Lec 4 : Nov. 13, 2025 (THU)

- 1-d cubic SNLS on $\mathbb{T}$

$$\begin{cases} i\partial_t u + \partial_x^2 u + |u|^2 u = \Phi\xi \\ u|_{t=0} = u_0 \in H^s(\mathbb{T}) \end{cases}, \qquad \Phi \in HS(L^2; H^s)$$

trans. inv.

$\Phi(e_n) = \phi_n e_n$

- Main difficulty: lack of (good) Strichartz estimates
  - Strichartz estimates on $\mathbb{T}^d$: $\|S(t)f\|_{L_t^q L_x^r} \lesssim \|f\|_{L_x^2}$?

$d=1$: $f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{inx}$

$$\Rightarrow S(t)f(x) = \sum_{n\in\mathbb{Z}} e^{-it n^2} \hat{f}(n) e^{inx}$$ ← periodic in time

  - Strichartz estimates on $\mathbb{T}^d$: only locally in time.
  - not as good as those on $\mathbb{R}^d$: need $q=r$.
  - Zygmund '74, Bourgain '93, '13 ← analytic number theory (HL circle method
    Bourgain-Demeter '15, Killip-Vişan '16.

For KdV : $\partial_t u + \partial_x^3 u + \partial_x (u^2) = 0$

'80 ~ Kenig-Ponce-Vega, oscillatory integral approach (on $\mathbb{R}$)

KdV: LWP in $H^s(\mathbb{R})$, $s > \frac{3}{4}$ (not on $\mathbb{T}$)

· Fourier restriction norm method: Bourgain '93

$X^{s,b}$ - spaces

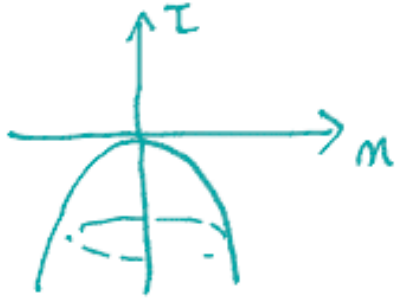
(wave: Klainerman-Machedon '93 hyperbolic Sobolev space

$$\|u\|_{X^{s,b}} = \| \langle n \rangle^s \langle \tau + |n|^2 \rangle^b \hat{u}(\tau, n) \|_{l_n^2 L_\tau^2 (\mathbb{Z}^d \times \mathbb{R})}$$

space-time F.T.

Linear Sch. eqn: $i\partial_t u + \Delta u = 0 \Rightarrow -(\tau + |n|^2)\hat{u}(\tau, n) = 0$

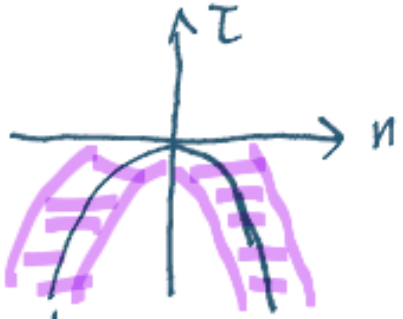
measure supp on the paraboloid $\tau + |n|^2 = 0$



· The weight $\underline{\langle \tau + |n|^2 \rangle^b}$ $(b > 0)$ "penalizes" functions whose Fourier support is away from $\tau + |n|^2 = 0$

$\Rightarrow$ In running a contraction argument in $\overline{B}_R \subset X^{s,b}$, we do NOT see functions whose Fourier supp is away from $\tau + |n|^2 = 0$.

i.e. $u \in \overline{B}_R \subset X^{s,b}$ $\rightsquigarrow$ u is close to being a linear soln



· suitable for perturbative study:

deterministic NLS

$$u(t) = S(t)u_0 + \underbrace{i\int_0^t S(t-t')|u|^2 u(t')\,dt'}_{\text{"perturbation"}}$$

- The Fourier restriction norm method has been effective in the low regularity study of nonlinear dispersive PDEs.

ex: Bourgain '93

- 1-d cubic NLS on $\mathbb{T}$: LWP in $L^2(\mathbb{T})$. ← improved from $s > 1/2$
- KdV on $M = \mathbb{R}$ or $\mathbb{T}$: LWP in $L^2(M)$ ← $s > \frac{3}{4}$ on $\mathbb{R}$; $s > \frac{3}{2}$ on $\mathbb{T}$

  (Kenig-Ponce-Vega: LWP in $H^{-\frac{3}{4}+}(\mathbb{R})$
  in $H^{-\frac{1}{2}}(\mathbb{T})$ Colliander-Keel-Staffilani-Takaoka-Tao '03

