Lec 7: Dec 19, 2024 (THU)

Sec 1.7: Basic scattering theory

- Energy-subcritical defocusing NLS on $\mathbb{R}^d$ ($s_c < 1$)

$$i\partial_t u + \Delta u = |u|^{p-1}u$$

- LWP in H^1 & conservation of

$$\text{mass}: M(u) = \int |u|^2 dx$$

$$\text{Hamiltonian}: H(u) = \frac{1}{2}\int |\nabla u|^2 dx + \frac{1}{p+1}\int |u|^{p+1} dx$$

$\Rightarrow$ GWP in H^1

Q: long-time behavior? scattering?

? $\exists\, u_\pm \in H^1(\mathbb{R}^d)$ s.t. $\|u(t) - S(t)u_\pm\|_{H^1} \to 0$ as $t \to \pm\infty$.

Does u behave asymptotically like a linear soln as $t \to \pm\infty$?

(If so, u must spread out in space and its height must decay to 0.
($\Leftarrow$ dispersive estimate.

· Virial identity & Morawetz estimate

I_{virial} = "force"

$$i\partial_t u + \Delta u = \lambda |u|^{p-1} u, \qquad \lambda = 1 : \text{defocusing}, \quad \lambda = -1 : \text{focusing}$$

Virial potential : $V_a(t) = \int_{\mathbb{R}^d} a(x) |u(x)|^2 dx.$

$$\Rightarrow \partial_t^2 V_a = 4 \int \mathrm{Re}\,(\partial_k u \partial_j \bar{u})\, \partial_k \partial_j a + 2\lambda \frac{p+1}{p-1} \int |u|^{p+1} \Delta a - \int |u|^2 \Delta^2 a.$$

↘ $\partial_t V = \int a\, 2\mathrm{Re}\,(\partial_t u\, \bar{u})\, dx$; ↑ use the eqn

· Virial identity: $a(x) = |x|^2$, mass-critical $p = 1 + 4/d$

$$\Rightarrow \partial_t^2 V(t) = 16\, H(u)$$

⇒ If $H(u_0) < 0$, then the soln must blow up in finite time.

↑ possible only for the focusing case (see Lec 2).

· **Morawetz estimate**: $a(x) = |x|$, $\partial_j a = \frac{x_j}{|x|}$, $\Delta a = \frac{d-1}{|x|}$

$$\partial_t^2 V_a = 4 \int \frac{|\not\nabla u|^2}{|x|}\, dx \geq 0$$

$$+ 2\lambda \frac{p-1}{p+1} \int \frac{|u|^{p+1}}{|x|}\, dx$$

$$\Delta^2 a = \begin{cases} -\frac{(d-1)(d-3)}{|x|^3} \leq 0, & d>3 \\ -8\pi\delta_0, & d=3. \end{cases}$$

$\frac{1}{4\pi|x|}$ is the fundamental soln to $\Delta u = 0$ $(d=3)$

$\lambda = 1$

$$-\int |u|^2 \Delta^2 a \geq 0$$

$$|\not\nabla u|^2 = |\nabla u|^2 - \Big|\underbrace{\frac{x}{|x|}\cdot \nabla u}_{\text{radial component}}\Big|^2$$

↑ angular component of the gradient

↖ See Lec 15-1b from my course note '16

On the other hand,

$$\partial_t^2 V_a = \partial_t \, \mathrm{Im} \int \nabla u \cdot \frac{x}{|x|}\, \bar{u}\, dx$$

$\lambda = 1$:

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \frac{|u|^{p+1}}{|x|}\, dx\, dt \lesssim \sup_{t_1 \leq t \leq t_2} \Big| \int_{\mathbb{R}^d} \nabla u \cdot \frac{x}{|x|}\, \bar{u}\, dx \Big|$$

$$\lesssim \sup_{t_1 \leq t \leq t_2} \|u(t)\|_{\dot{H}^{1/2}}^2 \quad \text{or} \quad M^{\frac{1}{2}}(u_0)\, H^{\frac{1}{2}}(u_0)$$

· Send $t_1 \to -\infty$ & $t_2 \to +\infty$.

