Lec 5 : Dec 10, 2024 (TUE)

· Strichartz estimate for the Schrödinger equation on $\mathbb{R}^d$

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad 2 \le q, r \le \infty, \ (q, r, d) \neq (2, \infty, 2)$$

$$\| S(t) f \|_{L_t^q L_x^r (\mathbb{R}_t \times \mathbb{R}_x^d)} \lesssim \| f \|_{L_x^2}$$

$$\Rightarrow \quad q = r = \frac{2(d+2)}{d} \qquad \frac{2 \cdot d}{2(d+2)} + \frac{d \cdot d}{2(d+2)} = \frac{d}{2}$$

① $$\| S(t) f \|_{L_{t,x}^{\frac{2(d+2)}{d}}} \lesssim \| f \|_{L^2}.$$

② $$\| S(t) P_{\le N} f \|_{L_{t,x}^{\infty} (\mathbb{R}_t \times \mathbb{R}_x^d)} \overset{\text{Bernstein's ineq}}{\lesssim} N^{d/2} \| S(t) P_{\le N} f \|_{L_t^\infty L_x^2}$$

↑ proj onto $\{ |\xi| \le N \}$

$$\le N^{d/2} \| f \|_{L_x^2}$$

① & ② $\Longrightarrow$ interpolate $\| S(t) P_{\le N} f \|_{L_{t,x}^p (\mathbb{R}_t \times \mathbb{R}_x^d)} \lesssim N^{\frac{d}{2} - \frac{d+2}{p}} \| f \|_{L_x^2}$ $\quad \frac{2(d+2)}{2} \le p \le \infty$

On $\mathbb{T}^d$?

$$S(t)f(x) = \sum_{n\in\mathbb{Z}^d} e^{-4\pi^2 i|n|^2 t}\, e^{2\pi i n\cdot x}\, \hat{f}(n).$$

$d=1$: $S(t)f$ is periodic in time.

(For $d\geq 2$, take $f(x) = f(x_1,\cdots,x_d) = f(x_1)$.

$\Rightarrow$ dispersive & Strichartz estimates cannot hold globally in time.

(dispersive estimate for $P_{\leq N} f$ holds up to time $O(N^{-1})$

Strichartz estimate on $\mathbb{T}^d$:

$$\| S(t) P_{\leq N} f \|_{L^p_{t,x}([0,1]\times\mathbb{T}^d)} \lesssim N^{\frac{d}{2}-\frac{d+2}{p}} \|f\|_{L^2_x}$$

for $p > \frac{2(d+2)}{d}$ — local in time

When $p = \frac{2(d+2)}{d}$, we have N^{ε}.

- Bourgain '93. analytic number theory.
- Bourgain-Demeter '15, incidence geom
- Killip-Vişan '16, ε-removal.

$d=2$: $(\log N)^{\frac{1}{4}}$ Heo-Kwak FM Pi '24.

· Strichartz estimate for the wave eqn on $\mathbb{R}^d$:

wave admissible: $\frac{1}{q} + \frac{d-1}{2r} \le \frac{d-1}{4}$, $q, r, d \ge 2$.

$d=1$: wave is NOT dispersive

scaling condition

$0 \le s \le 1$

$(q,r), (\tilde q, \tilde r)$ admis.

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - s = \frac{1}{\tilde q'} + \frac{d}{\tilde r'} - 2, \qquad r, \tilde r < \infty, \quad (q,r,d) \neq (2,\infty,3)$$

$$\Leftrightarrow \quad \frac{1}{q} + \frac{1}{\tilde r} = \frac{d}{2} - (1-s)$$

(q,r): s-admissible

$(\tilde q, \tilde r)$: $(1-s)$-admissible

Consider $\begin{cases} \partial_t^2 u - \Delta u = F \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases}$.

$\dot{\mathcal{H}}^s = \dot H^s \times \dot H^{s-1}$

$$\Rightarrow \boxed{\begin{aligned} &\|(u, \partial_t u)\|_{L_t^\infty \dot{\mathcal{H}}_x^s} + \|u\|_{L_t^q L_x^r} \\ &\lesssim \|(u_0, u_1)\|_{\dot{\mathcal{H}}_x^s} + \|F\|_{L_t^{\tilde q'} L_x^{\tilde r'}} \end{aligned}}$$

See Vişan's Oberwolfach notes for the proof.

