Lec 2: Oct 30, 2025 (THU)

$$\Psi(t) = \int_0^t S(t-t')\,\xi(dt')$$

$$= \int_0^t S(t-t')\,dW(t')$$

$$= \sum_{n\in\mathbb{Z}^d} e^{in\cdot x} \underbrace{\int_0^t e^{-i(t-t')|n|^2}\,dB_n(t')}_{\text{Wiener integral}}$$

· **Wiener integral:** Given $f \in L^2([a,b])$ deterministic

$$I(f) = \int_a^b f(t)\,dB(t) \qquad B = \text{B.M.}$$

· constructed as a limit of Riemann sums in $L^2(\Omega)$

see Kuo, chap 2.

· mean 0 Gaussian r.v. $\quad \mathbb{E}[I(f)] = 0$

· $\mathrm{Var}(I(f)) = \mathbb{E}[|I(f)|^2] = \|f\|^2_{L^2([a,b])}$

$I : L^2([a,b]) \longrightarrow L^2(\Omega)$ is an isometry. (2)

- $\mathbb{E}[I(f)\,\overline{I(g)}] = \langle f, g\rangle_{L^2}$

$$\Rightarrow \quad \Psi(t) = \sum_{n\in\mathbb{Z}^d} e^{in\cdot x} \underbrace{\int_0^t e^{-i(t-t')|n|^2}\, dB_n(t')}_{\sim N_{\mathbb{C}}(0,t)}$$

⇐ dilated spatial white noise for each fixed t.

$$\in H^{-\frac{d}{2}-}(\mathbb{T}^d) \quad \text{a.s.} \quad (\text{for each fixed } t)$$

Note: For $f \in C^1$, we can define $I(f)$ as a Paley-Wiener-Zygmund integral. (by IBP):

$$I(f) = \int_a^b f\, dB = f(b)B(b) - \int_a^b f'(t)B(t)\,dt$$

Recall $B(0) = 0$.

Sec 4.1: Stochastic convolutions

· Stoch. convolution for SNLS:

$$\begin{cases} i\partial_t \Psi + \Delta \Psi = (i)\Phi \Xi \\ \Psi|_{t=0} = 0. \end{cases}$$

so that we do not need to keep i in Ψ.

$$\Psi(t) = \int_0^t S(t-t')\Phi \, dW(t')$$

$$= \sum_{n \in \mathbb{Z}^d} \int_0^t S(t-t')\Phi(e_n)\, dB_n(t')$$

$n \in \mathbb{N}$ on $\mathbb{R}^d$

$$\begin{cases} e_n = e^{in\cdot x} \text{ on } \mathbb{T}^d. \\ \{e_n\}_{n\in\mathbb{N}} = \text{O. N. B. of } L^2(\mathbb{R}^d) \end{cases}$$

· If Φ is translation invariant (spatial homogeneous)

i.e. Fourier multiplier operator (Grafakos vol. 1. Chap 2

$$\begin{cases} \tau_y f(x) = f(x-y) \\ \Rightarrow \Phi(\tau_y f) = \tau_y \Phi(f). \end{cases}$$

On $\mathbb{T}^d$

$$\Psi(t) = \sum_{n\in\mathbb{Z}^d} \phi_n e_n \int_0^t e^{-i(t-t')|n|^2} dB_n(t)$$

· **Lemma 4.1: Kolmogorov continuity criterion**

$\{X_t\}$ with values in a metric space

Suppose $\mathbb{E}\left[d(X_{t_1}, X_{t_2})^p \right] \le C_0 |t_1 - t_2|^{1+\alpha}$

for some $p \ge 1$, $\alpha > 0$.

Then, $P\left(\sup_{t_1 \neq t_2} \frac{d(X_{t_1}, X_{t_2})}{|t_1 - t_2|^{\alpha/p} - \varepsilon} \ge \lambda \right) \le \frac{C_1}{\lambda^p}$, $\forall\, 0 < \varepsilon < \frac{\alpha}{p}$

$\Rightarrow$ X_t is a.s. $(\frac{\alpha}{p} - \varepsilon)$- Hölder continuous.

ex: B.M. $\mathbb{E}\left[|B(t_2) - B(t_1)|^2 \right] = |t_2 - t_1|$

($B(t_2) - B(t_1)$: Gaussian)

$\Rightarrow \mathbb{E}\left[|B(t_2) - B(t_1)|^p \right] \sim p^{p/2} |t_2 - t_1|^{p/2}$, $p < \infty$

($\alpha = 1 + \frac{p}{2} - 1$)

$\Rightarrow \frac{\alpha}{p} = \frac{\frac{p}{2} - 1}{p} = \frac{1}{2} - \frac{1}{p} \to \frac{1}{2}-$ as $p \to \infty$ $\Rightarrow$ BM is $(\frac{1}{2}-)$- Hölder conti a.s.

