Lec 6 : Dec. 17, 2024 (TUE)

Dispersive estimate: $\|S(t)f\|_{L^\infty(\mathbb{R}^d)} \lesssim \frac{1}{|t|^{d/2}} \|f\|_{L^1(\mathbb{R}^d)}$, $t \neq 0$.

- easily follows from Young's inequality & the explicit formula for the fundamental soln:

$$K_t(x) = \mathcal{F}^{-1}\left(e^{-4\pi^2 i t|\xi|^2}\right)(x) = \int_{\mathbb{R}^d} e^{-4\pi^2 i t|\xi|^2} e^{2\pi i x\cdot\xi}\, d\xi$$

$$= \frac{1}{(4\pi i t)^{d/2}} e^{-|x|^2/4it} \quad (\to \delta_0 \text{ as } t \to 0).$$

GOAL: Prove the dispersive estimate <u>without</u> using the explicit formula.

- $d = 1$: last time (by IBP).

The proof for $d \geq 2$ is also based on IBP but we need to work harder, (using the tools introduced at the end of the last lecture.

$$K_t(x) = \int_{\mathbb{R}^d} e^{i(-4\pi^2|\xi|^2 t + 2\pi x\cdot\xi)}\, d\xi.$$

$$P_{\le N} K_t(x) = \int_{\mathbb{R}^d} e^{i(-4\pi^2|\xi|^2 t + 2\pi x\cdot\xi)} \varphi\left(\frac{\xi}{N}\right) d\xi \Leftarrow |\xi| \lesssim N$$

$$P_N K_t(x) = \int_{\mathbb{R}^d} e^{i(-4\pi^2|\xi|^2 t + 2\pi x\cdot\xi)} \psi\left(\frac{\xi}{N}\right) d\xi \Leftarrow |\xi| \sim N$$

N, dyadic.

$\psi(\cdot) = \varphi(\cdot) - \varphi(2\cdot)$

↑ supported on an annulus $\{|\xi| \sim 1\}$.

WTS:

$$\|K_t\|_{L^\infty_x} \lesssim \frac{1}{|t|^{d/2}}$$

Fix $t > 0$.

① $N_1 \in 2^{\mathbb{Z}}$, dyadic, largest s.t. $N_1 \le 100\, t^{-1/2}$ $\quad \xi = t^{1/2}\xi \Rightarrow |\xi| \lesssim 1.$

$$|P_{\le N_1} K_t(x)| = \frac{1}{|t|^{d/2}} \left| \int_{|\xi|\lesssim 1} e^{i(-4\pi^2|\xi|^2 + 2\pi \frac{x}{t^{1/2}}\cdot\xi)} \varphi\left(\frac{\xi}{t^{1/2}}\right) d\xi \right|$$

$$\lesssim \frac{1}{|t|^{d/2}}, \text{ unif in } x \in \mathbb{R}^d.$$

Fix $x \in \mathbb{R}^d$.

(2) N_2, dyadic, largest s.t. $N_1 < N_2 \leq 100\left|\frac{x}{t}\right|$.

(If no such N_2, skip this case.

- $r = |\xi| \gtrsim N_1$

$\Rightarrow |x| r \gtrsim N_1^2 t \sim 1$ b/c $N_1 \sim t^{-1/2}$

FACT: f, radial, i.e. $f(x) = f_0(|x|)$ For us, $|x| r \gtrsim 1$.

$$\Rightarrow \widehat{f}(\xi) = \frac{2\pi}{|\xi|^{\frac{d-2}{2}}} \int_0^\infty f_0(r)\, J_{\frac{d-2}{2}}(2\pi|\xi| r)\, r^{d/2}\, dr$$

($\widehat{f}$: radial)

J_ν = Bessel function of order ν

(See App B of Grafakos vol 1

$$J_\nu(r) = \sqrt{\frac{2}{\pi r}} \cos\left(r - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) + R_\nu(r) \quad \text{as } r \to \infty$$

$\text{Re}\,\nu > -1/2$

$$|R_\nu(r)| = O(r^{-3/2}).$$

Main contribution:

$N_1 \lesssim r \lesssim N_2$

$$I_t^{\pm}(x) = \frac{c}{|x|^{\frac{d-1}{2}}} \int e^{-4\pi^2 i t r^2 \pm 2\pi i |x| r} \Psi(r)\, r^{\frac{d-1}{2}}\, dr$$

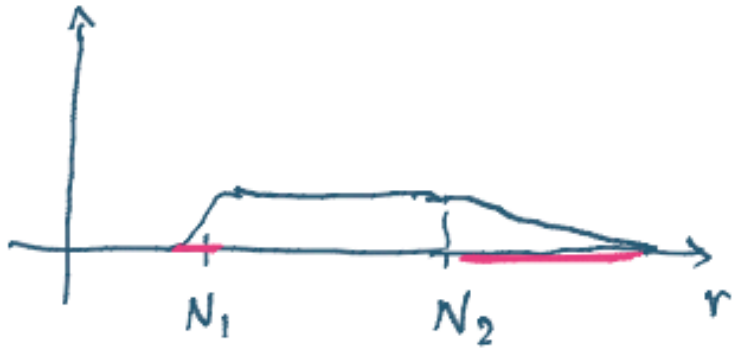
$i t \phi(r)$

- $\Psi(r) = \sum_{N_1 < N \le N_2} \psi\left(\frac{r}{N}\right) = \varphi\left(\frac{r}{N_2}\right) - \varphi\left(\frac{2r}{N_1}\right)$

$$\Rightarrow \Psi'(r) = \frac{1}{N_2}\varphi'\left(\frac{r}{N_2}\right) - \frac{2}{N_1}\varphi'\left(\frac{2r}{N_1}\right)$$