· $r = \infty$ case: See Fang-Wang, Nonlin. Anal. '06.

$$S(t) = \frac{\sin t|\nabla|}{|\nabla|}, \qquad |\nabla| = \sqrt{-\Delta}$$

$$u(t) = \partial_t S(t) u_0 + S(t) u_1 + \int_0^t S(t-t') F(t') dt'.$$

$$= \cos(t|\nabla|) u_0 + \frac{\sin(t|\nabla|)}{|\nabla|} u_1 + \underbrace{\int_0^t \frac{\sin((t-t')|\nabla|)}{|\nabla|} F(t') dt'}_{=: I(F)(t)}.$$

- $\| I(F) \|_{L_t^\infty \dot{H}_x^s} \lesssim \| F \|_{L_t^1 \dot{H}_x^{s-1}}$.
- $\| I(F) \|_{L_t^q L_x^r} \leq \int_{\mathbb{R}} \left\| \frac{\sin((t-t')|\nabla|)}{|\nabla|} F(t') \right\|_{L_t^q L_x^r} dt'$

$$\lesssim_{\text{homog. Str. est.}} \| F \|_{L_t^1 \dot{H}_x^{s-1}}$$

$$\Rightarrow \boxed{(\text{LHS}) \text{ of Stri esti} \lesssim \| (u_0, u_1) \|_{\dot{\mathcal{H}}_x^s} + \| F \|_{L_t^1 H_x^{s-1}}}$$

energy estimate. (& stri. esti)

· Rmk: The same Strichartz estimates hold on $\mathbb{T}^d$
(but up to a finite time T, with const depending T.)

ex: 3-d cubic NLW: $\partial_t^2 u - \Delta u = u^3$. on $\mathbb{R}^3$ (or $\mathbb{T}^3$)

Scaling critical regularity $s_{crit} = 1/2$.

(There is also a critical reg. w.r.t. Lorentzian Invariance conformal

$$\Gamma(u)(t) = \partial_t S(t) u_0 + S(t) u_1 + \int_0^t \frac{\sin((t-t')|\nabla|)}{|\nabla|} u^3(t')\, dt'.$$

(i) LWP in $\dot{\mathcal{H}}^1$: $\vec{\Gamma}[u] = (\Gamma(u), \partial_t \Gamma[u])$

$$\|\vec{\Gamma}[u]\|_{L_T^\infty \dot{\mathcal{H}}_x^1} \le C_1 \|(u_0, u_1)\|_{\dot{\mathcal{H}}^1} + \int_0^T \underbrace{\|u^3(t')\|_{L_x^2}}_{\le \|u(t')\|_{L_x^6}^3 \overset{Sob}{\lesssim} \|u(t')\|_{\dot{H}_x^1}^3} dt'$$

$$\le C_1 \|(u_0, u_1)\|_{\dot{\mathcal{H}}^1} + T \|\vec{u}\|_{L_T^\infty \dot{\mathcal{H}}_x^1}^3$$

(ii) LWP in $\dot{H}^{1/2}$ ⇐ critical space.

$(q, r) = (4,4)$ is $\frac{1}{2}$- admissible

$\| \quad$

$(\tilde{q}, \tilde{r})$

$\frac{1}{4} + \frac{3-1}{2\cdot 4} \le \frac{3-1}{4}$ ✓

$\frac{1}{4} + \frac{3}{4} = \frac{3}{2} - \frac{1}{2}$

Let $X(T) = C_T \dot{H}^s_x \cap C^1_T \dot{H}^{s-1}_x \cap L^4_{T,x}$

$\le \| u \|^3_{L^4_{T,x}} \le \| u \|^3_{X(T)}$.

$\Rightarrow \| \Gamma(u) \|_{X(T)} \lesssim \| (u_0, u_1) \|_{\dot{H}^{1/2}} + \| u^3 \|_{L^{4/3}_{T,x}}$

↑ NO T^θ.

$\Rightarrow$ LWP for small data

Proceeding as in the case of 2-d cubic NLS, we also have LWP for large data. ($T = T(u_0, u_1)$, small $\Leftarrow \| \partial_t S(t) u_0 + S(t) u_1 \|_{L^4_{T,x}} \le \eta \ll 1$

Aside: heat equation $\partial_t u = \Delta u$.

Schauder estimate on $\mathbb{R}^d$: $1 \le p \le q \le \infty$, $\alpha \ge 0$.

(*) $$\| |\nabla|^\alpha e^{t\Delta} f \|_{L^q(\mathbb{R}^d)} \lesssim t^{\underbrace{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})-\frac{\alpha}{2}}_{\ge 0}} \|f\|_{L^p(\mathbb{R}^d)}, \quad t>0.$$

Pf: $e^{t\Delta} f = K_t * f$. $\widehat{K_t}(\xi) = e^{-t|\xi|^2}$.

$= \widehat{K}(t^{1/2}\xi)$, $K = K_{t=1}$.

$\Rightarrow K_t(x) = \frac{1}{t^{d/2}} K(\frac{x}{t^{1/2}})$.