Morawetz estimate:

$\lambda = 1$, defocusing

$$\int_{\mathbb{R}_t}\int_{\mathbb{R}^d_x} \frac{|u(t,x)|^{p+1}}{|x|}\,dx\,dt \lesssim \sup_t \|u(t)\|^2_{\dot{H}^{1/2}} \; M(u_0)^{\frac{1}{2}} H^1(u_0)^{\frac{1}{2}}$$

- useful in the radial case

· By repeating this derivation centered at y ($d=3$),

$$\partial_t \int \nabla u(x)\cdot \frac{x-y}{|x-y|}\,\bar{u}(x)\,dx = 2\int \frac{|\nabla_y u(x)|^2}{|x-y|}\,dx + 2\lambda\frac{p-1}{p+1}\int \frac{|u(x)|^{p+1}}{|x-y|}\,dx + 4\pi|u(y)|^2$$

≥ 0, drop

· Multiply by $|u(y)|^2$ and $\int \cdot\, dy$.

Interaction Morawetz estimate: CKSTT, CPAM '04.

$$\int_{\mathbb{R}_t}\int_{\mathbb{R}^3} |u(y)|^4\,dy\,dt \lesssim \sup_t \|u(t)\|^2_{L^2}\,\|u(t)\|^2_{\dot{H}^{1/2}} \quad \text{or} \quad M(u_0)^{\frac{3}{2}} H(u_0)^{\frac{1}{2}}.$$

· Scattering for the defocusing cubic NLS on $\mathbb{R}^3$

$i\partial_t u + \Delta u = |u|^2 u$ $\quad$ scrit $= 1/2$ (energy-subcritical)

LWP in $H^1(\mathbb{R}^3)$ (see HW soln).

$\Rightarrow$ GWP in $H^1(\mathbb{R}^3)$ $\quad$ GWP in $H^{1/2}$ is OPEN.

WTS: $\exists\, u_+ \in H^1(\mathbb{R}^3)$ s.t. $\|u(t) - S(t)u_+\|_{H^1} \to 0$ as $t \to +\infty$.

$$\| u(t) - S(t)u_+ \|_{H^1} = \| S(t)(S(-t)u(t) - u_+) \|_{H^1}$$

$S(-t)u(t) = u_0 - i\int_0^t S(-t')|u|^2 u(t')\,dt'$

? ↓ u_+

$\Rightarrow$ Suffices to make sense (in H^1) of

$$\int_0^\infty S(-t')|u|^2 u(t')\,dt'.$$

· Existence of wave operator:

$$\Omega_+ : u_+ \in H^1(\mathbb{R}^3) \longmapsto u_0 \in H^1(\mathbb{R}^3)$$

Given $u_+ \in H^1(\mathbb{R}^3)$, can we find $u_0 \in H^1(\mathbb{R}^3)$ s.t. the corresp. soln u (with $u|_{t=0} = u_0$) scatters to $S(t)u_+$ as $t \to +\infty$?

Rmk: If Ω_+ exists, it is injective

($\Leftarrow$ follows from the uniqueness claim of the WP theory.

· If Ω_+ is invertible, we say that asymptotic completeness holds (= scattering).

