Lec 1 : Oct. 23, 2025 (THU)

Dispersive PDEs:

- **Nonlinear Schrödinger eqn (NLS):**

$$\begin{cases} i\partial_t u + \Delta u \mp |u|^{p-1}u = 0 \\ u|_{t=0} = u_0 \in H^s(M) \end{cases}$$

$M = \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ or $\mathbb{R}^d$

- **Nonlinear wave eqn (NLW):**

$$\begin{cases} \partial_t^2 u - \Delta u \pm |u|^{p-1}u = 0 \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \in \mathcal{H}^s(M) := H^s(M) \times H^{s-1}(M) \end{cases}$$

- $\|f\|_{H^s(\mathbb{T}^d)} = \|\langle\nabla\rangle^s f\|_{L^2_x} = \|\langle n\rangle^s \hat{f}(n)\|_{\ell^2_n}$

  Bessel potential of order $-s$

$$= \left(\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s} |\hat{f}(n)|^2\right)^{1/2}$$

$\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$

$|\nabla| = \sqrt{-\Delta}$

$\langle\nabla\rangle = \sqrt{1-\Delta}$

Goal: Study these eqns with rough random initial data and/or stochastic forcing.

- initiated by Bourgain '94 -
  Tzvetkov, Burq-Tzv, Oh mid '00.
  more recently: Gubinelli-Koch-Oh, Bringmann, Deng-Nahmod-Yue.
- of physical importance.
- at the intersection of harmonic analytic approach to dispersive PDEs & stoch. analysis.

See also the course note from 2021 (← random data / additive noise), 2022 (↑ multiplicative noise)

· NLS & NLW are Hamiltonian PDEs.

Hamiltonian for NLS: $H(u) = \frac{1}{2}\int |\nabla u|^2 dx \pm \frac{1}{p+1}\int |u|^{p+1} dx$

- for NLW: $E(\vec{u}) = E(u, \partial_t u) = H(u) + \frac{1}{2}\int (\partial_t u)^2 dx$

· NLS: L^2 = symplectic space (symplectic form) with $\omega(f,g) = \mathrm{Im}\int f\,\overline{g}\,dx$.

$J = i$.

(Gâteaux deriv at u in the direction of Φ)

$$\omega(\Phi, J\nabla H) \overset{\text{def}}{=} dH|_u(\Phi) \overset{\text{def}}{=} \frac{d}{d\varepsilon} H(u+\varepsilon\Phi)\Big|_{\varepsilon=0}$$

· linear case: $H(u) = \frac{1}{2}\int |\nabla u|^2 dx \Rightarrow dH_u(\Phi) = \mathrm{Re}\langle \nabla u, \nabla \Phi\rangle$

$$\overset{\text{IBP}}{=} \mathrm{Re}\langle \Phi, -\Delta u\rangle$$
$$= \mathrm{Im}\langle \Phi, i\Delta u\rangle$$
$$= \omega(\Phi, i\Delta u)$$

$\Rightarrow \partial_t u = J\nabla H = i\Delta u$

- can repeat a similar computation for the full Hamiltonian to write NLS as $\partial_t u = J\nabla H$.

Formally, $\partial_t u = i\frac{\partial H}{\partial \bar{u}}$

· NLW: Symplectic space $= \dot{\mathcal{H}}^{1/2} = \dot{H}^{1/2} \times \dot{H}^{-1/2}$

$$\langle \vec{f}, \vec{g} \rangle_{\dot{\mathcal{H}}^{1/2}} = \int (|\nabla| f_1) g_1 \, dx = \int (|\nabla|^{-1} f_2) g_2 \, dx$$

· almost symplectic structure: $J = \begin{pmatrix} 0 & -|\nabla|^{-1} \\ |\nabla| & 0 \end{pmatrix}$.

$J : \dot{\mathcal{H}}^{1/2} \to \dot{\mathcal{H}}^{1/2}$ and $J^2 = -Id$.