- $\phi(r) = -4\pi^2 r^2 \pm 2\pi \frac{|x|}{t} r$

$\Rightarrow |\phi''(r)| \gtrsim 1.$



Cor to van der Corput (Lec 5, p.21)

$$\Rightarrow |I_t^{\pm}(x)| \lesssim \frac{1}{|t|^{1/2}} \frac{1}{|t|^{\frac{d-1}{2}}} \left(1 + \int_{N_1 \lesssim r \lesssim N_2} |\Psi'(r)|\, dr\right)$$

$$\sim \frac{1}{|t|^{d/2}}$$

$$\frac{r^{\frac{d-1}{2}}}{|x|^{\frac{d-1}{2}}} \lesssim \frac{N_2^{\frac{d-1}{2}}}{N_2^{\frac{d-1}{2}} |t|^{\frac{d-1}{2}}}$$

$$\int_{N_1 \lesssim r \lesssim N_2} |\Psi'(r)|\, dr \lesssim \sum_{j=1}^{2} \frac{1}{N_j} \int_{r \sim N_j} |\varphi'(\tfrac{r}{N_j})|\, dr \lesssim 1.$$

error:

$$P_N \mathcal{E}(x) = \frac{c}{|x|^{\frac{d-2}{2}}} \int_{r\sim N} e^{-4\pi^2 i t r^2} \psi\left(\frac{r}{N}\right) R_{\frac{d-2}{2}}(2\pi|x| r)\, r^{\frac{d}{2}}\, dr$$

ch of var: $r = N\tau$

$$= c\, \frac{N^{\frac{d+2}{2}}}{|x|^{\frac{d-2}{2}}} \int_{\tau\sim 1} e^{-4\pi^2 i t N^2 \tau^2} \psi(\tau) \underbrace{R_{\frac{d-2}{2}}(2\pi N|x|\tau)}_{O\left(\frac{1}{N^{3/2}|x|^{3/2}}\right)} \tau^{d/2}\, d\tau.$$

$$\Rightarrow |P_N \mathcal{E}(x)| \lesssim \frac{N^{\frac{d+2}{2}}}{|x|^{\frac{d-2}{2}}} \frac{1}{N^{3/2}|x|^{3/2}}$$

$$= \frac{N^{\frac{d-1}{2}}}{|x|^{\frac{d+1}{2}}} \lesssim \frac{1}{|t|^{\frac{d+1}{2}} N}$$

$|x| \gtrsim N|t|$

$$\Rightarrow \sum_{N_1 < N \le N_2} |P_N \mathcal{E}(x)| \lesssim \frac{1}{|t|^{\frac{d+1}{2}}} \frac{1}{N_1} \sim \frac{1}{|t|^{d/2}}$$

geometric series

$N_1 \sim t^{-1/2}$

③ $N > N_2 \gg \frac{|x|}{t}$ $\qquad \xi = N\zeta$

$$P_N K_t(x) = N^d \int_{\mathbb{R}^d} e^{-4\pi^2 i t N^2 |\zeta|^2} e^{2\pi i N x\cdot\zeta} \psi(\zeta)\, d\zeta$$

$$= \psi(|\zeta|)$$

polar coord

$$= N^d \int_{S^{d-1}} \int_0^\infty e^{-4\pi^2 i t N^2 r^2} e^{2\pi i N r\, x\cdot\omega} \psi(r)\, r^{d-1}\, dr\, d\sigma(\omega)$$

$r \sim 1$

$-itN^2\phi(r)$

surface meas on S^{d-1}

$$\phi(r) = 4\pi^2 r^2 - 2\pi r \frac{x}{tN}\cdot\omega \qquad \left|\frac{x}{tN}\right| \ll 1$$

$$\Rightarrow |\phi'(r)| \sim 1.$$

$$e^{-itN^2\phi(r)} = \frac{\partial_r e^{-itN^2\phi(r)}}{-itN^2\phi'(r)}$$

IBP d times

$$\Rightarrow |P_N K_t(x)| \sim \frac{N^d}{(|t|N^2)^d}$$

$N_2 > N_1 \sim t^{-1/2}$

$$\Rightarrow \sum_{N>N_2} |P_N K_t(x)| \sim \sum_{N>N_2} \frac{1}{|t|^d} N^d \sim \frac{1}{|t|^d} N_2^d \lesssim \frac{1}{|t|^{d/2}} \qquad \square$$