· Pf of existence of wave op: $u_+ = u_0 - i\int_0^\infty S(-t')|u|^2u(t')\,dt'$

$$\Longrightarrow S(-t)u(t) = u_0 - i\int_0^t S(-t')|u|^2u(t')\,dt'.$$

Terminal value problem

$$u(t) = S(t)u_+ + i\int_t^\infty S(t-t')|u|^2u(t')\,dt' =: \tilde{\Gamma}(u)(t)$$

(u_+: value at $t=+\infty$)

$t = +\infty \xrightarrow{①} t = T \xrightarrow{②} t = 0$

① well-posed on $[T, \infty)$ small data theory

$\widetilde{S}^1 = L^5_{t,x} \cap L^{10/3}_t W^{1,10/3}_x$ $\qquad \frac{2}{q} + \frac{d}{r} = \frac{d}{2}$, $d=3$

- $(\frac{10}{3}, \frac{10}{3})$, admis
- $(5, \frac{30}{11})$, admis

$\|u\|_{L^5_{t,x}} \lesssim \|u\|_{L^5_t W^{1,30/11}_x}$

↑ Sob (not sharp) $\quad \left(\frac{11}{30} - \frac{1}{5} = \frac{1}{6} = \frac{1/2}{3}\right.$

By the Strichartz estimate,

$$\|S(t)u_+\|_{\widetilde{S}^1(\mathbb{R}_+)} \lesssim \|u_+\|_{H^1} < \infty$$

$\Rightarrow$ By DCT (or the monotone conv. thm), $\forall \varepsilon > 0, \exists\, T \gg 1$ s.t.

(*) $\qquad \|S(t)u_+\|_{\widetilde{S}^1([T,\infty))} \le \varepsilon$

$$\Rightarrow \underline{\|\widehat{\Gamma}(u)\|_{\widetilde{S}^1([T,\infty))}} \overset{(*)}{\underset{\text{Str}}{\leq}} \varepsilon + C\,\|\langle\nabla\rangle |u|^2 u\|_{L_{t,x}^{10/7}([T,\infty))}$$

frac. Leibniz

$$\leq \|u\|^2_{L_{t,x}^{5}} \|\langle\nabla\rangle u\|_{L_{t,x}^{10/3}([T,\infty))} \qquad \tfrac{7}{10} = \tfrac{1}{5}+\tfrac{1}{5}+\tfrac{3}{10}$$

$$\leq \|u\|^3_{\widetilde{S}^1([T,\infty))}$$

$$\underline{\leq 2\varepsilon} \quad \text{for } u \in \overline{B_{2\varepsilon}} \subset \widetilde{S}^1([T,\infty))$$

Similarly, $\|\widehat{\Gamma}(u)-\widehat{\Gamma}(v)\|_{\widetilde{S}^1([T,\infty))} \leq$

$$C\left(\|u\|^2_{\widetilde{S}^1([T,\infty))} + \|v\|^2_{\widetilde{S}^1([T,\infty))}\right) \times \|u-v\|_{\widetilde{S}^1([T,\infty))}$$

$\leq 1/2$ for $\varepsilon \ll 1$

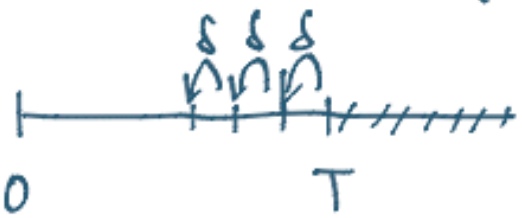
$$\leq \tfrac{1}{2}\|u-v\|_{\widetilde{S}^1([T,\infty))}$$

$$\forall\, u, v \in \overline{B_{2\varepsilon}}$$

$\Rightarrow$ $\tilde{F}$ is a contraction on $\overline{B_{2\varepsilon}} \subset \tilde{S}'([T,\infty))$, $\varepsilon \ll 1$

$\Rightarrow$ u on $[T,\infty)$ (also show $u \in C([T,\infty); H^1)$

(2)



Iteratively Apply LWP in $H^1(\mathbb{R}^3)$ and the conservation of $M(u)$ & $H(u)$ to u on $[0,T]$

□

· Summary: (1) $[T,\infty)$: small data theory

(2) $[0,T]$: iteration of (large data) LWP.