$\alpha = 0$: $\|K_t\|_{L^r_x} = t^{-d/2} \|K(\frac{\cdot}{t^{1/2}})\|_{L^r_x} = t^{-d/2} t^{d/2r} c_r \sim t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}$

$\Rightarrow \|e^t f\|_{L^q} = \|K_t * f\|_{L^q} \overset{\text{Young}}{\le} \|K_t\|_{L^r} \|f\|_{L^p}$, $\boxed{\frac{1}{q}+1 = \frac{1}{r}+\frac{1}{p}}$

$t^{\overset{?}{-}\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}$

$\alpha > 0$: $|\nabla|^\alpha e^{t\Delta} f = (|\nabla|^\alpha K_t) * f$.

$$\widehat{|\nabla|^\alpha K_t}(\xi) = |\xi|^\alpha e^{-t|\xi|^2} = t^{-\alpha/2} \underbrace{(t^{1/2}|\xi|)^\alpha e^{-t|\xi|^2}}_{= \widehat{G}(t^{1/2}\xi)} \quad G \in \mathcal{S}(\mathbb{R}^d)$$

$$\Rightarrow \| |\nabla|^\alpha K_t \|_{L^r} \sim t^{-\alpha/2} t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}$$

Then, apply Young's inequality

On $\mathbb{T}^d$? If $u_0 \equiv$ const c, $\partial_t \hat{u}(0) = -|n|^2 \hat{u}(n)\big|_{n=0} = 0$.

$\Rightarrow$ Schauder estimate on $\mathbb{T}^d$ does not hold for $t \gg 1$.

(unless $p = q$.)

Schauder estimate on $\mathbb{T}^d$: $\circledast$ holds for $0 < t \leq 1$

- can't use the scaling argument (used for the $\mathbb{R}^d$-case).

Poisson summation formula: f on $\mathbb{R}^d$.

$$|f(x)| + |\hat{f}(x)| \lesssim \langle x \rangle^{-d-\varepsilon} \quad \text{for some } \varepsilon > 0.$$

Then,
$$\sum_{n \in \mathbb{Z}^d} \hat{f}(n)\, e^{2\pi i n \cdot x} = \underbrace{\sum_{n \in \mathbb{Z}^d} f(x+n)}_{= \text{periodization of } f}$$

Pf: Let $F(x) = \sum_{n \in \mathbb{Z}^d} f(x+n)$, func on $\mathbb{T}^d$.

$$\hat{F}(n) = \int_{\mathbb{T}^d} \sum_{n \in \mathbb{Z}^d} f(x+n)\, e^{-2\pi i n \cdot x}\, dx$$

$$= \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i n \cdot x}\, dx = \underbrace{\hat{f}(n)}_{\mathcal{F}_{\mathbb{R}^d}(f)(n)}$$

□

· Pf of Schauder esti on $\mathbb{T}^d$:

$$\widehat{K_t^{\mathbb{T}^d}}(n) = e^{-t|n|^2} = \mathcal{F}_{\mathbb{R}^d}(K_t)(n).$$

$$\| |\nabla|^\alpha K_t^{\mathbb{T}^d} \|_{L^r(\mathbb{T}^d)}$$

$$= \| \sum_{n\in\mathbb{Z}^d} \mathcal{F}_{\mathbb{R}^d}(|\nabla|^\alpha K_t)(n)\, e^{2\pi i n\cdot x} \|_{L^r(\mathbb{T}^d)}$$

$$\overset{\text{Poisson}}{=} \| \sum_{n\in\mathbb{Z}^d} |\nabla|^\alpha K_t(x+n) \|_{L^r(\mathbb{T}^d)}$$

$$\| \langle m\rangle^{-\beta} \|_{\ell^{r'}_m} \| \langle m\rangle^{\beta} |\nabla|^\alpha K_t(x+n) \|_{\ell^r_n}$$

$\| \langle m\rangle^{-\beta} \|_{\ell^{r'}_m} \lesssim 1$ for $\beta r' > d$ $\left(\Longleftrightarrow \beta > d(1-\frac{1}{r})\right.$

$$\lesssim \| |\nabla|^\alpha K_t(x) \|_{L^r(\mathbb{R}^d)} + \| |x|^\beta |\nabla|^\alpha K_t(x) \|_{L^r(\mathbb{R}^d)}$$

$$\sim t^{-\frac{\alpha}{2}} t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}$$

$|x| \sim |\frac{x}{t^{1/2}}|^\beta$ for $0<t\le 1$.