· symplectic form $\omega(\vec{f}, \vec{g}) = \langle \vec{f}, J\vec{g} \rangle_{\dot{\mathcal{H}}^{1/2}}$

Then, (NLW) $\iff \partial_t \begin{pmatrix} u \\ v \end{pmatrix} = J\nabla E$, $\quad v = \partial_t u$

where $\omega(\vec{\Phi}, J\nabla H) = dE|_{(u,v)}(\vec{\Phi}) = \frac{d}{d\varepsilon} E(\vec{u} + \varepsilon\vec{\Phi})\Big|_{\varepsilon=0}$. $\quad \vec{u} = (u, v)$

Formally, $\partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \mp \begin{pmatrix} 0 \\ |u|^{p-1}u \end{pmatrix}$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{\partial E(u,v)}{\partial (u,v)}$$

· Given a Hamiltonian system on $\mathbb{R}^{2N}$

$$\partial_t q_j = \frac{\partial H}{\partial p_j}, \quad \partial_t p_j = -\frac{\partial H}{\partial q_j}, \quad j = 1, \cdots, N$$

$$H(q,p) = H(q_1, \cdots, q_N, p_1, \cdots, p_N)$$

① Liouville thm: $div\, X = \sum_{j=1}^{N} \left(\frac{\partial}{\partial q_j} \frac{\partial H}{\partial p_j} + \frac{\partial}{\partial p_j} \left(-\frac{\partial H}{\partial q_j} \right) \right) = 0$

$\Rightarrow dq\,dp = \prod_{j=1}^{N} dq_j\, dp_j$ is invariant under the flow.

② Hamiltonian is conserved under the flow.

$$\partial_t H(q(t), p(t)) = \sum_{j=1}^{N} \left(\frac{\partial H}{\partial q_j} \partial_t q_j + \frac{\partial H}{\partial p_j} \partial_t p_j \right) = 0$$

$\partial_t q_j = \frac{\partial H}{\partial p_j}$, $\partial_t p_j = -\frac{\partial H}{\partial q_j}$

$\Rightarrow$ Gibbs measure ρ:

$$d\rho = Z^{-1} e^{-\beta \underbrace{H(q,p)}_{②}} \underbrace{dq\,dp}_{①}$$

$\beta > 0$ = inverse temperature

is invariant under the dynamics.

- Bourgain '94, '96 : study of invariant Gibbs measures for NLS on $\mathbb{T}^d$

co-dim'l Hamiltonian system

Friedlander '85, McKean '95

Lebowitz-Rose-Speer '88 (see the notes from last year)

NLS: $i\partial_t u + \underline{(1-\Delta)}\, u \mp |u|^{p-1}u = 0$

↑ to avoid an issue at the freq 0.

$$H(u) = \frac{1}{2}\|u\|_{H^1}^2 \pm \frac{1}{p+1}\int |u|^{p+1}dx.$$

· NO ∞-dim'l Lebesgue meas du !!

$$\Rightarrow \quad d\rho = Z^{-1} e^{-H(u)}\, du$$

$$= Z^{-1} e^{\mp \frac{1}{p+1}\int |u|^{p+1}dx}\, e^{-\frac{1}{2}\|u\|_{H^1}^2}\, du$$

↑ Gaussian meas on $\mathcal{D}'(\mathbb{T}^d)$

on $H^s(\mathbb{T}^d)$, $s < 1-\frac{d}{2}$

$W^{s,r}(\mathbb{T}^d)$, $1 \le r \le \infty$.

· − sign defocusing case: $d=1$, trivial
$d=2$, see the notes from 2024/2025

$d=3$, $p=4$: $\overline{\Phi}^4_3$-meas

· + sign: focusing case

$d=1$, LRS '88. Bourgain '94, Oh-Sosoe-Tolomeo '22

$d \ge 2$, Oh-Okamoto-Tolomeo '24, '25, Oh-Seong-Tolomeo '24

- Gaussian measure : $d\mu_s = Z^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du$

  - truncated version : $d\hat{u}(n)$ = Lebes meas on $\mathbb{C}$.