· Scattering (= asymptotic completeness)

Goal: Make sense of $\int_0^\infty S(-t)|u|^2 u(t)\,dt$.

$$\left\| \int_0^\infty S(-t)|u|^2 u(t)\,dt \right\|_{H^1} \lesssim_{\text{dual Str.}} \|\langle\nabla\rangle(|u|^2 u)\|_{L_{t,x}^{10/7}}$$

$$\lesssim \|u\|_{L_{t,x}^{5}}^2 \|\langle\nabla\rangle u\|_{L_{t,x}^{10/3}}$$

$$\lesssim \|u\|_{S^1}^3$$

where $\|u\|_{S^1} \approx \max\left(\|u\|_{L_t^\infty H_x^1},\ \|\langle\nabla\rangle u\|_{L_t^2 L_x^b} \right)$ $\quad 2 \le r \le \frac{2d}{d-2} = b.$

(By interpolation, $\|\langle\nabla\rangle u\|_{L_t^q L_x^r} \lesssim \|u\|_{S^1}$ $\quad (q,r)$, admis.

$\Rightarrow$ (**) $\|u\|_{S^1(\mathbb{R}^+)} < \infty$ implies scattering

Strong space-time bound.

If (**) holds, then

$$\| \int_T^\infty S(-t) |u|^2 u(t)\, dt \|_{H^1} \lesssim \| u \|_{L_t^5([T,\infty); L_x^5)} \| \langle \nabla \rangle u \|_{L_t^{10/3}([T,\infty); L_x^{10/3})}$$

$$\longrightarrow 0 \quad \text{by DCT.}$$

Claim: "weak" space-time bound:

$$\| u \|_{L_{t,x}^q} \lesssim 1 \quad \text{for some } q \in [\tfrac{10}{3}, 10]$$

implies the strong space-time bound (**) (and hence scattering)

Rmk: · interaction Morawetz $\Rightarrow$ weak space-time bd $q = 4$

· Morawetz for u, radial $\Rightarrow$ $q = 5$.

$$\iint |u|^5 dx\, dt = \iint |x||u| \cdot \frac{|u|^4}{|x|} dx \le \| |x| u \|_{L_{t,x}^\infty} \iint \frac{|u|^4}{|x|} dx\, dt$$

$$\le C(\| u_0 \|_{H^1})$$

radial Sobolev ineq.

Morawetz ($3+1$)

· **Radial Sobolev ineq:** f, radial

$$\| |x|^s f \|_{L^\infty(\mathbb{R}^d)} \lesssim \| f \|_{H^1(\mathbb{R}^d)}, \quad \frac{d}{2} - 1 \le s \le \frac{d-1}{2}$$

$\frac{d-1}{2} = 1$ when $d=3$.

Pf: Assume f is real-valued.

$d=3$

$$-\partial_r (r^2 f^2(r)) = -2r f^2 - 2 r^2 f \partial_r f \le -r^2 f \partial_r f \quad (-2rf^2 \le 0)$$

Integrate $[r, \infty)$ ← integration in the polar coord.

$$r^2 f^2(r) \lesssim -\int_r^\infty r^2 f \partial_r f \, dr \overset{C\text{-}S}{\lesssim} \| f \|_{L^2(\mathbb{R}^3)} \| \nabla f \|_{L^2(\mathbb{R}^3)}$$

$$\le \| f \|^2_{H^1(\mathbb{R}^3)}$$

r^2 ($d=3$)

· **other endpt:**

$$-\partial_r \left(r^{2(d-2)} f^4(r) \right) \lesssim -r^{2(d-2)} f^3 \partial_r f$$

$$\Rightarrow r^{2(d-2)} f^4(r) \lesssim \| f \|^3_{L^6(\mathbb{R}^3)} \| f \|_{\dot{H}^1(\mathbb{R}^3)}$$

$$\lesssim \| f \|^4_{\dot{H}^1(\mathbb{R}^3)}$$

□

· Pf of Claim: Given $\varepsilon > 0$, divide $\mathbb{R}_+ = \bigcup_{j=1}^{J} I_j$ $\quad I_j = [t_j, t_{j+1})$ s.t.

