Lec 3: Nov 03, 2025 (MON)

Prop 4.1.5: $\Phi \in HS(L^2(\mathbb{R}^d); H^s(\mathbb{R}^d))$, $s \in \mathbb{R}$

$1 \le q < \infty$, $2 \le r < \infty$ and $r \le \frac{2d}{d-2}$, $d \ge 3$

($2^* = \frac{2d}{d-2}$, $H^1(\mathbb{R}^d) \subset L^{\frac{2d}{d-2}}(\mathbb{R}^d)$)

Then, $\Psi \in \underbrace{L^q_T W^{s,r}_x(\mathbb{R}^d)}_{\uparrow L^q([-T,T]; W^{s,r}(\mathbb{R}^d))}$, a.s.

for any finite $T > 0$.

de Bouard-Debussche '03
Oh-Pocovnicu-Wang '20

- Wave case: Brun-Li-Liu, JDE'24.

• Strichartz estimates on $\mathbb{R}^d$

See Sec 1.4 from 2024.

We say that (q,r) is admissible if

$$2 \le q, r \le \infty, \quad (q,r,d) \neq (2,\infty,2),$$

$$\boxed{\frac{2}{q} + \frac{d}{r} = \frac{d}{2}}.$$

For $d \ge 3$, $r \le \frac{2d}{d-2}$

① (homog): If (q,r) is admissible, then

$$\| S(t) f \|_{L_t^q(\mathbb{R}; L_x^r(\mathbb{R}^d))} \lesssim \| f \|_{L_x^2}$$

② (dual): $$\| \int_{\mathbb{R}} S(-t) F(t) dt \|_{L_x^2} \lesssim \| F \|_{L_t^{q'} L_x^{r'}}$$

③ (nonhomog): $$\| \int_0^t S(t-t') F(t') dt' \|_{L_t^q L_x^r} \lesssim \| F \|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

(q,r), $(\tilde{q},\tilde{r})$, admissible.

Pf of Prop 4.1.5: $\Psi(t,x) = \int_0^t S(t-t') \Phi \, dW(t')$

$$= \sum_{n \in \mathbb{N}} \underbrace{\int_0^t S(t-t') \Phi(e_n)(x) \, dB_n(t')}_{= \text{Wiener integral}}$$

$\Rightarrow \mathbb{E}\left[|\langle \nabla \rangle^s \Psi(t,x)|^2 \right]$

$$= \sum_{n \in \mathbb{N}} \int_0^t |S(t-t') \langle \nabla \rangle^s \Phi(e_n)(x)|^2 \, dt'$$

$\Rightarrow \underline{\| \langle \nabla \rangle^s \Psi(t,x) \|_{L^p(\Omega)}} \sim p^{1/2} \| \langle \nabla \rangle^s \Psi(t,x) \|_{L^2(\Omega)}$

$$= p^{1/2} \left\| \left(\int_0^t |S(\underbrace{t-t'}_{= \tau}) \langle \nabla \rangle^s \Phi(e_n)(x)|^2 \, dt' \right)^{1/2} \right\|_{\ell^2_n}$$

(*)

$$= \underline{p^{1/2} \| S(\tau) \langle \nabla \rangle^s \Phi(e_n)(x) \|_{\ell^2_n L^2_\tau([0,t])}}$$

$q, r < \infty$. Take $p \geq q \vee r$ $(= \max(q,r))$

$$\Big\| \|\Psi\|_{L_T^q W_x^{s,r}} \Big\|_{L^p(\Omega)} = \Big\| \|\langle\nabla\rangle^s \Psi\|_{L_T^q L_x^r} \Big\|_{L^p(\Omega)}$$

$$\overset{\text{Mink}}{\leq} \Big\| \|\langle\nabla\rangle^s \Psi(t,x)\|_{L^p(\Omega)} \Big\|_{L_T^q L_x^r}$$

$$\overset{\circledast}{\lesssim} p^{1/2} T^{1/q} \Big\| \|S(\tau)\langle\nabla\rangle^s \Phi(e_n)(x)\|_{\ell_n^2 L_\tau^2([0,T])} \Big\|_{L_x^r}$$

$$\overset{\text{Mink}}{\leq} p^{1/2} T^{1/q} \Big\| \|S(\tau)\langle\nabla\rangle^s \Phi(e_n)\|_{L_T^2 L_x^r} \Big\|_{\ell_n^2}$$