- Basic properties: see the course notes from 2024

① interaction representation: $v(t) = S(-t)u(t)$

$$\Rightarrow \|u\|_{X^{s,b}} = \|\langle n\rangle^s \langle \tau + |n|^2\rangle^b \hat{u}(\tau,n)\|_{l^2_n L^2_\tau}$$

$$= \|\langle n\rangle^s \langle \tau\rangle^b \hat{v}(\tau,n)\|_{l^2_n L^2_\tau} = \|\langle \nabla\rangle^s \langle \partial_t\rangle^b v\|_{L^2_{t,x}}$$

② $b > 1/2$: $X^{s,b} \subset C_t H^s_x$

↖ essentially — Riemann-Lebesgue lemma

$H^b_t \subset \mathcal{F}^{-1}_t(L^1) \subset C_0$ (for $b > \frac{1}{2}$)

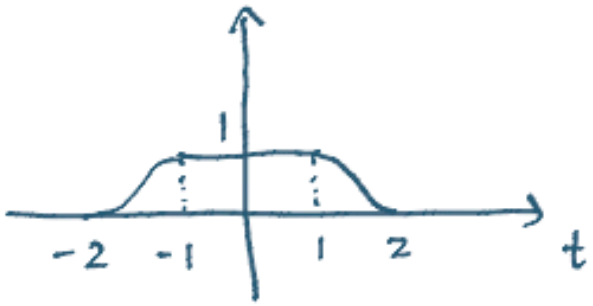
③ $\forall s, b \in \mathbb{R}$

$$\| \eta(t) S(t) f \|_{X^{s,b}} \lesssim \| f \|_{H^s}$$

↑ smooth cutoff function in time

$\eta \equiv 1$ on $[-1,1]$

$\operatorname{supp} \eta \subset [-2,2]$



· Pf: $\mathcal{F}_x(S(t)f)(n) = e^{-it|n|^2} \hat{f}(n) \Rightarrow \widehat{\eta(t) S(t) f}(\tau, n) = \hat{\eta}(\tau + |n|^2) \hat{f}(n)$

$$\Rightarrow \| \eta(t) S(t) f \|_{X^{s,b}} = \| \langle n \rangle^s \langle \tau + |n|^2 \rangle^b \hat{\eta}(\tau + |n|^2) \hat{f}(n) \|_{l^2_n L^2_\tau}$$

$$= \| \eta \|_{H^b_t} \| f \|_{H^s_x}$$

Q: $S(t)f \in X^{s,b}$? (without a cutoff in time.)

Ans: **NO!!**

$$\widehat{S(t)f}(\tau, n) = \delta(\tau + |n|^2)\,\hat{f}(n)$$

(Dirac delta)

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④ $$\left\| \eta(t) \int_0^t S(t-t') F(t')\,dt' \right\|_{X^{s,b}} \lesssim \|F\|_{X^{s,b-1}}$$

$\forall\, s \in \mathbb{R}$

but $b > 1/2$

$0 < T \le 1$ $$\left\| \eta\left(\tfrac{t}{T}\right) \int_0^t S(t-t') F(t')\,dt' \right\|_{X^{s,b}} \lesssim T^{\varepsilon} \|F\|_{X^{s,b-1+\varepsilon}}$$

(5) Transference:

If $\| S(t) f \|_{L_t^q L_x^r} \lesssim \| f \|_{H_x^s}$,

then $\| u \|_{L_t^q L_x^r} \lesssim \| u \|_{X^{s,b}}$, $b > 1/2$

(6) periodic L^4-Strichartz estimate on $\mathbb{T}$.

- Zygmund '74: $\| \eta(t) S(t) f \|_{L^4_{t,x}} \lesssim \| f \|_{L^2}$

$$\xRightarrow{\text{transf}} \quad \| \eta(t) u \|_{L^4_{t,x}} \lesssim \| u \|_{X^{0,b}}, \quad b > 1/2$$

- Bourgain '93: $\| u \|_{L^4_{t,x}} \lesssim \| u \|_{X^{0,\frac{3}{8}}}$

See Lec 12 from 2024.

$$\|\Phi u\|_{X^{s,b}} \lesssim \|u\|_{X^{s,b}} \qquad \forall s,b \in \mathbb{R}$$

Φ, smooth cutoff in time

$$\|\mathbb{1}^{(t)}_{[t_1,t_2]} u\|_{X^{s,b}} \lesssim \|u\|_{X^{s,b}} \qquad -\tfrac{1}{2} < b < \tfrac{1}{2}$$

$$\left(\Leftarrow \|\mathbb{1}^{(t)}_{[t_1,t_2]} f(t)\|_{H^b_t} \lesssim \|f\|_{H^b_t} \right.$$