$$d\mu_{s,N} = Z_N^{-1} \, e^{-\frac{1}{2}\sum_{|n|\le N} \langle n\rangle^{2s} |\hat{u}(n)|^2} \prod_{|n|\le N} d\hat{u}(n)$$

$$= Z_N^{-1} \prod_{|n|\le N} e^{-\frac{1}{2}\langle n\rangle^{2s}|\hat{u}(n)|^2} d\hat{u}(n)$$

$$= |g_n|^2 \qquad \sim_n dg_n$$

Q: $N \to \infty$? u_N under $\mu_{s,N}$ (i.e. Law $(u_N) = \mu_{s,N}$ on $\mathbb{C}^{2N+1}$)

$$\Rightarrow \boxed{u_N = \sum_{|n|\le N} \frac{g_n}{\langle n\rangle^s} e^{in\cdot x}}$$

$\{g_n\}_{|n|\le N}$ = family of indep. standard complex-valued Gaussian r.v.'s

- $\mathbb{E}\left[\|u_N\|_{H^\sigma}^2\right] = \mathbb{E}\left[\sum_{|n|\le N} \frac{|g_n|^2}{\langle n\rangle^{2(s-\sigma)}}\right]$

$$\sim \sum_{|n|\le N} \frac{1}{\langle n\rangle^{2(s-\sigma)}} \lesssim 1, \text{ unif. in } N\in\mathbb{N}$$

if $2(s-\sigma) > d \iff \boxed{\sigma < s - \frac{d}{2}}$

- $u_N \to u$, a.s. in $H^\sigma(\mathbb{T}^d)$, $\sigma < s - \frac{d}{2}$

$\Rightarrow \underline{\mu_s = \mathrm{Law}(u)}$ $\quad u \in H^s(\mathbb{T}^d)$

- (H^s, H^σ, μ_s) = abstract Wiener space.

↑ Cameron-Martin space

$\underline{W^{\sigma,r}\ ?}$

$\underline{r<\infty}$: $\quad \| \, \| u_N \|_{W^{\sigma,r}} \|_{L^p(\Omega)} \leq \| \, \| \overbrace{\langle \nabla \rangle^\sigma u_N(x)}^{\sum_{|n|\leq N} \frac{g_n}{\langle n \rangle^{s-\sigma}} e^{in\cdot x}} \|_{L^p(\Omega)} \, \|_{L^r_x}$

$p \geq r$ finite

$\left(\|f\|_{W^{\sigma,r}} = \|\langle \nabla \rangle^\sigma f\|_{L^r} \right.$

$$\lesssim p^{1/2} \, \| \, \| \sum_{|n|\leq N} g_n \frac{e^{in\cdot x}}{\langle n \rangle^{s-\sigma}} \|_{L^2(\Omega)} \, \|_{L^r_x}$$

$$\| \, \mathbb{1}_{|n|\leq N} \frac{e^{in\cdot x}}{\langle n \rangle^{s-\sigma}} \|_{\ell^2_n}$$

throw away

$\sim p^{1/2}$ unif in $N \in \mathbb{N}$. if $\sigma < s - \frac{d}{2}$.

$\underline{r=\infty}$: Use Sobolev and repeat the computation above

$$\|u_N\|_{W^{\sigma,\infty}} \lesssim \|u_N\|_{W^{\sigma+\varepsilon,r}} \qquad \varepsilon r > d$$

(Bényi - Oh '13.

· Gibbs measure for NLS: $d\rho = Z^{-1} e^{\mp \frac{1}{p+1}\int |u|^{p+1} dx} d\mu_1$

$d\mu_1 = Z^{-1} e^{-\frac{1}{2}\|u\|_{H^1}^2} du$

↑ (massive) Gaussian free field on $\mathbb{T}^d$.