Choose $\tilde{q} \geq 2$ s.t. $(\tilde{q}, r)$ is admissible.

$$\overset{\text{Hölder in } \tau}{\lesssim} p^{1/2} T^{\theta} \Big\| \|S(\tau)\langle\nabla\rangle^s \Phi(e_n)\|_{L_T^{\tilde{q}} L_x^r} \Big\|_{\ell_n^2}$$

$$\overset{\text{Str.}}{\lesssim} p^{1/2} T^{\theta} \Big\| \|\Phi(e_n)\|_{H^s} \Big\|_{\ell_n^2} = p^{1/2} T^{\theta} \|\Phi\|_{HS(L^2; H^s)}. \qquad \square$$

Rmk: ① By Chebyshev, we have (see the proof of Lem 1.8.1 in Lec 8 from 2024.)

$$P\left(\|\Psi\|_{L_T^q W_x^{s,r}} > \lambda \right) \le C e^{-c \lambda^2 / T^{2/\theta} \|\Phi\|^2_{HS(L^2;H^s)}}.$$

② On $\mathbb{T}^d$, if Φ is not trans. invariant, we also need to use the Strichartz estimates on $\mathbb{T}^d$ (much weaker than those on $\mathbb{R}^d$, often with a derivative).

Sec 4.2: 1-d SNLS & SNLW

SNLS: $i\partial_t u + \Delta u + |u|^{k-1}u = \Phi\xi, \quad k \in 2\mathbb{N}+1$

SNLW: $\partial_t^2 u + (1-\Delta)u + u^k = \Phi\xi, \quad k \geq 2$, integer.

- **SNLS on $\mathbb{T}^d$ or $\mathbb{R}^d$**: algebra case $s > \frac{d}{2}$

$$\|fg\|_{H^s} \lesssim \|f\|_{H^s}\|g\|_{H^s}, \quad s > \frac{d}{2}$$

$$\Gamma_{u_0,\Phi}(u) = S(t)u_0 + i\int_0^t S(t-t')|u|^{k-1}u(t')\,dt' - i\Psi$$

WTS: $\exists!\ u \in \overline{B_R} \subset C_T H^s_x$ s.t. $u = \Gamma(u)$

$$\|\Gamma[u]\|_{C_T H^s_x} \le \|u_0\|_{H^s} + \left\| \int_0^t S(t-t')|u|^{k-1}u(t')\,dt' \right\|_{C_T H^s_x} \quad (7)$$

Mink
$$\le \sup_{|t|\le T} \int_0^t \underbrace{\|S(t-t')|u|^{k-1}u(t')\|_{H^s_x}}_{\lesssim \|u(t')\|^k_{H^s_x}} dt'$$

$$\le \|u\|^k_{C_T H^s_x}$$

$$+ \|\Psi\|_{C_T H^s_x} \Leftarrow \text{finite a.s. (Prop 4.1.2 \& 4.1.3)}$$

$\Rightarrow$ $0 < T \le 1$.

$$\|\Gamma[u]\|_{C_T H^s_x} \le \|u_0\|_{H^s} + C_1 T \|u\|^k_{C_T H^s_x} + \|\Psi\|_{C_1 H^s_x}$$

$$\le \frac{R}{2} + C_1 T R^k \le R, \quad \forall u \in \overline{B}_R \subset C_T H^s_x \qquad = C([0,1]; H^s_x)$$

by setting $R = 2\left(\|u_0\|_{H^s} + \|\Psi\|_{C_1 H^s_x}\right)$

R is random, a.s. finite

$$T \le \frac{1}{2C_1 R^{k-1}}.$$

$\Rightarrow$ $T > 0$, a.s.