· We say that $\Phi : H_1 \to H_2$ is Hilbert-Schmidt

if $\|\Phi\|_{HS(X;Y)} = \left(\sum_{n \in \mathbb{N}} \|\Phi(e_n)\|_Y \right)^{1/2} < \infty$

· $\{e_n\}_{n \in \mathbb{N}}$ = O.N.B. of X $\quad H_1, H_2$, Hilbert space

· If Φ is translation invariant, i.e.

$\Phi(e_n) = \phi_n e_n$ (on $\mathbb{T}^d$)

A special case of γ-radonifying operators in the Banach space setting.

(see Appendix of Forlano-Oh-Wang '20)

then

$$\|\Phi\|_{HS(L^2(\mathbb{T}^d); H^s(\mathbb{T}^d))} \simeq \left(\sum_{n \in \mathbb{Z}^d} \|\phi_n e_n\|_{H^s}^2 \right)^{1/2} = \|\langle n \rangle^s \phi_n\|_{l^2_n} = \|\check{\phi}\|_{H^s}$$

Prop 4.2: $\Phi \in HS(L^2(\mathbb{T}^d); H^s(\mathbb{T}^d))$, trans. inv.

Then, $\Psi(t) = \underline{\int_0^t S(t-t')\Phi\, dW(t')} \in C_t^{\frac{\alpha}{2}-} W_x^{s-\alpha, r}(\mathbb{T}^d)$

a.s. $\underline{1 \le r \le \infty}$, $\alpha > 0$.

only on $\mathbb{T}^d$.

Also, $\Psi \in C_t H_x^s$, a.s.

$$\sum_{n\in\mathbb{Z}^d} \phi_n e_n \underbrace{\int_0^t e^{-i(t-t')|n|^2} dB_n(t')}_{\text{independent for } n\in\mathbb{Z}^d}$$

Pf: $0 < t_1 \le t_2$

$$\mathbb{E}\left[\Psi(t_1, x_1)\overline{\Psi(t_2, x_2)}\right]$$

indep

Wiener isom.

$$= \sum_{n\in\mathbb{Z}^d} |\phi_n|^2 e_n(x_1 - x_2) \underbrace{\int_0^{t_1} e^{-i(t_1-t_2)|n|^2} dt'}_{= t_1 e^{-i(t_1-t_2)|n|^2}}$$

$$\int_0^{t_1} + \int_{t_1}^{t_2}$$

↑ indep of $\int_0^{t_1} f\, dB_n$

$$\left(\mathbb{E}\left[\int_0^{t_1} f\, dB_n \overline{\int_0^{t_2} g\, dB_n}\right] = \int_0^{t_1} f\bar{g}\, dt. \right.$$

$\Leftarrow$ Apply $\langle \nabla_{x_1}\rangle^s$ and $\langle \nabla_{x_2}\rangle^s$

$$\mathbb{E}\left[\langle\nabla_{x_1}\rangle^s \Psi(t_1,x_1)\,\overline{\langle\nabla_{x_2}\rangle^s \Psi(t_2,x_2)}\right]$$

$$= \sum_{n\in\mathbb{Z}^d} |\Phi_n|^2 \langle n\rangle^{2s} \underbrace{e_n(x_1-x_2)}\cdot t_1 \underbrace{e^{-i(t_1-t_2)|n|^2}}$$

$\to 1$

- Now, set $x_1 = x_2$ and $t_1 = t_2$

$$\|\underbrace{\langle\nabla\rangle^s \Psi(t,x)}_{\text{Gaussian}}\|_{L^p(\Omega)} \sim p^{1/2} \|\langle\nabla\rangle^s \Psi(t,x)\|_{L^2(\Omega)}$$

$$= p^{1/2} t^{1/2} \|\Phi\|_{HS(L^2;H^s)}$$

$r<\infty$, $r\le p<\infty$

$$\| \|\Psi(t)\|_{W^{s,r}_x} \|_{L^p(\Omega)} = \| \|\langle\nabla\rangle^s \Psi(t)\|_{L^r_x} \|_{L^p(\Omega)}$$

$$\overset{\text{Mink}}{\le} \| \|\langle\nabla\rangle^s \Psi(t,x)\|_{L^p(\Omega)} \|_{\underbrace{L^r(\mathbb{T}^d)}}$$

$$\sim p^{1/2} t^{1/2} \|\Phi\|_{HS(L^2;H^s)}$$

throw away since $|\mathbb{T}^d| \sim 1$.

$$\Rightarrow \quad \Psi(t) \in \begin{cases} W^{s,r}(\mathbb{T}^d), & \text{a.s.} \quad 1 \le r < \infty \\ W^{s-,\infty}(\mathbb{T}^d), & \text{a.s.} \quad r = \infty \end{cases} \quad \text{(by Sobolev}$$

$$\text{Also, } \quad P\left(\|\Psi(t)\|_{W^{s,r}} > \lambda \right) \le C e^{-c \lambda^2 / t \|\phi\|_{HS}^2}, \quad r < \infty$$

Next, we study the temporal regularity.