Sec 1.b : Local smoothing & maximal function estimates

Prop (local smoothing estimate) $d=1$

$$\| D^{1/2} S(t) f \|_{L^\infty_x L^2_t} \sim \| f \|_{L^2_x} \qquad D = |\partial_x|$$

Note: $S(t) = e^{it\partial_x^2}$ is unitary on H^s & time reversible

$\Rightarrow$ No global gain of differentiability (unlike the heat eqn

Pf: $D^{1/2} S(t) f(x) = \int |\xi|^{1/2} e^{ix\xi - it\xi^2} \hat f(\xi)\, d\xi \qquad \eta = \xi^2$

$$\sim \int |\eta|^{-\frac14} e^{ix\eta^{1/2} - it\eta} \hat f(\eta^{1/2})\, d\eta \qquad \tfrac12 \tfrac{1}{\eta^{1/2}} d\eta = d\xi$$

- change of viewpt : view t as a spatial var, x as a temporal var. $\to\ i\partial_t u = D^{1/2} u$

$\Rightarrow \| D^{1/2} S(t) f(x) \|_{L^2_t} \overset{\text{Plancherel in } t}{\sim} \| |\eta|^{-1/4} e^{ix\eta^{1/2}} \hat f(\eta^{1/2}) \|_{L^2_\eta}$

$= \left(\int \frac{1}{|\eta|^{1/2}} |\hat f(\eta^{1/2})|^2 d\eta \right)^{1/2} \overset{\text{undo ch of var}}{\sim} \| \hat f(\xi) \|_{L^2_\xi} = \| f \|_{L^2_x}$ ☐

Note: ① For $d \geq 2$, see HW soln.

② ∃ proof without using F.T. (short note by Tao)

"A physical space proof of ...".

③ Another version of local smoothing estimate, $d \geq 1$

$$\left(\int_{\mathbb{R}} \int_{\mathbb{R}^d} \left| |\nabla|^{1/2} S(t) f(x) \right|^2 \underbrace{e^{-|x|^2}}_{\text{spatial localization}} \, dx \, dt \right)^{1/2} \lesssim \|f\|_{L^2_x}$$

$\Leftarrow$ see Vişan's note (or my course note '16).

Prop: Maximal function estimate, $d=1$

$$\|S(t)f\|_{L_x^4 L_t^\infty} \lesssim \|D^{\frac{1}{4}} f\|_{L_x^2}$$

$$\| \sup_{t\in\mathbb{R}} |S(t)f| \|_{L_x^4}$$

↑ maximal function

Q: $u(t,x) = S(t)f(x) \longrightarrow f(x)$, a.e.?

Carleson '80, $f \in H^{\frac{1}{4}}(\mathbb{R})$

Dahlberg-Kenig '82, false if $s < \frac{1}{4}$

↷ Bourgain, ...

Du-Guth-Li, '17, Ann. Math Du-Zhang '19 $s > \frac{d}{2(d+1)}$

· Pf of a.e. conv ($d=1$, $s \geq \frac{1}{4}$).

$$\limsup_{t \searrow 0} |S(t)f(x) - f(x)| \leq \sup_t |S(t)(f-g)(x)|$$

$$+ |f(x) - g(x)| \qquad g, \text{ smooth}$$

$$+ \limsup_{t \searrow 0} |S(t)g(x) - g(x)|$$

$$= 0$$

· Given $\alpha > 0$, set

$$A_\alpha = \{ x \in \mathbb{R} : \limsup_{t \searrow 0} |S(t)f(x) - f(x)| > \alpha \}$$

$$\subset \{ \sup_t |S(t)(f-g)(x)| > \frac{\alpha}{2} \} \cup \{ |f(x) - g(x)| > \frac{\alpha}{2} \} =: B_\alpha \cup C_\alpha.$$

$$\Rightarrow |A_\alpha| \leq |B_\alpha| + |C_\alpha| \overset{\text{Chebyshev}}{\leq} \left(\frac{2}{\alpha}\right)^4 \| \sup_t |S(t)(f-g)| \|_{L^4_x}$$

$$+ \left(\frac{2}{\alpha}\right)^4 \| f - g \|_{L^4_x}$$

Max-func. $\lesssim$ Sob. $\frac{1}{\alpha^4} \| f - g \|_{\dot{H}^{\frac{1}{4}}} < \frac{\varepsilon}{\alpha^4}$ by choosing suitabe $g = g_\varepsilon$

$\Rightarrow$ Since the choice of $\varepsilon > 0$ is arbitrary, $|A_\alpha| = 0$, $\forall\, \alpha > 0$. $\square$

· **Pf of maximal function estimate:**

WTS: $\| \underbrace{D^{-\frac{1}{4}} S(t) f}_{= T} \|_{L_x^4 L_t^\infty} \lesssim \| f \|_{L_x^2}$

$\Leftrightarrow$ dual: $\| \int_{\mathbb{R}} D^{-\frac{1}{4}} S(-t') F(t') \, dt' \|_{L_x^2} \lesssim \| F \|_{L_x^{4/3} L_t^1}$

$\Leftrightarrow TT^*$: $\| \int_{\mathbb{R}} D^{-1/2} S(t-t') F(t') \, dt' \|_{L_x^4 L_t^\infty} \lesssim \| F \|_{L_x^{4/3} L_t^1}$.

