Lec 9 : Mar. 04, 2025 (TUE)

Prop 1.8.5: $W(t)(f_0, f_1) = \cos(t|\nabla|) f_0 + \frac{\sin(t|\nabla|)}{|\nabla|} f_1$

Let $(f_0, f_1) \in \mathcal{H}^0 = L^2 \times H^{-1}$.

Then, given any finite $q, r \geq 2$, we have

$$P\left(\| W(t)(f_0^\omega, f_1^\omega) \|_{L_t^q L_x^r (I \times \mathbb{T}^d)} > \lambda \right) \qquad I = [a, b]$$

$$\leq C \exp\left(- c \frac{\lambda^2}{\max(1, a^2, b^2) |I|^{2/q} \| (f_0, f_1) \|_{\mathcal{H}^0}} \right)$$

for any $\lambda > 0$.

Prop 1.8.7: $s > 0$, $2 \leq r \leq \infty$.

$$P\left(\| W(t)(f_0^\omega, f_1^\omega) \|_{L_T^\infty L_x^r} > \lambda \right)$$

$$\leq C(1+T) \exp\left(- c \frac{\lambda^2}{\max(1, T^2) \| (f_0, f_1) \|_{\mathcal{H}^s}^2} \right)$$

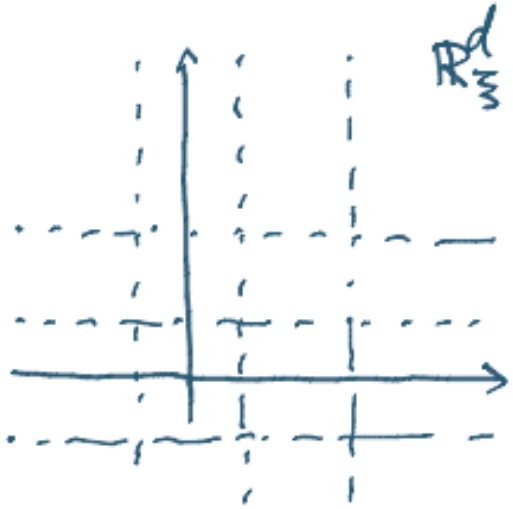
(See Prop 3.3 in Oh-Pocovnicu, JMPA '16.

- On $\mathbb{T}^d$, $f(x) = \sum_{n \in \mathbb{Z}^d} \hat{f}(n)\, e^{in\cdot x}$

$$\Rightarrow \quad f^{\omega}(x) = \sum_{n \in \mathbb{Z}^d} g_n\, \hat{f}(n)\, e^{in\cdot x}$$

- <u>On $\mathbb{R}^d$?</u> $f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi)\, e^{i\xi\cdot x}\, d\xi$

$$= \sum_{n \in \mathbb{Z}^d} \int_{Q_n} \hat{f}(\xi)\, e^{i\xi\cdot x}\, d\xi.$$



$Q = (-\frac{1}{2}, \frac{1}{2}]^d$

$Q_n = n + Q$, $n \in \mathbb{Z}^d$

$\mathbb{R}^d_\xi = \bigcup_{n \in \mathbb{Z}^d} Q_n$

Wiener decomposition.

For practical purpose,

let $\psi \in C_c^\infty(\mathbb{R}^d; [0,1])$ s.t.

- $\operatorname{supp}\psi \in [-1,1]^d$.

and $\sum_{n \in \mathbb{Z}^d} \underbrace{\psi(\xi - n)}_{\text{smoothed version of } 1_{Q_n}} \equiv 1, \quad \forall\, \xi \in \mathbb{Z}^d$

$$\Rightarrow \quad f = \sum_{n\in\mathbb{Z}^d} \underbrace{\Psi(D-n)f}_{= \int \Psi(\xi-n)\hat{f}(\xi)\, e^{ix\cdot\xi}\, d\xi.}$$

Wiener randomization of a function on $\mathbb{R}^d$

$$f^\omega = \sum_{n\in\mathbb{Z}^d} g_n \Psi(D-n)f$$

Rmk: Why unit scale? ⇐ a matter of choice.