$|\nabla|^\alpha K_t(x) = G(\frac{x}{t^{1/2}})$

Note: When $p = q$, we have $r = 1$.

$$\| |\nabla|^\alpha K_t^{\mathbb{T}^d} \|_{L^1(\mathbb{T}^d)} = \| \sum_{n \in \mathbb{Z}^d} |\nabla|^\alpha K_t(x+n) \|_{L^1(\mathbb{T}^d)}$$
$$\lesssim t^{-\alpha/2} \quad \text{for any } t > 0.$$

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ex: $\partial_t u = \Delta u + u^3$ on $\mathbb{R}^d$ (or $\mathbb{T}^d$).

$$\mathcal{C}^s = B^s_{\infty,\infty} = \text{Hölder–Besov space.}$$

$$\|f\|_{\mathcal{C}^s} = \sup_{\substack{N \geq 1 \\ \text{dyadic}}} \|P_N f\|_{L^\infty_x}$$

P_N: L-P projector onto $\{|\xi| \sim N\}$.

$0 < s < 1$: $\mathcal{C}^s = \Lambda^s =$ Lipschitz space

$$\|f\|_{\Lambda^s} = \|f\|_{L^\infty} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^s}$$

- $\mathcal{E}^s$, $s \geq 0$, is an algebra.
- $\| |\nabla|^\alpha e^{t\Delta} f \|_{\mathcal{E}^s} \lesssim t^{-\frac{\alpha}{2}} \|f\|_{\mathcal{E}^s} \quad \forall s \in \mathbb{R}$.
- $\Gamma(u)(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-t')\Delta} u^3(t')\, dt'$.

Fix $t > 0$.

$\sigma = 0+$

$s < 0$

(A1)

$$t^\theta \|\Gamma(u)(t)\|_{\mathcal{E}^\sigma} \lesssim t^\theta t^{-\frac{\sigma - s}{2}} \|u_0\|_{\mathcal{E}^s} + t^\theta \int_0^t \|e^{(t-t')\Delta} u^3(t')\|_{\mathcal{E}^\sigma} dt'$$

$\theta - \frac{\sigma - s}{2} \geq 0$

Take sup $0 < t \leq T$

algebra b/c $\sigma > 0$

$$\lesssim t^\sigma \Big(\sup_{0 \leq t \leq T} (t')^\theta \|u(t')\|_{\mathcal{E}^\sigma} \Big)^3 \underbrace{(t')^{-3\theta} \big((t')^\theta \|u(t')\|_{\mathcal{E}^\sigma} \big)^3}$$

Need $3\theta < 1$

$\leq t^\sigma \|u\|^3_{X(T)}$ defined on the next page

$\frac{\sigma - s}{2} \leq \theta < \frac{1}{3}$

$\Rightarrow \quad s > -\frac{2}{3} + \sigma$.

Choose $\theta = \frac{\sigma - s}{2}$ with $\sigma = 0+$

$$\|\Gamma(u)\|_{C_T\mathcal{C}^s_x} \lesssim \|u_0\|_{\mathcal{C}^s} + \Big\| \int_0^t \cdots dt' \Big\|_{C_T\mathcal{C}^\sigma_x} \lesssim t^\gamma \|u\|^3_{X(T)}$$

(A2)

$$\|u\|_{X(T)} = \|u\|_{C_T\mathcal{C}^s_x} + \sup_{0<t\le T} t^{\frac{\sigma-s}{2}} \|u(t)\|_{\mathcal{C}^\sigma}$$

$\Rightarrow$ From (A1) & (A2), we obtain

$$\|\Gamma(u)\|_{X(T)} \lesssim \|u_0\|_{\mathcal{C}^s} + T^\gamma \|u\|^3_{X(T)}.$$

· A difference estimate follows from a similar computation

$\Rightarrow$ LWP in $\mathcal{C}^s$, $s > -2/3$ $\Leftarrow$ indep of $d \ge 1$

$$u \in C([0,T]; \mathcal{C}^s_x) \cap C((0,T]; \mathcal{C}^{0+}_x)$$

$\partial_t u = \Delta u + u^3$ is invariant under the scaling

$$u^\lambda(t,x) = \frac{1}{\lambda} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right).$$

$$\Rightarrow \quad \|u_0^\lambda\|_{\mathcal{C}^{-1}} = \|u_0\|_{\mathcal{C}^{-1}}.$$

indep of $d \geq 1$

$s_{crit} = -1$

↑ NOT L^2-based Sobolev space but L^∞-based Sobolev space (and $\mathcal{C}^s$.)

- Cubic NLH is ill-posed in $\mathcal{C}^s$, $s \leq -\frac{2}{3}$.

  (Chevyrev-Oh-Wang, F.E. '25+

↑ in negative regularity spaces, the scaling heuristics can go wrong.

