Lec 8 : Feb. 25, 2025 (TUE)

Sec 1.8 : Probabilistic well-posedness of 3-d NLW

$$(NLW)\quad \begin{cases} -\partial_t^2 u + \Delta u = u^3 \\ (u, \partial_t u)\big|_{t=0} = (u_0, u_1) \in \dot{\mathcal{H}}^s = \dot{H}^s \times \dot{H}^{s-1} \end{cases}$$

On $\mathbb{T}^3$ or $\mathbb{R}^3$: scrit $= 1/2$

In Lec 5, we proved LWP of NLW in

- $\dot{\mathcal{H}}^1$: Sobolev $\dot{H}^1 \subset L^6$
- $\dot{\mathcal{H}}^{1/2}$: Strichartz estimates

on $\mathbb{R}^3$ but the same proof works on $\mathbb{T}^3$ thanks to the finite speed of propagation.

Q: Well-posedness in $\dot{\mathcal{H}}^s$ for $s < 1/2$?

Q: Well-posedness in $\dot{H}^s$ for $s < 1/2$?

- 3-d cubic NLW is ill-posed in $\dot{H}^s$ for $s < 1/2$.
  - Christ-Colliander-Tao, arXiv '03
  - Lebeau, BSMF '05.
  - Tzvetkov's lecture note '19. "Random data wave equation".
- norm inflation: $\forall \varepsilon > 0$, $\exists$ (smooth) soln $(u, \partial_t u)$ to NLW s.t. $s < 1/2$

$$\| (u(0), \partial_t u(0)) \|_{\dot{H}^s} < \varepsilon \quad \text{BUT} \quad \| (u, \partial_t u) \|_{L^\infty([0,\varepsilon]; \dot{H}^s)} > \frac{1}{\varepsilon}$$

$\Rightarrow$ soln map is discontinuous at $(0,0) \in \dot{H}^s$

(in fact, everywhere in $\dot{H}^s$)

Idea: Construct solns with random initial data
(or randomized initial data).

$\Rightarrow$ probabilistic well-posedness.

· Given random initial data, construct a unique soln
almost surely (but NO continuous dependence).

- probabilistic continuous dependence, using conditional probability.
  · Burq-Tzvetkov, JEMS '14
  Pocovnicu, JEMS '17.

· This study was initiated by Bourgain '94, '96. (CMP)
in the context of invariant Gibbs measures for NLS.

· Burq-Tzvetkov, Inventiones '08. part 1.
· Colliander-Oh, Duke '12, ...

- Over the last 15 years, this study lying at the intersection of dispersive PDEs & probability theory has been very active.
  - Gubinelli-Koch-Oh, TAMS '18, JEMS '24
  - Bringmann, JEMS '24
  - Deng-Nahmod-Yue, Inventiones '22, Annals '24
  - BDNY, Inventiones '24.

Oh-Okamoto-Tolomeo MAMS '24, MEMS '25

- Randomization of initial data $(u_0, u_1) \in \dot{\mathcal{H}}^s \setminus \dot{\mathcal{H}}^{1/2}$

f on $\mathbb{T}^d$ $\qquad f(x) = \sum_{n \in \mathbb{Z}^d} \widehat{f}(n)\, e^{in\cdot x}$

$\Longrightarrow$ Randomization of f: $\quad f^\omega(x) = \sum_{n \in \mathbb{Z}^d} g_n(\omega)\, \widehat{f}(n)\, e^{in\cdot x}$

$g_n(\omega)$: random variable

· Let $\{g_n\}_{n\in\mathbb{Z}^d}$ be a collection of independent mean-zero random variables with distributions $\mu_n^{(r)}, \mu_n^{(i)}$ for real and imaginary parts.

($\mu_n^{(r)}, \mu_n^{(i)}$: assume indep)

s.t.
$$\int e^{r\cdot x}\, d\mu_n^{(r)}(x) \le e^{cr^2}, \quad \forall n\in\mathbb{Z}^d,\ r\in\mathbb{R}$$
$$\int e^{r\cdot x}\, d\mu_n^{(i)}(x) \le e^{cr^2}$$

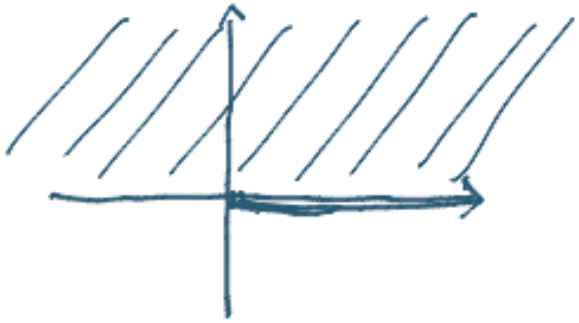
$\Leftarrow$ satisfied by standard $\mathbb{C}$-valued Gaussian r.v.'s

Bernoulli r.v.'s (or uniform distri. on $S^1\subset\mathbb{C}$).

