case

CASE 2:

(+++) $$\| \int_{t'<t} S(t-t')F(t)dt \|_{L_t^{\infty}L_x^2} \lesssim \|F\|_{L_t^{q'}L_x^{r'}}$$

$$\Big(\text{LHS of } (+++) \text{ at fixed } t\Big)^2 = \Big\langle \int_{t_1<t} S(t-t_1)F(t_1)\,dt_1,\ \int_{t_2<t} S(t-t_2)F(t_2)\,dt_2 \Big\rangle_{L^2_x} \qquad (12)$$

$$S(t)S(-t_1) \qquad S(t)S(-t_2)$$

$$= \int_{t_1<t} \underbrace{\Big\langle F(t_1),\ \int_{t_2<t} S(t_1-t_2)F(t_2)\,dt_2 \Big\rangle_{L^2_x}}_{\text{Hölder in } x}\, dt_1$$

$$\le \|F(t_1)\|_{L^{r'}_x} \Big\| \int_{t_2<t} S(t_1-t_2)F(t_2)\,dt_2 \Big\|_{L^r_x}$$

Hölder in t_1

$$\le \|F\|_{L^{q'}_t L^{r'}_x} \underbrace{\Big\| \int_{t_2<t} S(t_1-t_2)F(t_2)\,dt_2 \Big\|_{L^q_t L^r_x}}_{\lesssim \|F\|_{L^{q'}_t L^{r'}_x}}$$

$$\lesssim (RHS)^2 .$$

WTS (iii)

$\Leftrightarrow$ duality $\left| \iint_{t'<t} \langle S(t-t')F(t'), G(t) \rangle_{L^2_x} \, dt' \, dt \right|$

$\lesssim \| F \|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x} \| G \|_{L^{q'}_t L^{r'}_x}$.

(★)

· Fix $(\tilde{q}, \tilde{r})$.

case 1: (★) holds for $q = \tilde{q}$, $r = \tilde{r}$.

case 2: (★) holds for $q = \infty$, $r = 2$.

interpolation $\Rightarrow$ (★) holds for any $q \geq \tilde{q}$ (and $r = r(q)$ s.t. (q, r), admissible

For $q \le \tilde{q}$, use symmetry.

LHS of ⋆ $= \left| \int_{t'} \int_{t} \langle F(t), S(t'-t) G(t) \rangle_{L^2_x} \, dt \, dt' \right|$

$t > t'$

$$\mathbb{1}_{(t',\infty)}(t) = \mathbb{1}_{\mathbb{R}}(t) - \mathbb{1}_{(-\infty,t')}(t)$$

$\Rightarrow$ ⋆ holds for $q \le \tilde{q}$.

□

(2) Christ – Kiselev lemma

C-K, JFA '01
(Tao, CPDE '00)

Suppose $Tf(t) = \int_{\mathbb{R}} K(t,s) f(s)\, ds$

maps $L^p(\mathbb{R}; X)$ to $L^q(\mathbb{R}; Y)$, $1 < p < q < \infty$

$1 = p \le q \le \infty$

$p \le q = \infty$

Then,

$$\tilde{T}f(t) = \int_{s<t} K(t,s) f(s)\, ds$$

maps $L^p(\mathbb{R}; X)$ to $L^q(\mathbb{R}; Y)$.

- Endpt Strichartz with $q = \tilde{q} = 2$ is NOT covered by the C-K lemma.
  ($\Rightarrow \tilde{q}' = 2 = q$)
  but, this case was already treated in the proof of the endpt Strichartz estimate.

Pf: $1 < p < q < \infty$

For simplicity, assume that f is smooth and is nowhere zero.

- $\|f\|_{L^p} = 1$.

Set $\boxed{F(t) = \int_{s<t} |f(s)|^p \, ds}$ ⇐ strictly inc.

$\rightarrow F'(t) = |f(t)|^p$

⇒ F is an order-preserving bijection from $\mathbb{R}$ to $[0,1]$.