(See, for example, Choffrut-Pocovnicu, IMRN '18 for ill-posedness of fractional NLS

Sec 1.5 : Dispersive estimate.

$$\begin{cases} i\partial_t u + \Delta u = 0 \\ u|_{t=0} = f \end{cases}$$

$$\Rightarrow \quad S(t)f(x) = \frac{1}{(4\pi i t)^{d/2}} \int_{\mathbb{R}^d} \underbrace{e^{-\frac{|x-y|^2}{4it}}}_{\to K_t(x-y)} f(y)\,dy$$

$$\Rightarrow \quad \|S(t)f\|_{L^\infty_x} \lesssim \frac{1}{|t|^{d/2}} \|f\|_{L^1_x}, \qquad t \neq 0. \tag{$\dagger$}$$

Goal: Prove the dispersive estimate WITHOUT using the explicit formula

1-d case: $S(t)f = K_t * f$

$$K_t = \mathcal{F}^{-1}\left(e^{-4\pi^2 i t \xi^2}\right) = \int_{\mathbb{R}} e^{-4\pi^2 i t \xi^2} e^{2\pi i x \xi}\,d\xi.$$

Claim: $\|K_t(x)\|_{L^\infty_x} \lesssim \frac{1}{|t|^{1/2}}$ ($\Rightarrow$ ($\dagger$) follows from Young.

· Pf of Claim: $t > 0$. $\zeta = t^{1/2}\xi$

$$K_t(x) = \frac{1}{\sqrt{t}} \int_{\mathbb{R}} e^{-4\pi^2 i \zeta^2} e^{2\pi i \frac{x}{\sqrt{t}}\zeta} d\zeta$$

$$= \frac{1}{\sqrt{t}} K_1\left(\frac{x}{\sqrt{t}}\right) \quad \text{— } y \quad (t \text{ is fixed})$$

$$\Rightarrow \|K_t\|_{L^\infty_x} = \frac{1}{\sqrt{t}} \|K_1\|_{L^\infty_y}$$

Tool: Method of stationary phase (drop 2π)

$$K_1(x) = \int_{\mathbb{R}} e^{-i\xi^2 + ix\xi} d\xi = \int_{\mathbb{R}} e^{-i\Phi(\xi)} d\xi$$

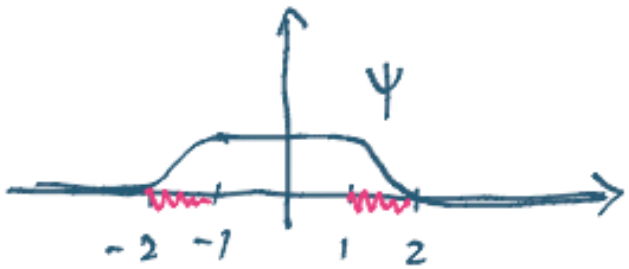
where $\Phi(\xi) = \xi^2 - x\xi$. (x is fixed)

Idea: Integration by parts: $e^{-i\Phi(\xi)} = \dfrac{\partial_\xi e^{-i\Phi(\xi)}}{-i\Phi'(\xi)}$ ← If $\Phi'(\xi)$ is not small, good

"$\int_{\mathbb{R}} e^{-i\Phi(\xi)} d\xi = \int \partial_\xi e^{-i\Phi(\xi)} \cdot \frac{1}{-i\Phi'(\xi)} d\xi$

$\overset{\text{IBP}}{=} \int e^{-i\Phi(\xi)} \partial_\xi \left(\frac{1}{i\Phi'(\xi)}\right) d\xi$" $\quad \Phi' = 2\xi - x$.

Let $\psi \in C^\infty(\mathbb{R}; [0,1])$ s.t. $\psi(\xi) = \begin{cases} 1, & |\xi| \le 1 \\ 0, & |\xi| \ge 2 \end{cases}$



$\Rightarrow K_1(x) = \int_{\mathbb{R}} e^{-i\phi(\xi)} \psi(2\xi - x)\, d\xi + \int_{\mathbb{R}} e^{-i\phi(\xi)} (1-\psi)(2\xi - x)\, d\xi$ ($\phi'(\xi)$)

$=: I(x) + II(x)$

- $|I(x)| \le \int_{|2\xi - x| \le 2} 1\, d\xi \sim 1$ $\quad \phi'(\xi) = 2\xi - x$
- $II(x) \overset{IBP}{=} \int_{\mathbb{R}} e^{-i\phi(\xi)} \partial_\xi \frac{(1-\psi)(2\xi - x)}{i\phi'(\xi)}\, d\xi$

$= \int_{|2\xi - x| \ge 1} e^{-i\phi(\xi)} \left\{ -2 \frac{(1-\psi)(2\xi - x)}{i(2\xi - x)^2} \right\} d\xi$ $\quad (1-\psi)' = \psi'$

$+ \int_{1 \le |2\xi - x| \le 2} e^{-i\phi(\xi)} \frac{(1-\psi)'(2\xi - x)}{i\phi'(\xi)}\, d\xi \Rightarrow |II(x)| \lesssim 1.$ $\square$

($\phi'(\xi) \gtrsim 1$)

• Glimpse of oscillatory integrals

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)}\psi(x)\,dx,$$

phase ϕ, real-valued

$\operatorname{supp}\psi \subset\subset (a,b)$, smooth.