· **Lemma:** $\underbrace{\left| \int_{\mathbb{R}} e^{i(x\xi - t\xi^2)} \frac{1}{|\xi|^{1/2}} d\xi \right|}_{= D^{-1/2} K_t(x)} \lesssim \frac{1}{|x|^{1/2}}$, unif in t.

· For now, we assume Lemma and prove the TT^* bound.

$$\left\| \int_{\mathbb{R}} D^{-1/2} S(t-t') F(t') dt' \right\|_{L_x^4 L_t^\infty} \qquad S(t)f = K_t *_x f$$

$$\int_y \int_{t'} D^{-1/2} K(t-t', x-y) F(t', y) dt' dy$$

convolution in time

Young in t

$$\lesssim \left\| \int_y \| D^{-1/2} K(\cdot, x-y) \|_{L_t^\infty} \| F(\cdot, y) \|_{L_t^1} dy \right\|_{L_x^4}$$

Apply Lemma

Lemma

$$\lesssim \left\| \frac{1}{|\cdot|^{1/2}} *_x \| F(t, \cdot) \|_{L_t^1} \right\|_{L_x^4} \qquad \text{H-L-S: } \frac{1}{4} + 1 = \frac{1}{2} + \frac{3}{4}$$

H-L-S

$$\lesssim \| F \|_{L_x^{4/3} L_t^1}$$

$\left\| \frac{1}{|x|^{1/2}} \right\|_{L^2(\mathbb{R})} = \infty$ but H-L-S works

□

· Pf of Lemma: WTS $\| D^{-1/2} K_t(x) \|_{L_t^\infty} \lesssim \frac{1}{|x|^{1/2}}$.

① $|\xi| \lesssim |x|^{-1}$.

$$(LHS) \lesssim \int_0^{|x|^{-1}} \frac{1}{|\xi|^{1/2}} d\xi \sim |\xi|^{1/2} \Big|_0^{|x|^{-1}} = \frac{1}{|x|^{1/2}}$$

② $|\xi| \gg |x|^{-1}$ (Assume $\xi > 0$). Ch of var: $\xi^2 = \eta$

(*) $$(LHS) \sim \int e^{\overbrace{ix\eta^{1/2} - it\eta}^{ix\Phi}} \frac{1}{\eta^{3/4}} d\eta.$$

$\Phi(\eta) = \eta^{1/2}$

$\Phi'(\eta) \sim \eta^{-1/2}$, $\Phi''(\eta) \sim \eta^{-3/2}$

$ix\Phi(\eta) - it\eta = ix\big(\underbrace{\Phi(\eta) - \frac{t}{x}\eta}_{=: \tilde{\Phi}}\big)$

(2.a) $\Omega_1 = \{ \eta \gg |x|^{-2} : \underbrace{|\phi'(\eta) - \frac{t}{x}|}_{=\Phi'} \le |\frac{t}{2x}| \}$

$\sim \eta^{-1/2}$

$\Rightarrow \phi'(\eta) \sim \frac{t}{x} \Rightarrow \eta \sim \left(\frac{x}{t}\right)^2$.

(*) $\sim \int e^{i \underbrace{x \min|\Phi''|}_{\lambda} \cdot \underbrace{\Phi / \min|\Phi''|}_{\text{2nd derivative} \ge 1}} \cdot |\phi''|^{1/2} d\eta$

Cor to van der Corput

$\lesssim (x \cdot \min|\Phi''|)^{-1/2} \{ \max|\phi''|^{1/2} + \int |\phi''|^{-1/2} |\phi'''| d\eta$

$\sim \left(x \cdot \frac{t^3}{x^3}\right)^{-1/2}$ $\quad \sim \left(\frac{t}{x}\right)^{3/2}$ $\quad \phi''' \sim \eta^{-5/2}$

$= x / t^{3/2}$

$\Phi(\eta) = \phi(\eta) - \frac{t}{x}\eta$

$\Rightarrow \Phi''(\eta) = \phi''(\eta)$

$\eta \sim \left(\frac{x}{t}\right)^2$

$\left(\frac{x}{t}\right)^{3/2} \left(\frac{t}{x}\right)^5 \left(\frac{x}{t}\right)^2 = \left(\frac{t}{x}\right)^{3/2}$

$\lesssim \frac{1}{|x|^{1/2}}$

(2.b) $\Omega_2 = \{ \eta \gg |x|^{-2} : |\Phi'(\eta) - \frac{t}{x}| > |\frac{t}{2x}| \}$.