- can introduce a similar randomization based on dilated cubes. (ex: Bényi-Oh-Pocovnicu, TAMS '15)

· translation on $\mathbb{R}^d_\xi$ ⟺ modulation $e^{ix\cdot\xi_0}$ on $\mathbb{R}^d_x$.

modulation space $\|f\|_{M^s_{p,q}} = \big\| \langle n\rangle^s \|\Psi(D-n)f\|_{L^p_x} \big\|_{\ell^q_n}$

Wiener amalgam space $\|f\|_{W^{p,q}} = \big\| \|\Psi(D-n)f\|_{\ell^q_n} \big\|_{L^p_x}$

Besov space $\|f\|_{B^s_{p,q}} = \big\| N^s \|P_N f\|_{L^p_x} \big\|_{\ell^q_{N\geq 1 \text{ dyadic}}}$

P_N: LP projector

Triebel-Lizorkin space $\|f\|_{F^s_{p,q}} = \Big\| \big\| \langle\nabla\rangle^s P_N f \big\|_{\ell^q_{N\geq 1 \text{ dyadic}}} \Big\|_{L^p_x}$

- On $\mathbb{T}^d$,

$$\|f\|_{M^{p,q}} = \|f\|_{W^{p,q}} = \|f\|_{FL^q} = \big\|\widehat{f}(n)\big\|_{\ell^q_n}.$$

Q: What about randomization based on the LP decomposition.

$$f = \sum_{N\geq 1 \text{ dyadic}} P_N f$$

$P_N f \to g_N$?

· Consider $g_N = \pm 1$.

$$f^{\omega} = \sum_{\substack{N \geq 1 \\ \text{dyadic}}} \pm P_N f .$$

BUT the LP theorem states $(1 < r < \infty)$

$$\| f^{\omega} \|_{L^r_x} \sim \| \Big(\sum_{\substack{N \geq 1 \\ \text{dyadic}}} |P_N f|^2 \Big)^{1/2} \|_{L^r_x}$$

$$\sim \| f \|_{L^r_x} .$$

$\Rightarrow$ No gain of integrability.

· Probabilistic Strichartz estimate on $\mathbb{R}^d$

· On $\mathbb{T}^d$, we used the fact that $|\mathbb{T}^d| \sim 1$.

· On $\mathbb{R}^d$, we use Bernstein's inequality: $1 \le p \le q \le \infty$.

$$\| P_N f \|_{L^q} \lesssim N^{\frac{d}{p} - \frac{d}{q}} \| P_N f \|_{L^p}$$

In fact

$$\| \mathcal{F}^{-1}(\mathbb{1}_A \cdot \widehat{f}) \|_{L^q} \lesssim |A|^{\frac{1}{p} - \frac{1}{q}} \| \mathbb{1}_A f \|_{L^p}$$

ξ_0 = center of A

$$\| e^{ix\cdot\xi_0} \mathcal{F}^{-1}(\mathbb{1}_A \cdot \widehat{f}) \|_{L^q}$$

$$= \mathcal{F}^{-1}(\mathbb{1}_{A-\xi_0} \cdot \widehat{f})$$

$A - \xi_0$ centered at $0 \in \mathbb{R}^d_\xi$.

$\Longrightarrow$ For any $2 \le r \le \infty$, $\| \Psi(D-n) f \|_{L^r_x} \lesssim \| \Psi(D-n) f \|_{L^2_x}$

· Let $p \geq \max(q, r)$, $2 \leq q, r < \infty$, $S(t) = e^{it\Delta}$

$$\| \| S(t) f^\omega \|_{L_T^q L_x^r} \|_{L^p(\Omega)} \overset{\text{Mink.}}{\leq} \| \| \sum_n \underbrace{\Psi(D-n) S(t) f(x)}_{= C_n} \cdot g_n \|_{L^p(\Omega)} \|_{L_T^q L_x^r}$$