· We first need / want to study (local) well-posedness of NLS with the GFF initial data

$$u_0 = \sum_{n\in\mathbb{Z}^d} \frac{g_n}{\langle n\rangle} e^{in\cdot x} \overset{\text{a.s.}}{\in} H^s(\mathbb{T}^d) \setminus H^{1-\frac{d}{2}}(\mathbb{T}^d)$$

$$s < 1 - \frac{d}{2}$$

$(< 0 \text{ for } d \geq 2)$

From page ⑨,

$\mathbb{E}\big[\|u\|_{H^{1-\frac{d}{2}}}^2\big] = \infty \Rightarrow P(u \notin H^{1-\frac{d}{2}}) > 0$

$\Rightarrow$ Kolmogorov's 0-1 law $\Rightarrow$ a.s. not in $H^{1-\frac{d}{2}}$

Rmk: Law$(u) = \mu_s$, i.e. $\boxed{u = \sum_{n \in \mathbb{Z}^d} \frac{g_n}{\langle n \rangle^s} e^{in\cdot x}} \in H^{s-\frac{d}{2}-\varepsilon}$ a.s.

$f \in H^s(\mathbb{T}^d)$, deterministic

$$\langle f, u \rangle_{H^s} = \sum_{n \in \mathbb{Z}^d} \langle n \rangle^s \widehat{f}(n) \cdot \overline{g_n}$$

$$\Rightarrow \| \langle f, u \rangle_{H^s} \|_{L^2(\Omega)} = \| f \|_{H^s} < \infty$$

- Although $u \notin H^s(\mathbb{T}^d)$ a.s., the inner product $\langle f, u \rangle_{H^s}$ is a.s. finite for each $f \in H^s(\mathbb{T}^d)$
- $\langle f, u \rangle_{H^s}$ is Gaussian for each $f \in H^s(\mathbb{T}^d)$

$\Leftarrow$ This is often used as a definition of a Gaussian probability meas. (In this case, μ_s is a Gaussian probability measure.

Gibbs measure for NLW: $\partial_t^2 u + (1-\Delta)u \pm |u|^{p-1}u = 0$ (13)

$$d\rho = Z^{-1} e^{-E(u,\partial_t u)} du\, d(\partial_t u)$$

$$= Z^{-1} e^{\mp \frac{1}{p+1}\int |u|^{p+1} dx} \underbrace{e^{-\frac{1}{2}\|u\|_{H^1}^2} du \otimes e^{-\frac{1}{2}\|\partial_t u\|_{L^2}^2} d(\partial_t u)}$$

base Gaussian meas on $(u, \partial_t u)$

$\mu_1 \otimes \mu_0 (u, \partial_t u)$

$\Rightarrow$ $\mathrm{Law}(u_0) = \mu_1$, $\boxed{\mathrm{Law}(u_1) = \mu_0}$

$$u_0 = \sum_{n\in\mathbb{Z}^d} \frac{g_n}{\langle n\rangle} e^{in\cdot x} \in H^{1-\frac{d}{2}-}(\mathbb{T}^d) \setminus H^{1-\frac{d}{2}}(\mathbb{T}^d) \quad \text{a.s.}$$

$$u_1 = \sum_{n\in\mathbb{Z}^d} h_n e^{in\cdot x} \in H^{-\frac{d}{2}-}(\mathbb{T}^d) \setminus H^{-\frac{d}{2}}(\mathbb{T}^d)$$

• u and $\partial_t u$ are real-valued.

$\Rightarrow$ u_0 and u_1 are real-valued

$\Leftrightarrow$ $\widehat{u_j}(-n) = \overline{\widehat{u_j}(n)}$, $j=1,2$, $\forall\, n \in \mathbb{Z}^d$

(*) $g_{-n} = \overline{g_n}$ & $h_{-n} = \overline{h_n}$, $\forall\, n \in \mathbb{Z}^d$

Define the index set $I = \left(\bigcup_{k=0}^{d-1} \mathbb{Z}^k \times \mathbb{N} \times \{0\}^{d-k-1} \right) \cup \{\vec{0}\}$

$$'' = \mathbb{Z}^d / 2''.$$

$\{(g_n, h_n)\}_{n \in I}$ = family of indep. std. $\mathbb{C}$-valued Gaussian r.v.'s

and extend onto $\mathbb{Z}^d$ by (*)

· μ_0 = (spatial) white noise on $\mathbb{T}^d$.