$\Rightarrow |\Phi' - \frac{t}{x}| \overset{\text{triang.}}{\geq} |\Phi'| - |\frac{t}{x}|$

$> |\Phi'| - 2|\Phi' - \frac{t}{x}|$

$\Phi' \sim \eta^{-1/2}$
$\Phi'' \sim \eta^{-3/2}$
$\Phi''' \sim \eta^{-5/2}$

$= \Phi'$ (on Ω_2)

$\Rightarrow |\Phi' - \frac{t}{x}| \gtrsim |\Phi'|$

$(*) = \int e^{i x \Phi(\eta)} \frac{1}{\eta^{3/4}} d\eta = \int \partial_\eta e^{i x \Phi(\eta)} \frac{|\Phi''|^{1/2}}{i \partial_\eta (x \Phi(\eta))} d\eta.$

IBP
$\leq \frac{1}{|x|} \int \left| \partial_\eta \left(\frac{|\Phi''|^{1/2}}{\partial_\eta \Phi(\eta)} \right) \right| d\eta \leq \frac{1}{|x|} \int \frac{1}{|\Phi'|^{1/2}} \frac{|\Phi'''|}{|\Phi' - \frac{t}{x}|} + \frac{|\Phi''|^{3/2}}{|\Phi' - \frac{t}{x}|^2} d\eta$

$\gtrsim |\Phi'|$.

$\sim -\frac{1}{|x|} \eta^{-\frac{1}{4}} \Big|_{|x|^{-2}}^{\infty} = \frac{1}{|x|^{1/2}}$

$\frac{3}{4} - \frac{5}{2} + \frac{1}{2} = -\frac{5}{4}$
$\Rightarrow \eta^{-5/4}$

$-\frac{9}{4} + 1 = -5/4$
$\Rightarrow \eta^{-5/4}$

$\int_{|x|^{-2}}^{\infty} \eta^{-\frac{5}{4}} d\eta$

$\square$

• Generalized KdV equation (gKdV)

$$\begin{cases} \partial_t u = \partial_x^3 u + \partial_x(u^5), & x \in \mathbb{R} \\ u|_{t=0} = u_0 \in L^2(\mathbb{R}) \end{cases}$$

We need to make up this derivative loss.

$$\partial_x(u^k): \quad s_{crit} = \frac{1}{2} - \frac{2}{k-1} \Rightarrow k = 5: \ s_{crit} = 0.$$

Airy eqn: $\begin{cases} \partial_t u = \partial_x^3 u \\ u|_{t=0} = u_0 \end{cases}$, $x \in \mathbb{R}$. $\quad S(t) = e^{\pm \partial_x^3}$

Lem A: (local smoothing) Kato smoothing.

$$\| \partial_x S(t) u_0 \|_{L_x^\infty L_t^2} \lesssim \| u_0 \|_{L_x^2}.$$

↑ one full derivative

Dual: $\sup_t \| \partial_x \int_{\mathbb{R}} S(t-t') F(t') dt' \|_{L_x^2} \lesssim \| F \|_{L_x^1 L_t^2}$

$$\| \partial_x \int_{\mathbb{R}} S(-t') F(t') dt' \|_{L_x^2}$$

$$\Rightarrow \quad \| \partial_x \int_0^t S(t-t') F(t')\, dt' \|_{L_t^\infty L_x^2}$$

$$\overset{\text{unitarity}}{=} \| \partial_x \int_0^t S(-t') F(t')\, dt' \|_{L_t^\infty L_x^2} \overset{\text{dual}}{\lesssim} \sup_t \| \mathbb{1}_{[0,t]}(t') F(t') \|_{L_x^1 L_{t'}^2}$$

$$\mathbb{1}_{[0,t]}(t')$$

$$\leq \| F \|_{L_x^1 L_t^2}$$

(K1)

- TT^*:

$$\| \partial_x^2 \int_{\mathbb{R}} S(t-t') F(t')\, dt' \|_{L_x^\infty L_t^2} \lesssim \| F \|_{L_x^1 L_t^2}$$

$$\Rightarrow \| \partial_x^2 \int_0^t S(t-t') F(t')\, dt' \|_{L_x^\infty L_t^2} \lesssim \| F \|_{L_x^1 L_t^2}$$

(K2)

↑ non-trivial Kenig-Ponce-Vega, CPAM '93 Thm 3.5

(Also, Linares-Ponce Lem 7.1, but see Thm 4.4

Lem B: (maximal func estimate)

$$\| S(t) f \|_{L_x^4 L_x^\infty} \lesssim \| f \|_{\dot{H}^{\frac{1}{4}}} \qquad \text{(same as the Schrödinger case}$$

Lem C: ① $\| S(t) f \|_{L_x^5 L_t^{10}} \lesssim \| f \|_{L_x^2}$

② $\| \int_0^t S(t-t') F(t') dt' \|_{L_x^5 L_t^{10}} \lesssim \| F \|_{L_x^{5/4} L_t^{10/9}}$

Pf: Lem A & B with Stein's interpolation

↳ T_z, operator analytically depending on z

$\{ 0 \le \mathrm{Re}\, z \le 1 \}$

Suppose $T_z : L^{p_j} \to L^{q_j}$, $1 \le p_j, q_j \le \infty$, $j = 0, 1$, when $\mathrm{Re}\, z = j$.

Then, T_θ is bounded from L^p to L^q.

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad 0 \le \theta \le 1.$$

· pf of Stein's interpolation: same as Riesz-Thorin from Lec 2.