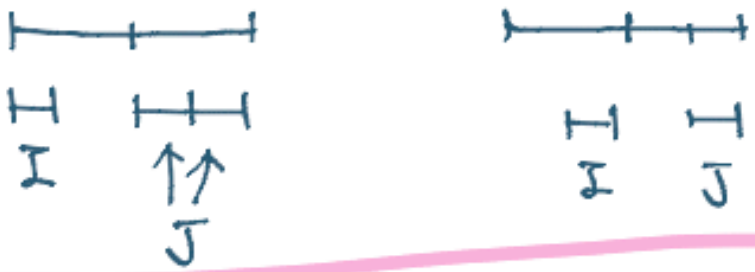
- $\mathcal{D}$ = family of dyadic intervals

$$\left[\frac{k-1}{2^j}, \frac{k}{2^j}\right), \quad j \in \mathbb{Z}_{\geq 0}, \quad k = 1, \cdots, 2^j.$$

- Given $I, J \in \mathcal{D}$, we write $I \lesssim J$ if
  - $|I| = |J|$.
  - NOT adjacent but have adjacent parents,
  - $\forall x \in I, y \in J \Rightarrow x < y$.

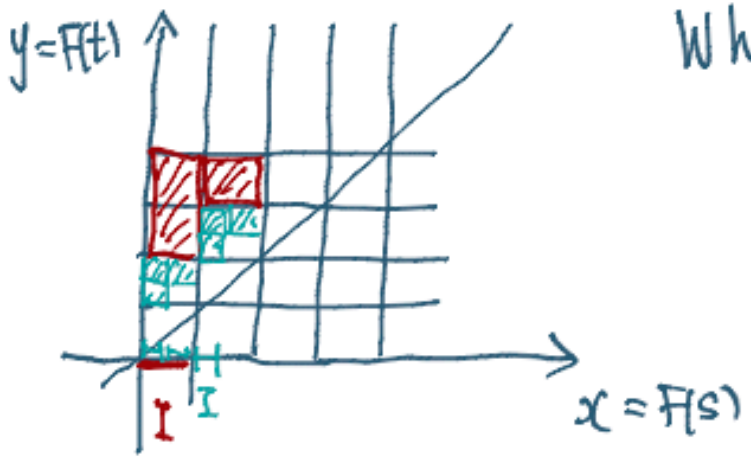
$\Rightarrow \forall\, x, y \in [0,1]$ with $x < y$,

$\exists!\ I, J \in \mathcal{D}$ s.t. $x \in I$, $y \in J$, $I \lesssim J$.



$$\int_{s<t} ds = \int_{F(s)<F(t)} ds = \sum_{\substack{I,J\in\mathcal{D}\\ I\lesssim J}} \mathbb{1}_{F^{-1}(J)}(t) \int_{F^{-1}(I)} ds$$

$F(s) = x$, $F(t) = y$



Whitney decomposition of $\{x<y\} \cap [0,1]^2$.

$$\Rightarrow \widetilde{T}f(t) = \sum_{\substack{I,J\in\mathcal{D}\\ I\lesssim J}} \mathbb{1}_{F^{-1}(J)}(t)\, T\left(\mathbb{1}_{F^{-1}(I)} f\right)(t).$$

- WTS: $\|\widetilde{T}f\|_{L^q} \lesssim 1 \left(= \|f\|_{L^p}\right)$.

Suffices to show, for some $\varepsilon > 0$,

(★★) $$\Big\| \sum_{\substack{I,J\in\mathcal{D}\\ I\lesssim J\\ |I|=2^{-j}}} \mathbb{1}_{F^{-1}(J)}\, T\left(\mathbb{1}_{F^{-1}(I)} f\right) \Big\|_{L^q} \lesssim 2^{-\varepsilon j}$$

unif. in $j \in \mathbb{Z}_{\geq 0}$.

- Fix $j \in \mathbb{Z}_{\geq 0}$. • Given $I \in \mathcal{D}$, $\exists$ at most two $J \in \mathcal{D}$ s.t. $I \lesssim J$.

(vice versa.