(A) $\|u\|_{L_{I_j}^q L_x^q} \le \varepsilon$ $\quad \left(\Leftarrow \|u\|_{L_t^q(\mathbb{R}_+; L_x^q)} < \infty \right.$

· $u(t) = S(t-t_j)u(t_j) - i\int_{t_j}^{t} S(t-t')|u|^2 u(t')\,dt', \quad t \in I_j$

(B) $\|u\|_{S^1(I_j)} \overset{\text{Str.}}{\lesssim} \|u(t_j)\|_{H^1} + \|u\|^2_{L_{I_j}^5 L_x^5} \|u\|_{L_{I_j}^{10/3} W_x^{1,10/3}}$

WTS: this is small. $\quad \le \|u\|_{S^1(I_j)}$

Note: $\|u\|_{L_{t,x}^{10/3}(I)} \le \|\langle\nabla\rangle u\|_{L_{t,x}^{10/3}(I)} \le \|u\|_{S^1(I)}$

$\|u\|_{L_{t,x}^{10}} \overset{\text{Sob}}{\lesssim} \|u\|_{L_I^{10} W_x^{1,30/13}} \le \|u\|_{S^1(I)}$ $\quad (10, \frac{30}{13})$, admis

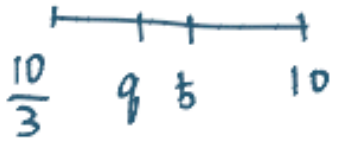
$\left(\frac{2}{10} + 3\cdot\frac{13}{30} = \frac{3}{2} \right.$

$\uparrow$

$\frac{13}{30} - \frac{1}{10} = 1/3 = s/d$

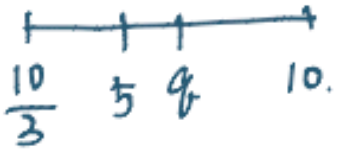
By interpolation,

$$\|u\|_{L^5_{t,x}(I_j)} \le \|u\|^{\theta}_{L^q_{t,x}(I_j)} \|u\|^{1-\theta}_{L^{10}_{t,x}(I_j)}$$



or

$$\|u\|^{\theta}_{L^q_{t,x}(I_j)} \|u\|^{1-\theta}_{L^{10/3}_{t,x}(I_j)}$$



(A)

$$\Rightarrow \quad \|u\|_{L^5_{t,x}(I_j)} \le C\,\varepsilon^{\theta}\,\|u\|^{1-\theta}_{S^1(I_j)} \qquad (C)$$

$\Rightarrow$ (B) & (C) implies

$$\|u\|_{S^1(I_j)} \le C\left(M(u_0)+H(u_0)\right)^{1/2} + C\,\varepsilon^{2\theta}\,\|u\|^{3-2\theta}_{S^1(I_j)} \qquad (D)$$

continuity arg.

$\Rightarrow$ $\|u\|_{S^1(I_j)} \lesssim 1$ $\Rightarrow$ sum over $j = 1, \cdots, J$

see next page

$\Rightarrow$ $\|u\|_{S^1(\mathbb{R}_+)} \lesssim 1$

$\Rightarrow$ scattering !!

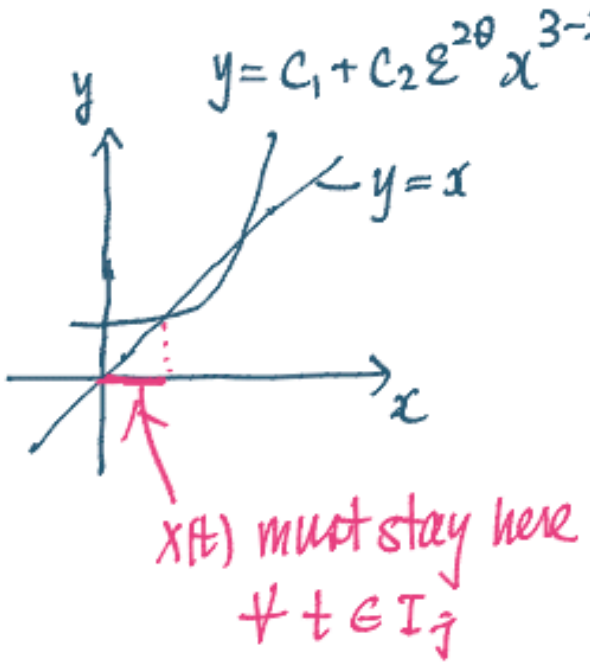
· **Continuity argument**: Let $X(t) = \|u\|_{S^1([t_j,t])}$, $t \in I_j = [t_j, t_{j+1})$