Lem 1.8.2 from Lec 8

$$\lesssim \sqrt{p} \, \| \| \Psi(D-n) S(t) f(x) \|_{l_n^2} \|_{L_T^q L_x^r}$$

Mink

$$\leq \sqrt{p} \, \| \underbrace{\| \Psi(D-n) S(t) f \|_{L_x^r}}_{\text{Bernstein}} \|_{L_T^q \underline{l_n^2}}$$

$$\lesssim \| \Psi(D-n) \cancel{S(t)} f \|_{\underline{L_x^2}}$$

$$\lesssim \sqrt{p} \, \| \| f \|_{L_x^2(\mathbb{R}^d)} \|_{L_T^q}$$

$$\lesssim \sqrt{p} \, T^{1/q} \| f \|_{L_x^2(\mathbb{R}^d)}$$

$\Rightarrow$ By Chebyshev, we obtain the same probabilistic Strichartz estimates as those on $\mathbb{T}^d$.

. On $\mathbb{R}^d$, we can combine this argument with the (deterministic) global-in-time Strichartz estimates.

Prop 1.8.8: Let (q,r) be Schrödinger admissible. $(q < \infty)$

Then, for any finite $\tilde{r} \geq r$, we have

$$P\left(\| S(t) f^\omega \|_{L_t^q L_x^{\tilde{r}}(\mathbb{R}\times\mathbb{R}^d)} > \lambda \right) \leq C \exp\left(- c \frac{\lambda^2}{\|f\|_{L_x^2}^2} \right)$$

(see Lem 2.3 in BOP '15.

· 3-d cubic NLW, $M = \mathbb{T}^3$ or $\mathbb{R}^3$.

$$\begin{cases} -\partial_t^2 u + \Delta u = u^3 \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \in \mathcal{H}^s(M) \setminus \mathcal{H}^{1/2}(M) \end{cases}$$

$s < \frac{1}{2} = s_{crit}$.

Randomize (u_0, u_1) and consider

$$\circledast \quad \begin{cases} -\partial_t^2 u + \Delta u = u^3 \\ (u, \partial_t u)|_{t=0} = (u_0^\omega, u_1^\omega) \end{cases}$$

Thm 1.8.9: Let $s \geq 0$. The Cauchy problem $\circledast$ is almost surely locally well-posed. More precisely, given small $T > 0$, $\exists\, \Omega_T$ s.t. ① $P(\Omega_T^c) \leq C \exp\left(-\frac{c}{T^\alpha}\right)$ for some $\alpha > 0$ $\to 0$ as $T \to 0$.

② $\forall\, \omega \in \Omega_T$, $\exists!$ soln u to $\circledast$ with $(u, \partial_t u)|_{t=0} = (u_0^\omega, u_1^\omega)$.

Rmk: $\begin{cases} -\partial_t^2 u_N + \Delta u_N = u_N^3 \\ (u_N, \partial_t u_N)\big|_{t=0} = (\pi_N u_0^\omega, \pi_N u_1^\omega) \end{cases}$

↑ freq proj onto $\{|\xi| \le N\}$.

Then, $(u_N, \partial_t u_N) \longrightarrow (u, \partial_t u)$ in $C([-T_\omega, T_\omega]; \mathcal{H}^s(M))$

McKean '95, Bourgain '96, Da Prato–Debussche '02.

Idea: First order expansion: Write

$u = \underline{z} + v$, where $\underline{z}^\omega(t) = W(t)(u_0^\omega, u_1^\omega)$

random linear soln.

Then, study the equation satisfied by v:

(+) $\begin{cases} -\partial_t^2 v + \Delta v = (v + \underline{z})^3 \\ (v, \partial_t v)\big|_{t=0} = (0, 0) \end{cases}$

↑ We can forget that z is random.

We only need some space-time control on z.