($\subset\subset$: cptly contained in.)

Lem A: Suppose $\phi'(x) \neq 0$, $\forall\, x \in [a,b]$

Then, $I(\lambda) = O(\lambda^{-N})$ as $\lambda \to \infty$, $\forall\, N \in \mathbb{N}$.

Pf: Let $Df(x) = \dfrac{1}{i\lambda\phi'(x)}\dfrac{df}{dx}$.

$$\Rightarrow D\,e^{i\lambda\phi(x)} = e^{i\lambda\phi(x)}$$

$$\int (Df)g\,dx = \int f\,D^T g\,dx$$

Transpose $D^T f = -\dfrac{d}{dx}\left(\dfrac{f}{i\lambda\phi'}\right)$

No bdry term since $\operatorname{supp}\psi \subset\subset [a,b]$

$$\Rightarrow I(\lambda) = \int_a^b e^{i\lambda\phi}\cdot\psi\,dx \overset{\text{IBP}}{=} \int_a^b e^{i\lambda\phi}\,(D^T)^N\psi\,dx$$

($D^N e^{i\lambda\phi}$; N times)

$$\Rightarrow |I(\lambda)| \lesssim_{N,\phi,\psi} \lambda^{-N}.$$

($|\phi'| \gtrsim 1$ on $[a,b]$) ☐

Prop B: (van der Corput lemma): Φ, real-valued, smooth on (a,b)

Suppose $\exists\, k \in \mathbb{N}$ s.t. $|\Phi^{(k)}(x)| \geq 1$, $\forall x \in [a,b]$

Then, $\left| \int_a^b e^{i\lambda\Phi(x)} dx \right| \leq C_k \lambda^{-1/k}$, ← Much worse decay than Lem A. due to the contributions from the endpts.

C_k indep of a, b

provided that (i) $k \geq 2$,

or (ii) $k = 1$ and Φ' is monotonic.

Pf: (ii) $k=1$: $\int_a^b e^{i\lambda\Phi} dx = \int_a^b D(e^{i\lambda\Phi}) \cdot 1 \, dx$

$$\overset{\text{IBP}}{=} \int_a^b e^{i\lambda\Phi} D^T(1)\, dx + \frac{e^{i\lambda\Phi}}{i\lambda\Phi'}\Big|_a^b$$

$\left|\frac{e^{i\lambda\Phi}}{i\lambda\Phi'}\right| \cdot |1| \leq 2/\lambda$

$$\left| \int_a^b e^{i\lambda\Phi} D^T(1) dx \right| = \frac{1}{\lambda} \left| \int_a^b e^{i\lambda\Phi} \frac{d}{dx}\left(\frac{1}{\Phi'}\right) dx \right|$$

$$\leq \frac{1}{\lambda} \int_a^b \left| \frac{d}{dx}\left(\frac{1}{\Phi'}\right) \right| dx$$

Φ' monotonic $\Rightarrow \frac{1}{\Phi'}$ is monotonic.

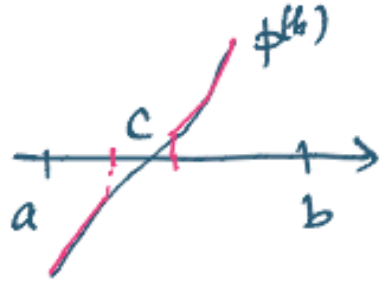
$$= \frac{1}{\lambda} \left| \int_a^b \frac{d}{dx}\left(\frac{1}{\Phi'}\right) dx \right| \overset{\text{F.T.C.}}{=} \frac{1}{\lambda} \left| \frac{1}{\Phi'(b)} - \frac{1}{\Phi'(a)} \right| \lesssim \frac{1}{\lambda}.$$

(i) $k \geq 2$: We proceed by induction on k.

Assume the bound for k. WLOG, assume $\phi^{(k+1)}(x) \geq 1$, $\forall x \in [a,b]$

· Let $x = c$ be the (unique) pt in $[a,b]$ s.t. $|\phi^{(k)}(x)|$ attains its min.

· If $\phi^{(k)}(c) = 0$, then $|\phi^{(k)}(x)| \geq \delta$ on $(c-\delta, c+\delta)^c$.



Write $\int_a^b = \int_a^{c-\delta} + \int_{c-\delta}^{c+\delta} + \int_{c+\delta}^b$.