Ⓓ on $[t_j, t]$: $\boxed{X(t) \le C_1 + C_2 \varepsilon^{2\theta} X(t)^{3-2\theta}}$. indep of j

Ⓔ

· $X(t)$ is continuous in t

(but $X(t) \not\to 0$ as $t \searrow t_j$).



At $t = t_j$, $X(t_j) \le C_1$

$\Rightarrow X(t) \lesssim 1$, $\forall\, t \in I_j$

by choosing $\varepsilon > 0$ small.

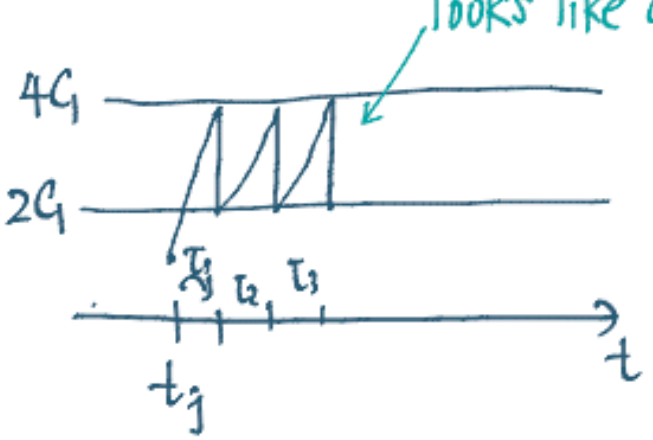
· **Bootstrap argument**: Ⓔ implies if $X(t) \le 4C_1$, then $X(t) \le 2C_1$ (for ε small).

· $X(t_j) \le C_1 \overset{\text{conti}}{\Rightarrow} \exists\, \tau_1$ s.t. $X(t_j + \tau_1) \le 4C_1 \overset{Ⓔ}{\Rightarrow} X(t_j + \tau_1) \le 2C_1$

$\overset{\text{conti}}{\Rightarrow} X(t_j + \tau_1 + \tau_2) \le 4C_1 \overset{Ⓔ}{\Rightarrow} X(t_j + \tau_1 + \tau_2) \le 2C_1 \Rightarrow \cdots$

We can iterate this process as long as (E) holds.

$\Rightarrow \quad X(t) \le 2C_1, \ \forall t \in I_j$



· In the energy-critical case, the local existence time (for LWP in H^1) depends on the profile of initial data and thus we can NOT conclude GWP in H^1 simply from the conservation of $M(u)$ and $H(u)$.

· Global well-posedness & scattering for the energy-critical def NLS

radial · Bourgain '99 (induction on energy), also Grillakis '99 (GWP)] $d=3$, $p=5$

general case · CKSTT '08

· $d \ge 4$: Ryckman-Visan '07, Visan '07.

- Kenig-Merle '06 (concentration compactness & rigidity method)
  - ↑ focusing case
  - proof by contradiction

- Mass-critical case
  - radial, Killip-Tao-Visan '09.
  - general case, Dodson, '16, etc.

- See the Oberwolfach lecture notes by Visan
  (and also the Clay lecture notes by Killip-Visan)

- On $\mathbb{T}^d$, there is NO scattering (see Appendix of CKSTT '10)