- Given $h \in \mathbb{R}$, let

$$\delta_h \Psi(t,x) = \Psi(t+h,x) - \Psi(t,x)$$

$$\Rightarrow \mathbb{E}\left[\delta_h \Psi(t,x) \overline{\delta_h \Psi(t,y)} \right]$$

$$= \mathbb{E}\left[\Psi(t+h,x) \overline{\Psi(t+h,y)} \right] - \mathbb{E}\left[\Psi(t,x) \overline{\Psi(t+h,y)} \right]$$

$$- \mathbb{E}\left[\Psi(t+h,x) \overline{\Psi(t,y)} \right] + \mathbb{E}\left[\Psi(t,x) \overline{\Psi(t,y)} \right]$$

$$= \sum_{n \in \mathbb{Z}^d} |\phi_n|^2 e_n(x-y) \underbrace{\left\{ (t+h) - t e^{ih|n|^2} - t e^{-ih|n|^2} + t \right\}}_{=: F_n(t,h)} \qquad h > 0$$

$$|F_n(t,h)| \le |h| + t\,|1 - e^{ih|n|^2}| + t\,|1 - e^{-ih|n|^2}| \qquad (9)$$

$$\overset{\text{MVT}}{\lesssim} |h| + t \underbrace{\min(1, h|n|^2)}_{(h|n|^2)^\alpha} \quad \forall\ 0 \le \alpha \le 1$$

$$\Rightarrow \|\langle\nabla\rangle^s \delta_h \Psi(t,x)\|_{L^p(\Omega)}$$

$$\sim p^{1/2} \|\langle\nabla\rangle^s \delta_h \Psi(t,x)\|_{L^2(\Omega)} \qquad |t| \le T.$$

$$\lesssim p^{1/2} T^{1/2} |h|^{\alpha/2} \|\Phi\|_{HS(L^2; H^{s+\alpha})}$$

$$\Rightarrow \| \|\delta_h \Psi(t)\|_{W^{s,r}_x} \|_{L^p(\Omega)} \le C_r p^{1/2} T^{1/2} |h|^{\alpha/2} \|\Phi\|_{HS(L^2;H^{s+\alpha})}$$

$r < \infty$, $p < \infty$

Lem 4.1

$$\Rightarrow \Psi \in C_t^{\frac{\alpha}{2} - \frac{1}{p} -} W_x^{s-\alpha, r} \quad \text{a.s.}$$

if $\Phi \in HS(L^2; H^s)$ (by replacing $s+\alpha$ by s).

take $p \gg 1$

$\Rightarrow$ $\Psi \in C_t^{\frac{\alpha}{2}-\varepsilon} W_x^{s-\alpha, r}$ a.s. if $\Phi \in HS(L^2; H^s)$

$r \leq \infty, \quad \alpha > 0$

$\underline{r=2}$: $\widetilde{\Psi}(t) = S(-t)\Psi(t) = \int_0^t S(-t')\Phi \, dW(t')$

$\Rightarrow \mathbb{E}\left[\widetilde{\Psi}(t_1, x_1)\overline{\widetilde{\Psi}(t_2, x_2)}\right] = \sum_{n \in \mathbb{Z}} |\Phi_n|^2 e_n(x_1 - x_2) \cdot t_1$ $\quad 0 < t_1 \leq t_2$

By repeating the computation,

$$\left\| \| f_h \widetilde{\Psi}(t) \|_{H_x^s} \right\|_{L^p(\Omega)} \leq C_T p^{1/2} \|\Phi\|_{HS(L^2; H^s)} \underline{|h|^{1/2}}$$

Lem 4.1 $\Rightarrow$ $\widetilde{\Psi} \in C_t^{\frac{1}{2}-} H_x^s$, a.s.

$S(t) f \in C_t H_x^s$, $f \in H^s$

$\Rightarrow$ $\Psi \in C_t^{\frac{1}{2}-} H_x^s$, a.s. since $S(t)$ is unitary on H^s

□

($\Psi(t+h) - \Psi(t) = S(t+h)\left[\widetilde{\Psi}(t+h) - \widetilde{\Psi}(t)\right]$

$+ \left(S(t+h) - S(t)\right)\widetilde{\Psi}(t)$

$\underbrace{\qquad}_{\to 0 \text{ as } h \to 0.}$

Note: Later in the course, I will present two more proofs of Prop 4.2.