(++)

Consider $\begin{cases} -\partial_t^2 v + \Delta v = (v+\underline{f})^3 \\ (v, \partial_t v)\big|_{t=0} = (v_0, v_1) \in \dot{\mathcal{H}}^1 \end{cases}$ $\underline{f}$: deterministic space-time func.

$$\Gamma(v) = W(t)(v_0, v_1) - \int_0^t \frac{\sin((t-t')|\nabla|)}{|\nabla|}(v+f)^3(t')\,dt'.$$

Set $\vec{\Gamma}(\vec{v}) = (\Gamma(v), \partial_t \Gamma(v))$. (also, $\vec{v} = (v, \partial_t v)$)

<u>WTS:</u> $\vec{\Gamma}$ is a contraction in $\overline{B_R} \subset C_T \dot{\mathcal{H}}^1_x$.

$$\partial_t \Gamma(v) = \partial_t W(t)(v_0, v_1) - \int_0^t \cos((t-t')|\nabla|)(v+f)^3(t')\,dt'.$$

- $\|\vec{\Gamma}(v)\|_{C_T\dot{\mathcal{H}}^1_x} \le C_1 \|(v_0, v_1)\|_{\dot{\mathcal{H}}^1} + C_2 \int_0^T \|v(t') + f(t')\|_{L^6_x}^3 \,dt'$.

  Sobolev $\le C_1 \|(v_0, v_1)\|_{\dot{\mathcal{H}}^1} + C_2' T \|\vec{v}\|^3_{C_T\dot{\mathcal{H}}^1_x} + C_2'' \|f\|^3_{L^3_T L^6_x}$.

- $\|\vec{\Gamma}(v) - \vec{\Gamma}(w)\|_{C_T\dot{\mathcal{H}}^1_x} \le C_3 T^\theta \left(\|\vec{v}\|^2_{C_T\dot{\mathcal{H}}^1_x} + \|\vec{w}\|^2_{C_T\dot{\mathcal{H}}^1_x} + \|f\|^2_{L^3_T L^6_x}\right) \|\vec{v} - \vec{w}\|_{C_T\dot{\mathcal{H}}^1_x}$

· $R = 2C_1 \|(v_0, v_1)\|_{\dot{\mathcal{H}}^1_x} + 1$

$\Rightarrow$ choose $T > 0$ small s.t. ($T = T(\|(v_0, v_1)\|_{\dot{\mathcal{H}}^1}, f) > 0$

$$C_2' T R^3 \le \frac{1}{4} R$$

$$C_2'' \|f\|^3_{L^3_T L^6_x} \le \frac{1}{4} R$$

$$C_3 T^\theta \left(2R^2 + \|f\|^2_{L^3_T L^6_x} \right) \le 1/2$$

$\Rightarrow$ The perturbed NLW (††) on p. 11 is locally well-posed (if f satisfies the condition above).

Back to (†): $(v_0, v_1) = (0, 0)$. and we have, from Prop 1.8.5.

$$\|z\|_{L^3_T L^6_x} \lesssim 1 \text{ on } \Omega_T \text{ with } P(\Omega_T^c) \lesssim e^{-\frac{c}{T^a}}.$$

□

· <u>Blowup alternative</u>: In view of the probabilistic Strichartz estimates on $[0,T]$ for any finite $T \gg 1$, we see that $\oplus$ on p. 10 is well-posed (for fixed ω) if

$$\sup_{|t|\le T} \| (v^\omega(t), \partial_t v^\omega(t)) \|_{\mathcal{N}} \le C_T^\omega < \infty.$$

<u>Rmk</u>:

$$\| v(t) \|_{L^2_x} \overset{v(0)=0}{=} \| \int_0^t \partial_t v(t')\, dt' \|_{L^2_x} \le \int_0^T \| \partial_t v(t') \|_{L^2_x} dt'$$

$$\le T \| \partial_t v \|_{C_T L^2_x}$$

Thm 1.8.10: Let $s > 0$. Given $(u_0, u_1) \in \mathcal{H}^s$, the defocusing cubic NLW on $\mathcal{M} = \mathbb{T}^3$ or $\mathbb{R}^3$ is almost surely globally well-posed w.r.t. the randomized initial data (u_0^ω, u_1^ω).

Rmk: ① This is a pathwise global well-posedness result.

(i.e. not by an invariant measure argument

Bourgain '94, '96, quasi- : Forlano-Tolomeo, JEMS '25+.

$\Leftarrow$ variation of a deterministic globalization argument.