Rmk: In many situations, it suffices to assume uniform (in n) boundedness of the kth moment $\mathbb{E}[|g_n|^k] \le C < \infty$ $\forall n\in\mathbb{Z}^d$.

Rmk: If f is real-valued, we have $\hat{f}(-n) = \overline{\hat{f}(n)}$, $n \in \mathbb{Z}^d$.

Let $I = \bigcup_{k=0}^{d-1} \mathbb{Z}^k \times \mathbb{Z}_+ \times \{0\}^{d-k-1}$ "$=$" $\mathbb{Z}^d/_2$



$\Rightarrow \mathbb{Z}^d = I \cup \{-I\} \cup \{0\}$.

Then, introduce $\{g_n\}_{n \in I \cup \{0\}}$

and set $g_{-n} = \overline{g_n}$, $\forall\, n \in \mathbb{Z}^d$. ($g_0$ is real-valued).

$\Rightarrow f^w(x) = \sum_{n \in \mathbb{Z}^d} g_n(w)\, \hat{f}(n)\, e^{in \cdot x}$ is <u>real-valued</u>.

Moral: If $f \in H^s \setminus H^{s+\varepsilon}$, then $f^\omega \in H^s \setminus H^{s+\varepsilon}$ a.s.

i.e. there is NO gain in differentiability.

BUT there is a gain in integrability

- If $f \in L^2$, then $f^\omega \in L^r$, $\forall\ 2 \le r < \infty$, a.s.

  (Paley-Zygmund 1930~. Khintchine's ineq.
  Kahane's book

Lem 1.8.1 (i) Given $f \in H^s(\mathbb{T}^d)$, $f^\omega = \sum g_n \hat{f}_m e^{in\cdot x}$.

Then, $P(\|f^\omega\|_{H^s} > \lambda) \le C \exp\left(-\lambda^2 / \|f\|_{H^s}^2\right) \to 0$ as $\lambda \to \infty$.

for any $\lambda > 0$. In particular, $f^\omega \in H^s$, a.s.

(ii) $f \in L^2(\mathbb{T}^d)$. Then, given any finite $r \ge 2$, we have

$$P(\|f^\omega\|_{L^r} > \lambda) \le C \exp\left(-\lambda^2 / \|f\|_{L^2}^2\right), \ \forall\ \lambda > 0.$$

$\Rightarrow$ $f^\omega \in L^r(\mathbb{T}^d)$, a.s.

Rmk: $f \in H^s \Rightarrow f^\omega \in W^{s,r}$, $\forall$ finite $r \geq 2$ a.s.

$f^\omega \in W^{s-\varepsilon,\infty}$

Lem 1.8.2: Let $\{g_n\}_{n\in\mathbb{Z}^d}$ be as above. Then, we have

$$\Big\| \sum_{n\in\mathbb{Z}^d} c_n g_n \Big\|_{L^p(\Omega)} \lesssim \sqrt{p}\, \| c_n \|_{\ell^2_n}, \quad \forall \text{ finite } p \geq 1.$$

Pf: We first prove

$$P\big(|\textstyle\sum c_n g_n| > \lambda \big) \leq C e^{-c\lambda^2/\|c_n\|^2_{\ell^2_n}}.$$

$$\int e^{t\sum c_n g_n} dP \overset{\text{indep}}{=} \prod_n \int e^{t c_n g_n} dP \leq \prod_n e^{c(tc_n)^2} = e^{ct^2 \sum c_n^2}$$

$$\Rightarrow P\big(\textstyle\sum c_n g_n > \lambda\big) \leq \dfrac{\mathbb{E}[e^{t\sum c_n g_n}]}{e^{t\lambda}} \leq e^{ct^2\sum c_n^2} / e^{t\lambda}$$

choose $t = \dfrac{\lambda}{2c\sum c_n^2}$ $\quad = e^{-\lambda^2/4c\sum c_n^2}$.