Similarly, we have

$$\| \Gamma(u) - \Gamma(v) \|_{C_T H^s_x} \le C_2 \left(\|u\|^{k-1}_{C_T H^s_x} + \|v\|^{k-1}_{C_T H^s_x} \right) \|u - v\|_{C_T H^s_x}$$

← No u_0 or Ψ.

$$\le \frac{1}{2} \|u - v\|_{C_T H^s_x}, \quad \forall\, u, v \in \overline{B_R}.$$

by setting $T \le \frac{1}{4 C_2 R^{k-1}}$.

$\Rightarrow$ $\Gamma_{u_0, \Psi}$ is a contration in $\overline{B_R} \subset C_T H^s_x$

$\Rightarrow$ By the Banach fixed pt thm, $\exists!\ u \in \overline{B_R}$ s.t. $u = \Gamma(u)$.

i.e. $u(t) = S(t) u_0 + i \int_0^t S(t - t') |u|^k u(t')\, dt' - i \Psi(t)$

on $[0, T_\omega]$, where T_ω is a.s. positive.

Thm 4.2.1: $s > \frac{d}{2}$, $\Phi \in HS(L^2; H^s)$.
Then, SNLS is locally well-posed in $H^s(M)$, $M = \mathbb{T}^d$ or $\mathbb{R}^d$.

Rmk: · The proof gives uniqueness of the solution only in $\overline{B_R} \subset C_T H^s_x$

⇐ By a standard continuity argument, we can upgrade uniqueness to hold in the entire class $C_T H^s_x$ (unconditional uniqueness).

· The question of uncond. uniqueness for stoch. disp. PDEs in the low regularity setting is mostly open

Oh - Sosoe - Wang '25

1-d SNLW on $\mathbb{T}$: $\partial_t^2 u + (1-\Delta)u + u^k = \xi$, $\Phi = Id$.

$$S(t) = \frac{\sin(t\langle\nabla\rangle)}{\langle\nabla\rangle}$$

Duhamel formulation (= mild formulation):

$$u(t) = \partial_t S(t) u_0 + S(t) u_1 - \int_0^t S(t-t') u^k(t') dt' + \Psi(t)$$

where $\Psi(t) = \Psi_{wave}(t) = \int_0^t S(t-t') dW(t')$

$$= \int_0^t \frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle} dW(t') \in C_t H^{\frac{1}{2}-}_x(\mathbb{T})$$

a.s.

$C_t W^{\frac{1}{2}-,\infty}_x(\mathbb{T})$.

Thm 4.22: $s \geq \frac{1}{2} - \frac{1}{2k}$.

Then, 1-d SNLW on $\mathbb{T}$ with an additive space-time white noise is locally well-posed in $\mathcal{H}^s(\mathbb{T}) := H^s(\mathbb{T}) \times H^{s-1}(\mathbb{T})$

$\ni (u_0, u_1)$.

$\vec{u} = (u, \partial_t u)$.

$$\Gamma(\vec{u}) = \underbrace{\partial_t S(t) u_0 + S(t) u_1}_{= \cos(t\langle\nabla\rangle) u_0 + \frac{\sin(t\langle\nabla\rangle)}{\langle\nabla\rangle} u_1} - \int_0^t S(t-t') u^k(t') dt' + \Psi(t)$$

Set $\vec{\Gamma}(\vec{u}) = (\Gamma(\vec{u}), \partial_t \Gamma(\vec{u}))$.

$$\Rightarrow \|\vec{\Gamma}(\vec{u})\|_{C_T \mathcal{H}^s_x} \leq \|(u_0, u_1)\|_{\mathcal{H}^s} + \int_0^T \underbrace{\|u^k(t')\|_{H^{s-1}_x}}_{\| u^k(t')\|_{L^2_x} = \|u(t')\|^k_{L^{2k}_x} \overset{\text{Sob}}{\lesssim} \|u(t)\|^k_{H^s_x}} dt' + \underbrace{\|\vec{\Psi}\|_{C_1 \mathcal{H}^s_x}}_{< \infty, \text{ a.s.}}$$