· By the inductive hypothesis, $i(\lambda\delta)\,\delta^{-1}\phi$

$$\left| \int_a^{c-\delta} + \int_{c+\delta}^b e^{i\lambda\phi}\, dx \right| \leq 2C_k (\lambda\delta)^{-1/k}.$$

· On the other hand,

$$\left| \int_{c-\delta}^{c+\delta} \cdots dx \right| \leq 2\delta.$$

equate them $\Rightarrow \delta \sim (\lambda\delta)^{-1/k}$

$\Leftrightarrow \delta \sim \lambda^{-1/k+1}$.

If $\phi^{(k)}(c) \neq 0$, we have $c = a$ (or $c = b$)

write $\int_a^b = \int_a^{a+\delta} + \int_{a+\delta}^b$ and proceed as before.

$\square$

Cor C: Same assumption as in Prop B.

$$\left| \int_a^b e^{i\lambda\phi(x)} \psi(x)\, dx \right| \le C_k \lambda^{-1/k} \left[|\psi(b)| + \int_a^b |\psi'(x)|\, dx \right].$$

Pf: Let $F(x) = \int_a^x e^{i\lambda\phi(y)}\, dy$.

$$\overset{\text{Prop B.}}{\Longrightarrow} \quad |F(x)| \le C_k \lambda^{-1/k}.$$

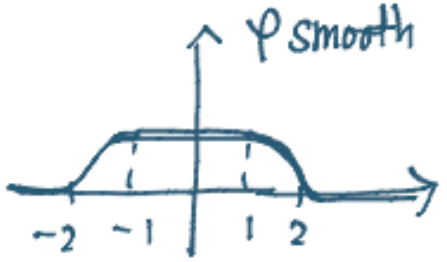
$$\Rightarrow \int_a^b e^{i\lambda\phi(x)} \psi(x)\, dx \underset{\text{IBP}}{=} F(b)\psi(b) - \cancel{F(a)\psi(a)} - \int_a^b F(x)\psi'(x)\, dx$$

($F(a) = 0$; $e^{i\lambda\phi(x)} = F'(x)$)

□

· Glimpse of the Littlewood-Paley decomposition

$$\varphi(\xi) = \begin{cases} 1, & |\xi| \le 1 \\ 0, & |\xi| \ge 2 \end{cases}$$



Given dyadic N

$$\left(\begin{array}{l} N = 2^j \text{ for some } j \in \mathbb{Z} \\ N \in 2^{\mathbb{Z}} \end{array} \right| \quad \begin{array}{l} \text{dyadic } N \ge 1 \\ \Leftrightarrow N \in 2^{\mathbb{Z}_{\ge 0}} \end{array}$$

define $P_{\le N} f = \mathcal{F}^{-1}\left(\varphi\left(\frac{\xi}{N}\right) \hat{f}(\xi) \right)$

$$\varphi\left(\frac{\xi}{N}\right) \begin{cases} 1, & |\xi| \le N \\ 0, & |\xi| \ge 2N \end{cases}$$

· Littlewood-Paley projector: $P_N f = P_{\le N} f - P_{\le \frac{N}{2}} f$

= freq localization into $\{|\xi| \sim N\}$

$\frac{1}{2} N \le |\xi| \le 2N$.

multiplier $\varphi\left(\frac{\xi}{N}\right) - \varphi\left(\frac{2\xi}{N}\right)$.

(People also use $P_j = P_{2^j}$, Q_j, Δ_j for L-P projectors.)

$\varphi_N(\xi) = \varphi\left(\frac{\xi}{N}\right)$ is L^∞_ξ-invariant

$\Rightarrow$ $\check{\varphi}_N$ is L^1_α-invariant.. So is $\check{\varphi}_N - \check{\varphi}_{N/2}$

$\Rightarrow$ (++)

$$\|P_{\le N} f\|_{L^p} \lesssim \|f\|_{L^p}$$
$$\|P_N f\|_{L^p} \lesssim \|f\|_{L^p}$$

) unif. in N

- $P_{\le N} f = \sum_{\substack{M \le N \\ \text{dyadic}}} P_M f$

- $f = \lim_{N \to \infty} P_{\le N} f = \sum_{N, \text{dyadic}} P_N f.$

Littlewood-Paley theorem: $1 < p < \infty$.

Then, $\|f\|_{L^p} \sim \left\| \left(\sum_{\substack{N \\ \text{dyadic}}} |P_N f|^2 \right)^{1/2} \right\|_{L^p} = \|f\|_{F^0_{2,p}}$

square function of f.