Similarly, $P\big(\textstyle\sum c_n g_n < -\lambda\big) \leq C e^{-c\lambda^2/\|c_n\|^2_{\ell^2_n}}$.

$$\Rightarrow \|\sum c_n g_n\|_{L^p(\Omega)}^p \le p \int_0^\infty \lambda^{p-1} e^{-c\lambda^2/\|c_n\|_{\ell^2_n}^2} d\lambda$$

$\zeta \sim \frac{\lambda}{\|c_n\|_{\ell^2_n}}$

Ch. of var

$$\lesssim p \|c_n\|_{\ell^2_n}^p \int_0^\infty \lambda^{p-1} e^{-\lambda^2/2} d\lambda$$

$$\lesssim (p-1)^{\frac{p-1}{2}}$$

$$p^{\frac{p}{2}} \quad p^{\frac{1}{2}} \quad \lesssim 1$$

$$\Rightarrow \|\sum c_n g_n\|_{L^p(\Omega)} \lesssim p^{1/2} \, p^{\frac{1}{2p}} \, \|c_n\|_{\ell^2_n}$$

□

Rmk: $P(|F^\omega| > \lambda) \le C e^{-c\lambda^{2/k}}$

$\Leftrightarrow \|F^\omega\|_{L^p(\Omega)} \lesssim p^{k/2} \quad \forall$ finite $p \ge 2$

See Lem 4.5 in Tsv. PTRF '10.

Proof of Lem 1.8.1:

(i) $2 \le p < \infty$

$$\| \|f^\omega\|_{H^s} \|_{L^p(\Omega)} \overset{\text{Mink}}{\le} \| \| \sum_n \underbrace{\langle n \rangle^s \hat{f}(n) e^{inx}}_{c_n(x)} g_n \|_{L^p(\Omega)} \|_{L^2_x}$$

$$\overset{\text{Lem 1.8.2}}{\lesssim} \sqrt{p} \, \| \underbrace{\| \langle n \rangle^s \hat{f}(n) \|_{\ell^2_n}}_{= \|f\|_{H^s}} \|_{L^2_x(\mathbb{T}^d)} \sim \sqrt{p} \, \|f\|_{H^s}$$

$\uparrow \; |\mathbb{T}^d| = 1.$

By Chebyshev,

$$P(\|f^\omega\|_{H^s} > \lambda) < \left(\frac{C_0 \, p^{1/2} \|f\|_{H^s}}{\lambda} \right)^p, \quad \forall \; p \ge 1.$$

Let $p = \left(\frac{\lambda}{C_0 e \|f\|_{H^s}} \right)^2$.

- If $p \ge 2$, then $P(\|f^\omega\|_{H^s} > \lambda) \le e^{-p} = e^{-c \lambda^2 / \|f\|^2_{H^s}}$.
- If $p < 2$, then choose c s.t. $c e^{-2} \ge 1$. Then,

$$P(\|f^\omega\|_{H^s} > \lambda) \le 1 \le c e^{-2} < e^{-c \lambda^2 / \|f\|^2_{H^s}}.$$

(ii) $\| \| f^\omega \|_{L^r_x} \|_{L^p(\Omega)} \overset{\text{Mink}}{\le} \| \| \sum_n \underbrace{\hat f(n) e^{in\cdot x}}_{c_n(x)} g_n \|_{L^p(\Omega)} \|_{L^r_x}$

$r \le p < \infty$

$\overset{\text{Lem 1.8.2}}{\lesssim} \sqrt{p}\, \| \hat f(n) \|_{\ell^2_n} = \sqrt{p}\, \|f\|_{L^2_x}.$

$\Rightarrow$ repeat the proof of (i).

□

- Q: gain of differentiability? NO!!

Lem 1.8.3: Suppose $f \in H^s \setminus H^{s+\varepsilon}$ for some $\varepsilon > 0$.

Suppose $\exists\, c > 0$ s.t.

$$\limsup_{|n| \to \infty} P(|g_n| \le c) \le 1 - \delta < 1$$

↙ satisfied if g_n is i.i.d. and $g_n \not\equiv 0$.