Instead of $\Phi(z) = \int_Y (Tf_z)\, g_z$, analytic in the strip

T_z needs to satisfy some growth condition.

by $\Phi(z) = \int_Y (T_z f_z) g_z$, analytic in the strip.

See Chap 1 of Grafakos vol. 1.

□

pf of ①: $T_z f = D^{-\frac{z}{4}} D^{1-z} S(t) f$. $\quad z \in S = \{0 \le \mathrm{Re}\, z \le 1\}$.

$D = |\partial_x| = \partial_x H$ (H: Hilbert transf $= -i\,\mathrm{sgn}\,\xi$)

Re z = 0: $T_{0+iy} = \partial_x S(t) D^{-\frac{5}{4}iy} H f$.

$\Rightarrow \|T_{0+iy} f\|_{L^\infty_x L^2_t} \overset{\text{loc smoothing}}{\lesssim} \|D^{-\frac{5}{4}iy} H f\|_{L^2_x} = \|f\|_{L^2_x}$ (Plancherel)

Re z = 1: $T_{1+iy} f = D^{-\frac{1}{4}} S(t) D^{-\frac{5}{4}iy} f$

$\Rightarrow \|T_{1+iy} f\|_{L^4_x L^\infty_t} \overset{\text{max. func. est.}}{\lesssim} \|f\|_{L^2_x}$

Here, we really need Stein's interpolation for the mixed Lebesgue spaces.

(see p. 26 ~)

Note: $S(t)f = T_{4/5} f$.

$$\frac{1}{5} = \frac{1-4/5}{\infty} + \frac{4/5}{4}$$

$$\frac{1}{10} = \frac{1-4/5}{2} + \frac{4/5}{\infty}$$

Stein

$$\Rightarrow \| S(t)f \|_{L_x^5 L_t^{10}} \lesssim \| f \|_{L_x^2}$$

$$\Rightarrow \text{dual}: \left\| \int_{\mathbb{R}} S(-t') F(t') \, dt' \right\|_{L_x^2} \lesssim \| F \|_{L_x^{5/4} L_t^{10/9}}$$

$$\Rightarrow TT^*: \left\| \int_{\mathbb{R}} S(t-t') F(t') \, dt' \right\|_{L_x^5 L_t^{10}} \lesssim \| F \|_{L_x^{5/4} L_t^{10/9}}$$

Replacing by $\int_0^t$ is non-trivial (involving $BMO_x(L_t^2)$, etc.)

KPV '93, Cor 3.8

$\Rightarrow$ (2)

□

· Small data GWP in $L^2(\mathbb{R})$

$$\Gamma_{u_0}(u)(t) = S(t)u_0 + \int_0^t S(t-t')\partial_x(u^5)(t')dt'.$$

NO T^ϕ (b/c critical)

$$\Rightarrow \|\Gamma[u]\|_{L_t^\infty L_x^2} \le \|u_0\|_{L_x^2} + \Big\|\partial_x \int_0^t S(t-t')\,u^5(t')\,dt'\Big\|_{L_t^\infty L_x^2}$$

(K1) on p.17

$$\lesssim \|u^5\|_{L_x^1 L_t^2} \le \|u\|_{L_x^5 L_t^{10}}.$$

· $\|\Gamma[u]\|_{L_t^5 L_x^{10}} \overset{\text{Lem C}}{\lesssim} \|u_0\|_{L_x^2} + \|u^4 \partial_x u\|_{L_x^{5/4} L_t^{10/9}}$

$$\overset{\wedge 1}{\|u\|^4_{L_x^5 L_t^{10}}} \quad \overbrace{\|\partial_x u\|_{L_x^\infty L_t^2}}^{\text{local smoothing space}}$$

· $\|\partial_x \Gamma[u]\|_{L_x^\infty L_t^2} \overset{\text{loc.-smoothing}}{\lesssim} \|u_0\|_{L_x^2} + \|u^5\|_{L_x^1 L_t^2}$

(K2) on p.17.

$$\le \|u\|^5_{L_x^5 L_t^{10}}$$

Set $X = L_t^\infty L_x^2 \cap L_x^5 L_t^{10} \cap \{ \| \partial_x u \|_{L_x^\infty L_t^2} < \infty \}$

$$\Rightarrow \| \Gamma(u) \|_X \le C_0 \| u_0 \|_{L_x^2} + C_1 \| u \|_X^5$$

Similarly, $\| \Gamma(u) - \Gamma(v) \|_X \le \underbrace{C_2 \left(\| u \|_X^4 + \| v \|_X^4 \right)}_{\le 1/2, \ \forall\, u, v \in \overline{B_R}} \| u - v \|_X$

Let $\| u_0 \|_{L^2} \le \eta \ll 1$.

$\overline{B_R} \subset X, \quad R = 2 C_0 \eta$

$$\Rightarrow \| \Gamma(u) \|_X \le \frac{1}{2} R + C_1 (2 C_0 \eta)^5 \le R \qquad \forall\, u \in \overline{B_R}$$

Thm 8: $\exists\ 0 < \eta \ll 1$ s.t. if $\| u_0 \|_{L^2} \le \eta$, then the L^2-critical quintic gKdV is GWP in $L^2(\mathbb{R})$.