- $\mathbb{1}_{F^{-1}(J)}(t)$ has disjoint supp

$$\left(\text{LHS of } \circledast\circledast\right)^q = \int \Big| \sum_{\substack{J\in\mathcal{D}\\ |J|=2^{-j}}} \mathbb{1}_{F^{-1}(J)}(t)\, T\Big(\sum_{I\subseteq J} \mathbb{1}_{F^{-1}(I)}(t)\Big)\Big|^q dt$$

disjoint supp

$$= \sum_{\substack{J\in\mathcal{D}\\ |J|=2^{-j}}} \int \Big| \mathbb{1}_{F^{-1}(J)}(t)\, T\Big(\sum_{I\subseteq J} \mathbb{1}_{F^{-1}(I)}(t)\Big)\Big|^q dt$$

- Given J, $\exists$ $O(1)$ many I s.t. $I\subseteq J$
- Then, switch the order of summations
- Given I, $\exists$ $O(1)$ many J s.t. $I\subseteq J$

$$\sim \sum_{\substack{I\in\mathcal{D}\\ |I|=2^{-j}}} \| T(\mathbb{1}_{F^{-1}(I)} f)\|_{L^q_t}^q$$

$$\lesssim \| \mathbb{1}_{F^{-1}(I)} f\|_{L^p_s}^q = \Big(\int_{F^{-1}(I)} |f(s)|^p ds\Big)^{q/p}$$

$$\overset{\text{Ch of var}}{=} \Big(\int_{x\in I} dx\Big)^{q/p} = 2^{-\frac{q}{p}j}$$

$$\lesssim 2^j 2^{-\frac{q}{p}j} = 2^{(1-\frac{q}{p})j}$$

$1-\frac{q}{p} < 0$

Note: $p = q$ fails. H = Hilbert transform

$$H(f)(t) = \text{p.v.} \int \frac{1}{t-s} f(s)\, ds.$$

$$\widetilde{H}(f)(t) = \int_{s<t} \frac{1}{t-s} f(s)\, ds. \quad \text{Take } f = \mathbb{1}_{[0,1]}$$

- $1 \le p \le q = \infty$.

Fix t. $|\widetilde{T}(f)(t)| = |T(\mathbb{1}_{(-\infty,t)} f)(t)|$ (fixed)

$$\lesssim \| \mathbb{1}_{(-\infty,t)} f \|_{L^p_s} \le \|f\|_{L^p_s}.$$

- $p = 1 \le q \le \infty$

$$T^t g(t) = \int K(s,t) g(s)\, ds.$$

$$\Rightarrow \widetilde{T^t}\, g(t) = \int_{s<t} K(s,t) g(s)\, ds.$$

T, bdd from L^1 to L^q $\Rightarrow$ T^t is bdd from $L^{q'}$ to L^∞, so is $\widetilde{T^t} = \widetilde{T}^t$

$\Rightarrow$ $\widetilde{T}$ is bdd from L^1 to L^q $\square$

What about

$$T^* f(t) = \sup_{\alpha \in \mathbb{R}} T\left(1_{(-\infty, \alpha)} f\right)(t) ?$$

↑ maximal function.

fixed ↙

$$x = F(s) < y = F(\alpha)$$

$\forall j \in \mathbb{Z}_{\geq 0}$, $\exists!\ J \in \mathcal{D}$ s.t. $F(\alpha) \in J$., $|J| = 2^{-j}$.

$$\Rightarrow T^* f(t) = \sup_{\alpha \in \mathbb{R}} \sum_{\substack{I, J \in \mathcal{D} \\ I \lesssim J}} 1_{F^{-1}(J)}(\alpha)\, T\left(1_{F^{-1}(I)} f\right)(t)$$

Suffices to show

(A) $$\left\| \sup_{\alpha \in \mathbb{R}} \left| \sum_{\substack{I, J \in \mathcal{D} \\ I \lesssim J, \\ |I| = 2^{-j}}} 1_{F^{-1}(J)}(\alpha)\, T\left(1_{F^{-1}(I)} f\right) \right| \right\|_{L^q_t} \lesssim 2^{-\varepsilon j}$$

↑ No summation in J.