Then, $P(f^\omega \in H^{s+\varepsilon}) = 0.$

Pf: $\int e^{-\|f^\omega\|_{H^{s+\varepsilon}}^2} dP \overset{\text{indep}}{=} \prod_n \int e^{-\langle n\rangle^{2(s+\varepsilon)} |\widehat{f}(n)|^2 |g_n|^2} d\mu_n$

$= \int_{-c}^{c} + \int_{|g_n|\geq c}$

Here, we assume g_n is real for simplicity.

$\leq \prod_n \Big\{ \mu_n(-c,c)$

$+ \underbrace{e^{-c^2\langle n\rangle^{2(s+\varepsilon)} |\widehat{f}(n)|^2}}_{=: \alpha_n} (1-\mu_n(-c,c)) \Big\}$

$= \prod_n \{ \mu_n(-c,c)(1-\alpha_n) + \alpha_n \}$

$\leq \prod_{|n|\gg 1} \{ (1-\delta)(1-\alpha_n) + \alpha_n \} = \prod_{|n|\gg 1} \{ 1 - \delta(1-\alpha_n) \} = 0.$ (*)

$\Rightarrow \|f^\omega\|_{H^{s+\varepsilon}} = \infty$, a.s.

Pf of ⊛: By assumption, we have $\sum \langle n\rangle^{2(s+\varepsilon)} |\hat{f}(n)|^2 = \infty$.

Then, we claim $\sum (1-\alpha_n) = \infty$.

$$\sum \left(1 - e^{-c^2 \langle n\rangle^{2(s+\varepsilon)} |\hat{f}(n)|^2}\right)$$

- If $\langle n\rangle^{2(s+\varepsilon)} |\hat{f}(n)|^2 \not\to 0$ as $|n| \to \infty$, then
$$\sum (1-\alpha_n) \geq \sum_{\infty\text{-many } n} (1-\varepsilon) = \infty.$$
- O.W. $\exists N$ s.t. $c^2 \langle n\rangle^{2(s+\varepsilon)} |\hat{f}(n)|^2 \leq 1$, $\forall |n| \geq N$
$$\Rightarrow \sum_{|n|\geq N} (1-\alpha_n) \geq C \sum_{|n|\geq N} \langle n\rangle^{2(s+\varepsilon)} |\hat{f}(n)|^2 = \infty.$$

Then, $\prod_n (1-\delta(1-\alpha_n))^{-1} = \prod_n \left(1 + \frac{\delta(1-\alpha_n)}{1-\delta(1-\alpha_n)}\right)$

$$\geq \prod_n (1+\delta(1-\alpha_n)) = \infty.$$

$e^{-x} \leq 1 - Cx$ for $0 < x \leq 1$ ($C \ll 1$)

($\prod_n (1+a_n) < \infty \iff \sum a_n < \infty$. $\square$)

Probabilistic Strichartz estimate

$S(t) = e^{it\Delta}$ = linear Schrödinger propagator.

Strichartz estimates on $\mathbb{R}^d$.

$$\| S(t) f \|_{L_t^q L_x^r (\mathbb{R} \times \mathbb{R}^d)} \lesssim \| f \|_{L_x^2}$$

$2 \le q, r \le \infty, \quad (q, r, d) \neq (2, \infty, 2)$

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$$

Prop 1.8.4: $f \in L^2(\mathbb{T}^d)$. Given $2 \le q, r < \infty$, we have

$$P\left(\| S(t) f^\omega \|_{L_t^q L_x^r ([-T,T] \times \mathbb{T}^d)} > \lambda \right) \le C \exp\left(- c \frac{\lambda^2}{T^{3/q} \|f\|_{L^2}^2} \right).$$

for any $\lambda > 0$ and $T > 0$. $(\to 0 \text{ as } \lambda \to \infty)$.

Rmk: · Set $\lambda = T^\theta \|f\|_{L^2_x}$, $\theta < 1/q$

$\Rightarrow \|S(t)f^\omega\|_{L^q_T L^r_x} \le T^\theta \|f\|_{L^2_x}$

outside a set of prob $\le C e^{-cT^{2(\theta-\frac{1}{q})}}$ $(\to 0 \text{ as } T \to 0)$

· Fix $T > 0$. Given $\varepsilon > 0$,

$$\|S(t)f^\omega\|_{L^q_T L^r_x} \le C_T \left(\log \tfrac{1}{\varepsilon}\right)^{1/2} \|f\|_{L^2_x}$$

outside a set of prob $< \varepsilon$.