· LWP in $L^2(\mathbb{R})$ for large data

Note that the last two estimates on p.21 does NOT involve the $L^\infty_t L^2_x$-norm (i.e. they are "closed" for the $L^5_x L^{10}_t$-norm & the local smoothing norm)

· $\|\Gamma(u)\|_{L^5_x L^{10}_T} \le \|S(t)u_0\|_{L^5_x L^{10}_T} + C\|u\|^4_{L^5_x L^{10}_T} \|\partial_x u\|_{L^\infty_x L^2_T}$

↑ We did not apply Lem C (to exploit smallness).

· $\|\partial_x \Gamma(u)\|_{L^\infty_x L^2_T} \le \|\partial_x S(t)u_0\|_{L^\infty_x L^2_T} + C\|u\|^5_{L^5_x L^{10}_T}$

Set $Y(T) = L^5_x L^{10}_T \cap \{\|\partial_x u\|_{L^\infty_x L^2_T} < \infty\}$.

$$\Rightarrow \|\Gamma(u)\|_{Y(T)} \le \|S(t)u_0\|_{Y(T)} + C\|u\|^5_{Y(T)} \quad \text{(K3)}$$

Similarly, $\|\Gamma(u)-\Gamma(v)\|_{Y(T)} \le C\left(\|u\|^4_{Y(T)} + \|v\|^4_{Y(T)}\right)\|u-v\|_{Y(T)}$ (K4)

Suppose that by choosing $T = T(u_0) > 0$ small, we have
↑ "depends on the profile of u_0".

(★) $$\|S(t)u_0\|_{Y(T)} = \|S(t)u_0\|_{L_x^5 L_T^{10}} + \|\partial_x S(t)u_0\|_{L_x^\infty L_T^2} \le \eta \ll 1.$$

Then, (K3) & (K4) $\Rightarrow$ Γ is a contraction on $\overline{B}_{2\eta} \subset Y(T)$.

- It remains to verify (★).
  - For the 2-d cubic NLS, an analogous claim followed from the Strichartz estimate & DCT.
  - For (★), we have the x-norms outside and such an argument does not work Instead

$$\|\partial_x S(t)u_0\|_{L_x^\infty L_T^2} \le \|\partial_x S(t) P_{>N} u_0\|_{L_x^\infty L_T^2} + \|\partial_x S(t) P_{\le N} u_0\|_{L_x^\infty L_T^2}$$

$$\lesssim \underbrace{\|P_{>N} u_0\|_{L_x^2}}_{\to 0 \text{ as } N \to \infty} + T^{1/2} N^{3/2} \|u_0\|_{L_x^2}$$

↙ local smoothing estimate

↙ · Hölder in t · switch the order of the norms · Bernstein

· First choose $N = N(u_0, \eta_f) = N(u_0)$ large and

η_f ← some small absolute const.

and then choose $T = T(N, \|u_0\|_{L^2}) = T(u_0)$ small.

$$\Rightarrow \quad \|\partial_x S(t) u_0\|_{L^\infty_x L^2_T} \leq \frac{\eta_f}{4} + \frac{\eta_f}{4} = \frac{\eta_f}{2}$$

By a similar argument, $\|S(t) u_0\|_{L^5_x L^{10}_T} \leq \eta_f/2$ $\Rightarrow$ ✡

$\Rightarrow$ ∃! soln u in $Y(T)$.

· Now, use the first estimate on p. 21 to show $u \in C_T L^2_x$
(but on $[0, T]$.)

Thm 9: The quintic gKdV is locally well-posed in $L^2(\mathbb{R})$

Rmk: In order to extend Thm 8 & Thm 9 to $H^s(\mathbb{R})$, $s>0$,
we need the fractional Leibniz rule (as on pp. 10-11 in Lec 2)
BUT with the x-norms OUTSIDE!! See Appendix of KPV '93.

· Appendix: Riesz-Thorin & Stein's interpolation for mixed Lebesgue spaces

In the proof of Lemma C on pp. 19-20, we applied Stein's interpolation, where the target space is the mixed Lebesgue spaces (but we only discussed the case where the target space is a single Lebesgue space.)

In the following, I will explain a proof for mixed Lebesgue spaces.

Setting: T_z, analytic (in z) family of operators in the strip $S=\{0\le \mathrm{Re} z\le 1\}$
(satisfying some growth condition).

Suppose $T_z : L^{p_j}(X) \to L^{q_j}(Y; L^{r_j}(\tilde{Y}))$, when $\mathrm{Re}\, z = j$.

$1 \le p_j, q_j \le \infty$, $j = 0, 1$.