$d\mu_0 = Z^{-1} e^{-\frac{1}{2}\|u\|_{L^2}^2} du$

· $\mathrm{Law}(u) = \mu_0$

$$\Rightarrow \quad u = \sum_{n \in \mathbb{Z}^d} g_n e^{in\cdot x} \in H^{-\frac{d}{2}-}(\mathbb{T}^d)$$

$$W^{-\frac{d}{2}-,\infty}(\mathbb{T}^d)$$

$$\mathcal{C}^{-\frac{d}{2}-}(\mathbb{T}^d)$$

· White noise u:

formal: $\mathbb{E}[u(x)\overline{u(y)}] = \delta(x-y)$

$$\left(= \mathbb{E}\Big[\sum_n g_n e^{in\cdot x} \sum_k \bar{g}_k e^{-ik\cdot y}\Big] = \sum_n e^{in\cdot(x-y)} \right.$$

$$\mathbb{E}[g_n^{\ell_1} \bar{g}_k^{\ell_2}] = \ell_1!\, 1_{n=k} 1_{\ell_1=\ell_2}$$

rigorous: $\mathbb{E}[u(f_1)\overline{u(f_2)}] = \langle f_1, f_2\rangle_{L^2} \quad \forall f_1, f_2 \in L^2$

$$\left(= \mathbb{E} \sum_n \hat{f}_1(n)\overline{g_n} \cdot \sum_k \overline{\hat{f}_2(k)}\, g_k = \sum_n \hat{f}_1(n)\overline{\hat{f}_2(n)} \right.$$

Chap 4: Stochastic dispersive PDEs

Stochastic NLS (SNLS): $i\partial_t u + \Delta u + \mathcal{N}(u) = \Phi\xi$

Stochastic NLW (SNLW): $\partial_t^2 u + (1-\Delta)u + \mathcal{N}(u) = \Phi\xi$.

- ξ = space-time white noise on $\mathbb{R}_t \times \mathbb{T}_x^d$
- Φ = Hilbert-Schmidt op from L^2 to H^s.

formal: $\mathbb{E}[\xi(t_1,x_1)\overline{\xi(t_2,x_2)}] = \delta(t_1-t_2)\delta(x_1-x_2)$.

rigorous: $\mathbb{E}[\xi(f_1)\overline{\xi(f_2)}] = \langle f_1, f_2\rangle_{L^2}$

$\forall\, f_1, f_2 \in L^2(\mathbb{R}\times\mathbb{T}^d)$

- Duhamel formulation for SNLS: $S(t) = e^{it\Delta}$

$$u(t) = S(t)u_0 - i\int_0^t S(t-t')N(u)(t')\,dt' - i\underbrace{\int_0^t S(t-t')\,\xi(dt')}_{=:\,\Psi}$$

suppose $\Phi = Id$

stochastic convolution

- Define $B_n(t) = \langle \xi, \mathbb{1}_{[0,t]} \cdot e^{in\cdot x} \rangle_{\mathcal{D}'-\mathcal{D}}$

B_n is a BM.

- $B_n(0) = 0$
- $B_n(t)$ = mean 0 Gaussian r.v. $\mathbb{E}[(B_n(t))^2] = t$
- indep increment on disjoint intervals.

- $\{B_n\}_{n\in\mathbb{Z}^d}$ = family of indep BM's.

(real-valued case: $B_{-n} = \overline{B_n}$)

L^2- cylindrical Wiener process

On $\mathbb{T}^d$

$$W(t) = \sum_{n \in \mathbb{Z}^d} B_n(t)\, e^{in\cdot x} \in W^{-\frac{d}{2}-,\infty}(\mathbb{T}^d) \quad \text{a.s.}$$

$\Leftarrow$ dilated spatial white noise

$$\Psi(t) = \int_0^t S(t-t')\, \Xi(dt') \qquad \Phi = Id$$

$$= \int_0^t S(t-t')\, dW(t')$$

$$= \sum_{n \in \mathbb{Z}} e^{in\cdot x} \underbrace{\int_0^t e^{-i(t-t')|n|^2}\, dB_n(t')}_{\text{= Wiener integral (next time)}}$$