(or $\le C_T K \|f\|_{L^2_x}$, prob $< e^{-cK^2}$)

Pf: Let $p \ge \max(q, r)$.

$$\| \|S(t)f^\omega\|_{L^q_T L^r_x} \|_{L^p(\Omega)} \overset{\text{Mink}}{\le} \| \underbrace{\|S(t)f^\omega\|_{L^p(\Omega)}}_{\overset{\text{Lem 1.8.1}}{\lesssim} \sqrt{p}\,\|f\|_{L^2_x}} \|_{L^q_T L^r_x}$$

$$\lesssim \sqrt{p}\, T^{1/q} \|f\|_{L^2_x}$$

$\Rightarrow$ Apply Chebyshev. $\square$

· The same works for the half wave operator $e^{\pm it|\nabla|}$

$\Rightarrow$ also for $\cos(t|\nabla|) = \dfrac{e^{it|\nabla|} + e^{-it|\nabla|}}{2}$.

Prop 1.8.5: (probabilistic Strichartz estimate for the wave equation).

$$W(t)(f_0, f_1) = \cos(t|\nabla|) f_0 + \frac{\sin(t|\nabla|)}{|\nabla|} f_1.$$

Let $(f_0, f_1) \in \mathcal{H}^0 = L^2 \times H^{-1}$.

Then, $P\left(\| W(t)(f_0^{\omega}, f_1^{\omega}) \|_{L_t^q L_x^r (I \times \mathbb{T}^d)} > \lambda \right)$ $\quad I = [a, b]$

$$\leq C \exp\left(- c \frac{\lambda^2}{\max(1, a^2, b^2) |I|^{2/q} \| (f_0, f_1) \|_{\mathcal{H}^0}^2} \right)$$

Pf: Repeat the proof of Prop 1.8.4, noting.

$$\left| \frac{\sin t|n|}{|n|} \right|_{n=0} = |t| \leq \max(1, |a|, |b|)$$

Q: $\max(q, r) = \infty$?

- $r = \infty$: Use Sobolev. $\|S(t) f^\omega\|_{L_T^q L_x^\infty} \lesssim \|S(t)\langle\nabla\rangle^s f\|_{L_T^q L_x^r}$

Prop 1.8.4. $\Rightarrow$

$$P\left(\|S(t) f^\omega\|_{L_T^q L_x^\infty} > \lambda\right) \le C \exp\left(-c \frac{\lambda^2}{T^{2/q} \|f\|_{H^s}^2}\right) \quad \forall r > d$$

$s > 0$

- $q = \infty$?

Prop 1.8.6: $2 \le r \le \infty$

$$P\left(\|S(t) f^\omega\|_{L_T^\infty L_x^r} > \lambda\right) \le C (1+T) \exp\left(-c \frac{\lambda^2}{\|f\|_{H^s}^2}\right)$$

$s > 0$

Pf: Assume $r < \infty$. and $T = 1$.

$$Z^\omega(t) = e^{it\Delta} f^\omega$$

- Given $k \in \mathbb{Z}_{\geq 0}$, $\{ t_{l,k} : l = 0, 1, \cdots, 2^k \}$ $\quad t_{l,k} = \frac{l}{2^k}$.

Fix $t \in [0,1]$

$$t = \sum_{j=1}^{\infty} \frac{a_j}{2^j} \qquad a_j \in \{0,1\}$$

Write

$$Z^\omega(t) = \sum_{k=1}^{\infty} \left(Z^\omega(t_{l_k,k}) - Z^\omega(t_{l_{k-1},k-1}) \right) + Z^\omega(0).$$

$$l_k = l_k(t) \in \{0, 1, \cdots, 2^k\}. \qquad t_{l_k,k} = \sum_{j=1}^{k} \frac{a_j}{2^j}$$

- $2 \leq r < \infty$.