→ essentially no summation in I.

$$\Big| \sum_{\substack{I,J\in\mathcal{D} \\ I\subsetneq J, \\ |I|=2^{-j}}} \mathbb{1}_{F^{-1}(J)}(\alpha)\, T\big(\mathbb{1}_{F^{-1}(I)} f\big) \Big|$$

$$\lesssim \mathbb{1}_{F^{-1}(J)}(\alpha) \Big| T\big(\mathbb{1}_{F^{-1}(I)} f\big) \Big|$$

$I = I(J)$.

- $\sup_{\alpha\in\mathbb{R}} \iff \max_{\substack{J \\ |J|=2^{-j}}} \iff \max_{\substack{I \\ |I|=2^{-j}}}$.

(A) $\lesssim \Big\| \underbrace{\max_{|I|=2^{-j}} \Big| T\big(\mathbb{1}_{F^{-1}(I)} f\big) \Big|}_{\lesssim \| \ \|_{l^q_I}} \Big\|_{L^q_t}$

$\lesssim 2^{-j(\frac{1}{p}-\frac{1}{q})}$

↑ Same as before.

· Back to NLS:

EX 1: 1-d cubic NLS.

$$\Gamma u(t) = S(t)u_0 \pm i\int_0^t S(t-t')\,|u|^2 u(t')\,dt'.$$

($\mathbb{1}_{[0,T]}(t)$)

$(q,r) = (8,4)$, admissible $\quad \frac{2}{8}+\frac{1}{4}=\frac{1}{2}$.

$$X_T = C_T L^2_x \cap L^8_T L^4_x$$

$$\|\Gamma u\|_{X_T} \overset{\text{Str.}}{\lesssim} \|u_0\|_{L^2_x} + \|\,|u|^2 u\|_{L_T^{\frac{8}{7}} L_x^{4/3}}$$

$$\overset{\text{Hölder}}{\leq} T^{1/2}\|u\|^3_{L^8_T L^4_x} \leq T^{1/2}\|u\|^3_{X_T}.$$

· A similar computation yields the difference estimate

uniqueness in X_T

Thm 5: 1-d cubic NLS is LWP in $L^2(\mathbb{R})$ — conditional uniqueness

$\Rightarrow$ GWP in $L^2(\mathbb{R})$ (by the mass conservation).

($\Leftarrow$ for persistence of regularity, use the fractional Leibniz rule.

EX 2: 2-d cubic NLS $\Leftarrow$ L^2-critical.

① small data GWP in $L^2(\mathbb{R}^2)$

$(q,r) = (4,4)$ admis. $\quad \frac{2}{4} + \frac{2}{4} = \frac{2}{2}$

$Y_T = C_T L^2_x \cap L^4_{T,x}$

NO T^θ!!

$$\Rightarrow \|\Gamma u\|_{Y_T} \le C_1 \|u_0\|_{L^2} + C_2 \underbrace{\| |u|^2 u \|_{L^{4/3}_{T,x}}}_{= \|u\|^3_{L^4_{T,x}} \le \|u\|^3_{Y_T}}.$$

Suppose $\|u_0\|_{L^2_x} = \eta_0 \ll 1$.

$u \in B_R$, $R = 2\eta_0$.

$\cap$

Y_T

$\le 4\eta_0^2 \|u\|_{Y_T}$

$C_2 \cdot 4\eta_0^2 \le 1/2 \Rightarrow$ hide on LHS.

T did NOT play any role.

$\Rightarrow$ the same argument holds for $T = \infty$, working in Y_∞.

Thm 6: $\exists\, \eta \ll 1$. If $\|u_0\|_{L^2} \le \eta$, then 2-d cubic NLS is GWP in $L^2(\mathbb{R}^2)$.

small data GWP

② LWP in $L^2(\mathbb{R})$.

Do NOT apply the Str. est.

$$\|\Gamma u\|_{L^4_{T,x}} \le \|S(t)u_0\|_{L^4_{T,x}} + C_2 \|u\|^3_{L^4_{T,x}}$$

- We have $\|S(t)u_0\|_{L^4(I_0, I; L^4_x)} \lesssim \|u_0\|_{L^2_x} < \infty$.