② Thm 1.8.10 also holds for $s = 0$

Burq-Tzvetkov, JEMS '14.

- Need a Yudovich-type argument giving a double exp bound. See Brezis-Gallouët ineq in Lec 2.

Pf: Fix $T \gg 1$.

$$E(\vec{v}) = \frac{1}{2}\int |\nabla v|^2 dx + \frac{1}{2}\int (\partial_t v)^2 dx + \frac{1}{4}\int v^4 dx.$$

- $E(v)$ is a conserved energy for the (unperturbed) NLW but is NOT conserved under

$$-\partial_t^2 v + \Delta v = (v+z)^3.$$

$$\Rightarrow \boxed{\partial_t E(\vec{v})} = \int \partial_t v \,(\partial_t^2 v - \Delta v + v^3)\, dx$$

$$= -(v+z)^3 = -v^3 - 3v^2 z - 3vz^2 - z^3$$

$$\overset{\text{C-S}}{\lesssim} \Big(\int (\partial_t v)^2 dx\Big)^{1/2} \Big(\int v^4 z^2 + z^6\, dx\Big)^{1/2}$$

$$\overset{\text{Young}}{\lesssim} E(\vec{v})^{1/2}\Big(\|z\|^2_{L^\infty_{T,x}} E(\vec{v})^{1/2} + \|z\|^3_{L^\infty_T L^6_x}\Big)$$

$$\overset{\text{Cauchy}}{\lesssim} \boxed{\Big(1 + \|z\|^2_{L^\infty_{T,x}}\Big) E(\vec{v}) + \|z\|^6_{L^\infty_T L^6_x}}, \quad \forall\, |t| \le T.$$

Gronwall $\Rightarrow$

$$E(\vec{v})(t) \lesssim \|z\|^{b}_{L^{\infty}_{T} L^{b}_{x}} \, e^{c t \left(1 + \|z\|^{2}_{L^{\infty}_{T,x}}\right)}, \quad \forall\, |t| \le T.$$

$\vee$

$$\|\vec{v}(t)\|^{2}_{\dot{H}^{1}_{x}}$$

- Our goal is to prove a.s. GWP.

  $\Leftarrow$ suffices to prove "almost a.s. GWP":

  $\forall\, T \gg 1$, small $\varepsilon > 0$, $\exists\, \Omega_{T,\varepsilon}$ with $P(\Omega^{c}_{T,\varepsilon}) < \varepsilon$.

  s.t. $\forall\, \omega \in \Omega_{T,\varepsilon}$, $\exists!$ soln $\vec{u}$ to NLW with $\vec{u}|_{t=0} = (u_0^{\omega}, u_1^{\omega})$ on $[-T, T]$.

Assume almost a.s. GWP.

① Given $\varepsilon > 0$, set $\Omega_\varepsilon = \bigcap_{j \in \mathbb{N}} \Omega_{2^j, \frac{\varepsilon}{2^j}}$.

· For $\omega \in \Omega_\varepsilon$, the soln u^ω exists globally in time and $P(\Omega_\varepsilon^c) \le \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon$.

② Set $\Sigma = \bigcup_{j \in \mathbb{N}} \Omega_{1/j}$.

· For $\omega \in \Sigma$, the soln u^ω exists globally in time.

$$P(\Sigma^c) = \inf_{j \in \mathbb{N}} \frac{1}{j} = 0.$$

$\Rightarrow$ a.s. GWP.

Goal: Prove almost a.s. GWP.

Fix $T \gg 1$ and small $\varepsilon > 0$.

Set $$A_\lambda = \{ \omega \in \Omega : \| z^\omega \|_{L_T^\infty L_x^6 \cap L_{T,x}^\infty} \le \lambda \}.$$

- Choose $\lambda = \lambda (\| (u_0, u_1) \|_{\mathcal{H}^s}, T) \gg 1$. ($s > 0$)

s.t. $P(A_\lambda^c) < \varepsilon/2$.