Sobolev space

$$\|f\|_{\dot{H}^s} \sim \Big\| \Big(\sum_{N,\text{dyadic}} N^{2s} |P_N f|^2 \Big)^{1/2} \Big\|_{L^2_x} = \|N^s P_N f\|_{\ell^2_n L^2_x}$$

$$\|f\|_{\dot{W}^{s,p}} \sim \Big\| \Big(\sum_{N,\text{dyadic}} N^{2s} |P_N f|^2 \Big)^{1/2} \Big\|_{L^p_x}.$$

$$\|f\|_{W^{s,p}} \sim \Big\| \Big(\sum_{N,\text{dyadic}} N^{2s} |P_N f|^2 \Big)^{1/2} \Big\|_{L^p_x} \qquad P_1 := P_{\leq 1}.$$

Besov space : $B^s_{p,q}$, $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$

$$\|f\|_{B^s_{p,q}} = \Big\| \, \|N^s P_N f\|_{L^p_x} \, \Big\|_{\ell^q_{N,\text{dyadic}}}$$

Triebel-Lizorkin space : $F^s_{p,q}$, $s \in \mathbb{R}$, $p, q < \infty$

$$\|f\|_{F^s_{p,q}} = \Big\| \, \|N^s P_N f(x)\|_{\ell^p_N} \, \Big\|_{L^q_x}.$$

$\overset{L-P}{\Rightarrow} F^s_{2,p} = W^{s,p}$ $\quad 1 < p < \infty$

· Bernstein's inequalities: $1 \le p \le q \le \infty$

(i) $\| P_{\le N} |\nabla|^s f \|_{L^p} \lesssim N^s \| P_{\le N} f \|_{L^p}$, $s \ge 0$

(ii) $\| P_N |\nabla|^s f \|_{L^p} \sim N^s \| P_N f \|_{L^p}$, $s \in \mathbb{R}$

(iii) $\| P_{\le N} f \|_{L^q} \lesssim N^{\frac{d}{p} - \frac{d}{q}} \| P_{\le N} f \|_{L^p}$

(iv) $\| P_N f \|_{L^q} \lesssim N^{\frac{d}{p} - \frac{1}{q}} \| P_N f \|_{L^p}$

} freq localized version of Sobolev's inequality $s = \frac{d}{p} - \frac{d}{q}$.

Pf: (ii) $\tilde{P}_N = P_{N/2} + P_N + P_{2N}$ ⇐ "fattened" freq support.

$\varphi(\frac{\xi}{2N}) - \varphi(\frac{4\xi}{N}) =: \psi(\frac{\xi}{N})$

$\Rightarrow P_N = \tilde{P}_N P_N$

$P_N |\nabla|^s f = |\nabla|^s \tilde{P}_N P_N f$

$\rightarrow N^s \left|\frac{\xi}{N}\right|^s \psi\left(\frac{\xi}{N}\right)$



Let $K_s = \mathcal{F}^{-1}\left(|\xi|^s \psi(\xi)\right)$

↑ supported away from $\xi = 0$.

$\Rightarrow$ $|\xi|^s \psi(\xi)$ is smooth $\Rightarrow$ K_s decays rapidly

$\Rightarrow \|K_s\|_{L^1_x} < \infty$

- $\mathcal{F}^{-1}\left(\left|\frac{\xi}{N}\right|^s \psi\left(\frac{\xi}{N}\right)\right) = \mathcal{F}^{-1}\left(\widehat{K_s}\left(\frac{\xi}{N}\right)\right) = N^d K_s(Nx) =: K_{s,N}(x)$

  $\Rightarrow \|K_{s,N}\|_{L^1} = \|K_s\|_{L^1} < \infty$

$\Rightarrow \|P_N |\nabla|^s f\|_{L^p} = N^s \|K_{s,N} * P_N f\|_{L^p} \overset{\text{Young}}{\lesssim} N^s \|P_N f\|_{L^p}$

This proves (LHS) $\lesssim$ (RHS).

- Let $g = |\nabla|^s f$. Then, $f = |\nabla|^{-s} g$.

  $\Rightarrow \|P_N f\|_{L^p} = \|P_N |\nabla|^{-s} g\|_{L^p} \lesssim N^{-s} \|P_N g\|_{L^p} = N^{-s} \|P_N |\nabla|^s f\|_{L^p}$

  This proves (LHS) $\gtrsim$ (RHS).

(i) $s>0$, $1\le p\le\infty$ $\qquad P_{\le N}|\nabla|^s f = P_{\le 2N}|\nabla|^s P_{\le N} f$.

$\Rightarrow \|P_{\le N}|\nabla|^s f\|_{L^p}$ Minkowski $\sum_{\substack{M\le 2N\\ \text{dyadic}}} \|P_M|\nabla|^s P_{\le N} f\|_{L^p}$

$\sim M^s \|P_M P_{\le N} f\|_{L^p}$ by (ii)

$\lesssim N^s \sup_{\substack{M\text{ dyadic}\\ M\le 2N}} \|P_M P_{\le N} f\|_{L^p}$

(ff) on p.23

$\lesssim N^s \|P_{\le N} f\|_{L^p}$

$\sum_{\substack{M\le 2N\\ \text{dyadic}}} M^s \lesssim N^s$, $s>0$

□