Then, T_θ is bounded from $L^p(X)$ to $L^q(Y; L^r(\tilde{Y}))$, $0 \le \theta \le 1$

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$$

Pf:

$f \in L^p(X)$, $g \in L^{q'}(Y; L^{r'}(\tilde{Y}))$

With $f = F e^{i\Phi}$, $F \ge 0$, Φ, real-valued, set

$$f_z = (F^{\frac{p}{p_0}})^{1-z} (F^{\frac{p}{p_1}})^{z} e^{i\Phi}, \quad z \in S.$$

↖ analytic in z

Then,

$$|f_z|^{p_0} = |f|^p \quad \text{for } \mathrm{Re}\, z = 0$$

$$|f_z|^{p_1} = |f|^p \quad \text{for } \mathrm{Re}\, z = 1.$$

(see p.2 of Lec 2.

$\Rightarrow$ (S1) $\|f_z\|_{L^{p_j}_x} = \|f\|^{p/p_j}_{L^p_x}$, $\mathrm{Re}\, z = j$, $j = 0, 1$

We assume that g is a simple function (in y). Namely,

$$g(y,\tilde{y}) = \sum_{j=1}^{J} \mathbb{1}_{E_j}(y)\, a_j(\tilde{y}),$$

- E_j, measurable set in Y, disjoint, $|E_j| < \infty$.
- $a_j(\tilde{y}) = A_j(\tilde{y})\, e^{i\psi_j(\tilde{y})}$, $A_j \geq 0$, ψ_j, real-valued.

Now, define g_z by

$$g_z(y,\tilde{y}) = \sum_{j=1}^{J} \mathbb{1}_{E_j}(y)\; A_j(\tilde{y})^{\frac{r'}{r_0'}(1-z)}\; A_j(\tilde{y})^{\frac{r'}{r_1'}z}\; e^{i\psi_j(\tilde{y})}$$

$$\times \underbrace{\|a_j(\tilde{y})\|_{L^{r'}_{\tilde{y}}}^{\,q'\left(\frac{1-z}{q_0'}+\frac{z}{q_1'}\right) - r'\left(\frac{1-z}{r_0'}+\frac{z}{r_1'}\right)}}_{=: H_z}$$

Note that

$$g_\theta(y,\tilde{y}) = g(y,\tilde{y}) \quad \text{b/c} \quad \frac{1}{q'} = \frac{1-\theta}{q_0'} + \frac{\theta}{q_1'}, \quad \frac{1}{r'} = \frac{1-\theta}{r_0'} + \frac{\theta}{r_1'}.$$

· $\underline{\mathrm{Re}\, z = j}$: $\|g_z(y,\vec{y})\|^{r_j'}_{L^{r_j'}_{\vec{y}}} = \sum_{j=1}^{J} \mathbb{1}_{E_j}(y) \|a_j(\vec{y})\|^{r'}_{L^{r'}_{\vec{y}}}$ (E_j's disjoint)

$\times\ |Hz|^{r_j'}$

$= \sum_{j=1}^{J} \mathbb{1}_{E_j}(y) \|a_j(\vec{y})\|^{r_j' \frac{q'}{q_j'}}_{L^{r'}_{\vec{y}}}$

$\Rightarrow$ (S2) $\boxed{\|g_z(y,\vec{y})\|_{L^{q_j'}_y L^{r_j'}_{\vec{y}}} = \left(\sum_{j=1}^{J} |E_j| \|a_j\|^{q'}_{L^{r'}_{\vec{y}}}\right)^{1/q'} = \|g\|^{q'/q_j'}_{L^{q'}_y L^{r'}_{\vec{y}}}}$

· Now, set $\Phi(z) = \int_{Y \times \vec{Y}} (T_z f_z)\, g_z$ $\Leftarrow$ analytic in z.

$\Rightarrow |\Phi(z)| \lesssim \|f_z\|_{L^{p_j}_x} \|g_z\|_{L^{q_j'}_y L^{r_j'}_{\vec{y}}}$ $\qquad \mathrm{Re}\, z = j,\ j = 0,1.$

$= \|f\|^{p/p_j}_{L^p_x} \|g\|^{q'/q_j'}_{L^{q'}_y L^{r'}_{\vec{y}}}$

(S1) & (S2)

By three line lemma,

$$|\Phi(\theta)| = \left| \int_{Y \times \widetilde{Y}} (T_\theta f) g \right|$$

$$\lesssim \left(\|f\|_{L^{p}_{x}}^{p/p_0} \, \|g\|_{L^{q'}_{y} L^{r'}_{\widetilde{y}}}^{q'/q_0'} \right)^{1-\theta}$$

$$\times \left(\|f\|_{L^{p}_{x}}^{p/p_1} \, \|g\|_{L^{q'}_{y} L^{r'}_{\widetilde{y}}}^{q'/q_1'} \right)^{\theta}$$

$$= \|f\|_{L^{p}_{x}} \, \|g\|_{L^{q'}_{y} L^{r'}_{\widetilde{y}}}$$

$$\left(\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q'} = \frac{1-\theta}{q_0'} + \frac{\theta}{q_1'} \right)$$

note $q' < \infty$

$\Rightarrow$ By the density of simple functions & duality, we conclude the boundedness of $T_\theta : L^{p}_{x} \to L^{q}_{y} L^{r}_{\widetilde{y}}$

$\square$ (simple functions $g = \sum_{j=1}^{J} 1_{E_j} a_j$, $|E_j| < \infty$, are NOT dense in L^∞.