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ex: $\Phi = \mathrm{Id} \Rightarrow \Phi \in HS(L^2; H^s)$, $s < -\frac{d}{2}$

$\Rightarrow \Psi \in C_t^{\frac{1}{2}-} H_x^{-\frac{d}{2}-}$, a.s., $s < -\frac{d}{2}$.

regularity of spatial white noise.

· Wave case:

$$\begin{cases} \partial_t^2 \Psi + (1-\Delta)\Psi = \xi, & \Phi = \mathrm{Id} \\ (\Psi, \partial_t \Psi)|_{t=0} = (0,0) \end{cases}$$

$$\Rightarrow \Psi_{\text{wave}}(t) = \int_0^t \frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle} \, dW(t')$$

$$\in C_t H_x^s, \quad \text{a.s.} \quad s < 1 - \frac{d}{2}$$

$$C_t W_x^{s,r} \quad 1 \le r \le \infty$$

$\mathbb{R}^d$ case:

Prop 4.3: $\Phi \in HS(L^2; H^s)$. (No translation invariance is assumed.)

$$\Rightarrow \Psi(t) = \int_0^t S(t-t')\Phi\, dW(t') \in C_t H^s_x(\mathbb{R}^1), \text{ a.s.}$$

$$= \sum_{n\in\mathbb{N}} \int_0^t S(t-t')\Phi(e_n)\, dB_n(t')$$

$\{e_n\}_{n\in\mathbb{N}}$ = O.N.B. of $L^2(\mathbb{R}^d)$

Lem 4.4: $T > 0$, $0 < \alpha < 1$, $q > \frac{1}{\alpha}$ — Da Prato-Zabczyk, Prop 5.9; Da Prato, "Kolm. eqn for SPDEs".

If $f \in L^q_T H^s_x$, then

$$F(t) = \int_0^t S(t-t')(t-t')^{\alpha-1} f(t')\, dt' \in C_T H^s_x.$$

and $\|F\|_{L^p_T H^s_x} \lesssim_{q,T} \|f\|_{L^q_T H^s_x}$

Pf: Factorization method

(*) $$\int_r^t (t-t')^{\alpha-1}(t'-r)^{-\alpha}\,dt' = \frac{\pi}{\sin \pi\alpha}, \quad \begin{matrix} 0<\alpha<1 \\ 0 \le r \le t' \le t. \end{matrix}$$

- Fix $0 < \alpha < \frac{1}{2}$.

$$\Psi(t) = \int_0^t S(t-r)\,\Phi\,dW(r)$$

$$\overset{(*)}{=} \frac{\sin \pi\alpha}{\pi} \int_0^t \left(\int_r^t (t-t')^{\alpha-1}(t'-r)^{-\alpha}\,dt' \right) S(t-r)\,\Phi\,dW(r)$$

stoch Fubini

$$= \frac{\sin \pi\alpha}{\pi} \int_0^t S(t-t')\,(t-t')^{\alpha-1} \times \underbrace{\int_0^{t'} S(t'-r)(t'-r)^{-\alpha}\,\Phi\,dW(r)}_{= f(t')}\,dt'$$

D.P.-Zab.
Oh-Wang-Zine '22
Appendix

In view of Lem 4.5, it suffices to show

(**) $$\mathbb{E}\left[\int_0^T \|f(t')\|_{H^s}^q \, dt'\right] \le C(T, q, \|\Phi\|_{HS}) < \infty$$

$$q > \frac{1}{\alpha}$$

$$f(t') = \int_0^{t'} S(t'-r)(t'-r)^{-\alpha} \Phi \, dW(r)$$

$$= \sum_{n\in\mathbb{N}} \int_0^{t'} S(t'-r)(t'-r)^{-\alpha} \Phi(e_n) \, dB_n(t')$$

$$\mathbb{E}[|f(t',x)|^2] = \sum_{n\in\mathbb{N}} \int_0^{t'} |S(t'-r)(t'-r)^{-\alpha} \Phi(e_n)(x)|^2 \, dr$$

$$\Rightarrow \| \|f(t')\|_{H^s} \|_{L^p(\Omega)} \lesssim p^{1/2} \underbrace{\left(\int_0^{t'} (t'-r)^{-2\alpha} dr\right)^{1/2}}_{\lesssim 1 \text{ for } \alpha < 1/2} \underbrace{\left(\sum_{n\in\mathbb{N}} \|\Phi(e_n)\|_{H^s}^2\right)^{1/2}}_{= \|\Phi\|_{HS(L^2;H^s)}}$$

$\Rightarrow$ (**)

□