$\overset{\text{DCT}}{\Rightarrow}$ $\lim_{T \to 0} \|S(t)u_0\|_{L^4_{T,x}} = 0$.

Given η_f as before, choose $T = T(u_0)$ s.t.

$$\|S(t)u_0\|_{L^4_{T,x}} \le \eta_f.$$

local existence time depends on the profile of u_0

$$\Rightarrow \begin{cases} \|\Gamma u\|_{L^4_{T,x}} \le \eta_f + \underbrace{8C_2\eta_f^3}_{\le \eta_f} \\ \|\Gamma u - \Gamma v\|_{L^4_{T,x}} \le \underbrace{4C_2\eta_f^2}_{\le 1/2} \|u-v\|_{L^4_{T,x}} \end{cases}$$

$\Rightarrow$ contraction in $B_{2\eta_f} \subset L^4_{T,x}$.

$\Rightarrow$ u in $L^4_{T,x}$, so need to show $u \in C_T L^2_x$.

$$\|u\|_{C_T L^2_x} \le \|u_0\|_{L^2_x} + C_2\|u\|^3_{L^4_{T,x}} < \infty \quad \forall.$$

in the critical sense

Thm 7: 2-d cubic NLS is LWP in $L^2(\mathbb{R}^2)$.

energy-critical

- See HW for LWP of 3-d cubic / quintic NLS in $H^1(\mathbb{R}^3)$

Rmk: • Since the local existence time $T = T(u_0)$ depends on the profile of u_0 (i.e. more info than the L^2-norm of u_0), we can NOT use the mass conservation to iterate the LWP argument and prove GWP in $L^2(\mathbb{R}^2)$.

- GWP & scattering of the defocusing cubic NLS in $L^2(\mathbb{R}^2)$
  Dodson, Duke Math J '16
- In the focusing case, there are finite time blowup solns (See Lec 2.)
- See the lecture notes by Killip-Vişan (Clay) and by Vişan (Oberwolfach) for further discussions
- 2-d cubic NLS is ill-posed in $H^s(\mathbb{R}^2)$, $s < 0$.
  Christ-Colliander-Tao, '03 (arXiv preprint),
  Oh, Funkcial Ekvac. '17, Kishimoto CPAA '19.

· Addendum: forbidden endpt $(q, r, d) = (2, \infty, 2)$.

(B) $$\Big\| \int_{t'<t} S(t-t') F(t') \, dt' \Big\|_{L^2_t L^\infty_x} \overset{?}{\lesssim} \|F\|_{L^2_t L^1_x}.$$

· By a limiting argument, we may assume that F is a measure supported on $\{x=0\}$

$$F(t,x) = f(t)\, \delta_0(x)$$

$$\Rightarrow \|F\|_{L^2_t L^1_x} = \|f\|_{L^2_t}.$$

(B) $\Rightarrow$ $$\Big\| \int_{t'<t} S(t-t') F(t') \big|_{x=0} \, dt' \Big\|_{L^2_t} \overset{(?)}{\lesssim} \|f\|_{L^2_t}$$

$S(t-t')\delta_0\big|_{x=0}$ = fundamental soln at $x=0$ $= \dfrac{c}{t-t'}$ $(d=2)$

$$\Big\| \int_{t'<t} \frac{1}{t-t'} f(t') \, dt' \Big\|_{L^2_t} \overset{(?)}{\lesssim} \|f\|_{L^2_t} = 1 \text{ for } \mathbb{1}_{[0,1]}$$

easily seen to be FALSE (just as seen on P.20.)

- For the failure of the homogeneous estimate (or the dual estimate) for $(q, r, d) = (2, \infty, 2)$, see

  Montgomery-Smith, Duke Math. J. '98

  on both the Schrödinger and wave equation.

- See also Lec 13-14 in my 2017 course on "Probabilistic perspectives in nonlinear dispersive PDEs". (lecture notes available from my website.)

  for the failure of the endpt Strichartz estimate for the wave eqn ($\because (q, r, d) = (2, \infty, 3)$.)

  (also the failure of the frequency localized estimate, using a Brownian motion.