$0 < T \leq 1$

$\frac{1}{2} - \frac{1}{2k} \leq s < \frac{1}{2}$

$s \geq \frac{1}{2} - \frac{1}{2k}$

$$\leq \underbrace{\left(\|(u_0, u_1)\|_{\mathcal{H}^s} + \|\vec{\Psi}\|_{C_1 \mathcal{H}^s_x} \right)}_{=: \frac{R}{2}} + \underbrace{C_1 T \|\vec{u}\|^k_{C_T \mathcal{H}^s_x}}_{\leq \frac{R}{2}, \ \forall \ \vec{u} \in \overline{B_R} \subset C_T \mathcal{H}^s_x}$$

$$\leq R.$$

for $T \leq \frac{1}{2 C_1 R^{k-1}}$

A similar computation yields

$$\|\vec{\Gamma}(\vec{u}) - \vec{\Gamma}(\vec{v})\|_{C_T\mathcal{H}^s_x} \le C_2 T \left(\|\vec{u}\|^{k-1}_{C_T\mathcal{H}^s_x} + \|\vec{v}\|^{k-1}_{C_T\mathcal{H}^s_x} \right) \|\vec{u} - \vec{v}\|_{C_T\mathcal{H}^s_x}$$

$$\le \frac{1}{2} \|\vec{u} - \vec{v}\|_{C_T\mathcal{H}^s_x}$$

$\forall\ \vec{u}, \vec{v} \in \overline{B_R}$, by choosing $T \le \frac{1}{4C_2R^{k-1}}$.

$\Rightarrow \vec{\Gamma}$ is a contraction in $\overline{B_R} \subset C_T\mathcal{H}^s_x$, $T = T_\omega > 0$, a.s.

□

Sobolev's ineq: $1 < p \le q < \infty$. $s \ge \frac{1}{p} - \frac{1}{q}$

$$\|f\|_{W^{s,q}} \lesssim \|f\|_{L^p}$$

· 1-d cubic SNLS on $\mathbb{R}$:

$$i\partial_t u + \partial_x^2 u + |u|^2 u = \Phi \xi$$

Thm 4.2.3: $s \geq 0$. $\Phi \in HS(L^2; H^s)$

Then, 1-d cubic SNLS on $\mathbb{R}$ is locally well-posed in $H^s(\mathbb{R})$.

Pf: We only consider $s = 0$.

· For $s > 0$, we can easily modify the proof, using the fractional Leibniz rule. (see Bényi-Oh-Zhao '25).

$$\Gamma(u) = S(t) u_0 + i \int_0^t S(t-t') |u|^2 u(t') dt' - i \Psi(t).$$

We need to use the Strichartz estimates.

- $(q,r) = (\infty, 2)$, $(8,4)$ is admissible

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad d = 1.$$

Set $X(T) = C_T L^2_x \cap L^8_T L^4_x$.

$$\|\Gamma(u)\|_{X(T)} \overset{\text{Strichartz}}{\lesssim} \|u_0\|_{L^2} + \|\,|u|^2 u\,\|_{L^{\frac{8}{7}}_T L^{4/3}_x} \lesssim T^{1/2} \|u\|_{L^8_T L^4_x}$$

$x: \frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$

$t: \frac{7}{8} = \frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$

$0 < T \le 1.$

$$+ \underbrace{\|\Phi\|_{X(1)}}_{< \infty, \text{ a.s. (Prop 4.1.3 \& 4.1.5)}}$$

$$\lesssim \left(\|u_0\|_{L^2} + \|\Phi\|_{X(1)} \right) + T^{1/2} \|u\|^3_{X(1)}$$

- Together with a difference estimate, we conclude LWP.

□

Rmk: Uniqueness holds only in $\overline{B_R} \subset C_T L^2_x \cap L^8_T L^4_x$.

$\Leftarrow$ conditional uniqueness.

- Next time, 1-d cubic NLS on $\mathbb{T}$.
  - Fourier restriction norm method ($X^{s,b}$-space).
- Lec 4 on Nov. 13 (THU).