$$\| F(t) \|_{L_x^r} \overset{LP}{\sim} \Big\| \Big(\sum_{\substack{N \geq 1 \\ \text{dyadic}}} |P_N F(t)|^2 \Big)^{1/2} \Big\|_{L_x^r}.$$

$$\overset{\text{Mink}}{\leq} \Big(\sum_{N \geq 1} \| P_N F(t) \|_{L_x^r}^2 \Big)^{1/2}.$$

$$\| z^\omega \|_{L_t^\infty([0,1];L_x^r)}$$

$$\lesssim \left[\sum_{N\geq 1} \left(\sum_{k=1}^{\infty} \max_{0\leq l_k\leq 2^k} \| P_N(z^\omega(t_{l_k,k}) - z^\omega(t_{l'_{k-1},k-1}) \|_{L_x^r} \right)^2 \right]^{1/2} + \| z^\omega(0) \|_{L_x^r}.$$

where l'_{k-1} is s.t. $|t_{l_k,k} - t_{l'_{k-1},k-1}| \leq 2^{-k}$

- $p \geq r$:

$$\left(\mathbb{E}\left[\| z^\omega \|_{L_1^\infty L_x^r}^p \right] \right)^{1/p}$$

$$\lesssim \left(\sum_{N\geq 1} \left\{ \sum_{k=1}^{\infty} \underbrace{\left(\mathbb{E}\left[\max_{0\leq l_k\leq 2^k} \| P_N(z^\omega(t_{l_k,k}) - z^\omega(t_{l'_{k-1},k-1}) \|_{L_x^r}^p \right] \right)^{1/p}}_{=: I_N} \right\}^2 \right)^{1/2}$$

(**)

$$+ \underbrace{\| z^\omega(0) \|_{L_\omega^p L_x^r}}_{\lesssim \sqrt{p}\, \| f \|_{L_x^2}}.$$

Set $q_k = \max(\log 2^k, p, r) \sim \log 2^k + p.$

$$I_N \le \sum_{k=1}^{\infty} \Big(\sum_{l_k=0}^{2^k} \mathbb{E}\big[\| P_N(z^\omega(t_{l_k,k}) - z^\omega(t_{l'_{k-1},k-1}) \|_{L^r_x}^{q_k} \big] \Big)^{1/q_k}$$

$$\lesssim \max_{0 \le l_k \le 2^k} \quad \Big(\Leftarrow (2^k+1)^{1/q_k} \lesssim 1.$$

$$\lesssim \sum_{k=1}^{\infty} \sqrt{q_k} \max_{0 \le l_k \le 2^k} \Big\| \underbrace{\big(e^{-i t_{l_k,k} |n|^2} - e^{-i t_{l'_{k-1},k-1} |n|^2} \big)}_{\lesssim \min(1, 2^{-k} N^2)} \hat{f}(n) \Big\|_{\ell^2_{|n| \sim N}}$$

$$\lesssim \sum_{k=1}^{\infty} \sqrt{q_k} \, \min(1, 2^{-k} N^2) \, \| P_N f \|_{L^2_x}$$

① $2^{-k} N^2 > 1$ ② $2^{-k} N^2 \le 1$

① $2^{-k}N^2 > 1$. Note $\sqrt{q_k} \lesssim \sqrt{\log N}\sqrt{p}$.

$$I_N \lesssim (\log N)^{3/2}\sqrt{p}\,\|P_N f\|_{L^2_x} \lesssim \sqrt{p}\,\|P_N f\|_{H^s}, \quad s>0.$$

② $2^{-k}N^2 \lesssim 1$.

$$I_N \lesssim \sqrt{p}\sum_{k\gtrsim \log N}\sqrt{\log 2^k}\;2^{-k}N\,\|P_N f\|_{L^2_x}$$

$$\lesssim \sqrt{p}\,(\log N)^{1/2}\|P_N f\|_{L^2_x} \lesssim \sqrt{p}\,\|P_N f\|_{H^s}, \quad s>0.$$

(**) p.20

$$\Rightarrow \left(\mathbb{E}\left[\|z^\omega\|^p_{L^\infty_I L^r_x}\right]\right)^{1/p} \lesssim \sqrt{p}\,\|f\|_{H^s}, \quad s>0.$$

$\Rightarrow$ Apply Chebyshev.

□.

Rmk: • The main idea is to replace $\sup_{0\le t\le 1}$ by $\max_{0\le l_k\le 2^k}$.

· A similar argument can be used to prove Kolmogorov's continuity criterion. See Appendix in Oh-Okamoto-Tzvetkov (See also Prop 2.4 in OOTz AIHP P&S, '24

· A straightforward modification yields the probabilistic Strichartz estimates for the wave equation when $\max(q,r)=\infty$ (by losing ε-regularity)