Namely, on A_λ, $E(\vec{v}^\omega)(t) \le C_T$, $\forall |t| \le T$. (C_T')

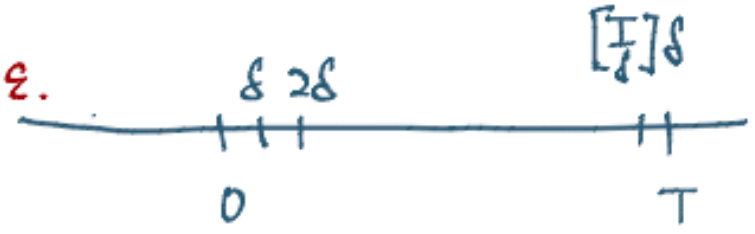
- Choose local existence time $\delta = \delta (\sup_{|t| \le T} \| \vec{v}(t) \|_{\mathcal{H}_x^1}) > 0$.

$$P\Big(\| z \|_{L_{I_j}^3 L_x^6}^3 \gtrsim C_T' \Big) \le \exp\Big(- \frac{C_T''}{\delta^{2/3} \| (u_0, u_1) \|_{\mathcal{H}^0}^2} \Big)$$

($B_j^{C''}$)

$I_j = [(j-1)\delta, j\delta]$.

$\forall\, \omega \in A_\lambda \cap \bigcap_{j=-[\frac{T}{\delta}]-1}^{[T/\delta]+1} B_j = \Omega_{T,\varepsilon}$,



$\exists!\ u = u^\omega$ to NLW on $[-T,T]$. $\left(u^\omega = z^\omega + v^\omega\right)$

$$P\left(\left(\bigcap_{j=-[\frac{T}{\delta}]-1}^{[T/\delta]+1} B_j\right)^c\right) \le \frac{T}{\delta} \sup_j P(B_j^c).$$

$$\le C\, T\, \delta^{-1} \exp\left(-c_T / \delta^{2/3} \|(u_0,u_1)\|_{\mathcal{H}^0}\right)$$

$$< \varepsilon/2 \quad \text{by choosing } \delta > 0 \text{ small.}$$

□

Rmk:

① The Gronwall argument breaks down for $-\partial_t^2 u + \Delta u = |u|^{p-1}u$, $p > 3$.

$$E(\vec{v}) = \frac{1}{2}\int |\nabla v|^2 + \frac{1}{2}\int (\partial_t v)^2 + \frac{1}{p+1}\int |v|^{p+1}.$$

$$\Rightarrow \partial_t E(\vec{v}) = \left(\int (\partial_t v)^2\right)^{1/2}\left(\int |v|^{2(p-1)} z^2 + \cdots\right) \Rightarrow$$

Need $z(p-1) \le p+1$

$\Leftrightarrow$ $p \le 3$.

② In computing the time derivative of $E(\vec{v})(t)$, we switched ∂_t and $\int_M \cdots dx$. How do we justify this?

Given $N \in \mathbb{N}$, consider $\begin{cases} -\partial_t^2 u_N + \Delta u_N = \pi_N u_N^3 \\ (u_N, \partial_t u_N)|_{t=0} = (\pi_N u_0, \pi_N u_1) \end{cases}$

freq proj onto $\{|\xi| \le N\}$

- $\vec{u}_N(0) \in C_x^\infty \Rightarrow \vec{u}_N \in C_T^1 C_x^\infty$

use the eqn $\Rightarrow$ $\vec{u}_N \in C_T^3 C_x^\infty$

$\Rightarrow \cdots \Rightarrow \vec{u}_N \in C_{T,x}^\infty \Rightarrow \vec{v}_N = \vec{u}_N - \vec{z}_N \in C_{T,x}^\infty$.

DCT $\Rightarrow$ We can justify the switching of ∂_t and $\int_M \cdots dx$ for $\vec{v}_N$ (for each fixed N).

- By a slight modification of the LWP argument, we have

$$\vec{v}_N = \vec{u}_N - \vec{z}_N \longrightarrow \vec{v} = \vec{u} - \vec{z} \text{ in } C^1_\delta \dot{\mathcal{H}}^1_x$$

local existence time

By iterating the local-in-time approximation, we conclude that the same Gronwall bound holds for the limiting soln $\vec{v}$.