Prop 4.2.4: $\Psi(t) = \int_0^t S(t-t')\Phi\, dW(t')$, $\Phi \in HS(L^2;H^s)$ trans. inv.

on $\mathbb{T}^d$

$$\left\| \|\mathbb{1}_{[0,T]}\Psi\|_{X^{s,b}} \right\|_{L^p(\Omega)} \lesssim_T p^{1/2} \|\Phi\|_{HS(L^2;H^s)}$$

$$\left(\text{Chebyshev} \Rightarrow P\left(\|\mathbb{1}_{[0,T]}\Psi\|_{X^{s,b}} > \lambda \right) \le C e^{-c_T \frac{\lambda^2}{\|\Phi\|^2_{HS(L^2;H^s)}}} \right.$$

Pf: Let $\widetilde{\Psi}(t) = S(-t)\Psi(t)$.

$$\widehat{\widetilde{\Psi}}(t) = \int_0^t S(-t')\Phi\, dW(t')$$

$$= \sum_{n\in\mathbb{Z}^d} \Phi_n e_n \int_0^t e^{it'|n|^2} dB_n(t')$$

F.T. in x $\Rightarrow$

$$\widehat{\widetilde{\Psi}}(t,n) = \Phi_n \int_0^t e^{it'|n|^2} dB_n(t')$$

F.T. int $\Rightarrow$

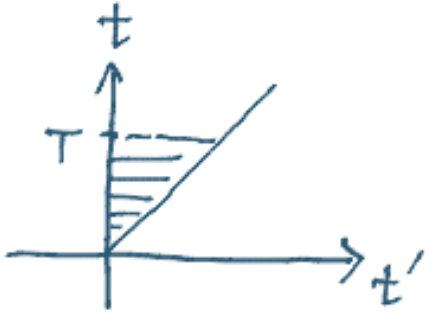
$$\widehat{\mathbb{1}_{[0,T]}\cdot\widetilde{\Psi}}(\tau,n) = \Phi_n \int_{\mathbb{R}} e^{-it\tau}\, \mathbb{1}_{[0,T]}(t) \int_0^t e^{it'|n|^2} dB_n(t')\, dt.$$

Stoch. Fubini

$$= \Phi_n \int_0^T e^{it'|n|^2} \int_{t'}^T e^{-it\tau} dt\, dB_n(t')$$

Φ_n: Wiener integral

$\int_{t'}^T e^{-it\tau} dt$: $\lesssim_T \min\left(1, \frac{1}{|\tau|}\right) \sim \frac{1}{\langle\tau\rangle}$



$$\Rightarrow \quad \| \widehat{\mathbb{1}_{[0,T]} \cdot \Phi}(\tau, n) \|_{L^p(\Omega)} \sim p^{1/2} \| \cdots \|_{L^2(\Omega)}$$

(*)

$$\lesssim p^{1/2} \left(T^{3/2} \vee T^{1/2} \right) \frac{|\Phi_n|}{\langle \tau \rangle}$$

$$\overset{p \geq 2}{\Rightarrow} \left\| \| \mathbb{1}_{[0,T]} \cdot \Phi \|_{X^{s,b}} \right\|_{L^p(\Omega)} = \left\| \| \langle n \rangle^s \langle \tau \rangle^b \widehat{\mathbb{1}_{[0,T]} \cdot \Phi}(\tau, n) \|_{l^2_n L^2_\tau} \right\|_{L^p(\Omega)}$$

$$\overset{\text{Mink}}{\leq} \left\| \langle n \rangle^s \langle \tau \rangle^b \| \widehat{\mathbb{1}_{[0,T]} \cdot \Phi}(\tau, n) \|_{L^p(\Omega)} \right\|_{l^2_n L^2_\tau}$$

(*)

$$\lesssim C_T p^{1/2} \underbrace{\| \langle n \rangle^s \Phi_n \|_{l^2_n}}_{= \| \Phi \|_{HS(L^2; H^s)}} \underbrace{\| \langle \tau \rangle^{b-1} \|_{L^2_\tau}}_{< \infty \text{ iff } b < 1/2}$$

□

Thm 4.2.5: Let $s \geq 0$.

Then, 1-d cubic SNLS is locally well-posed in $H^s(\mathbb{T})$.

Pf: Duhamel formulation (= mild formulation)

$$u(t) = \eta(t) S(t) u_0 + i \eta\left(\frac{t}{T}\right) \int_0^t S(t-t') |u|^2 u(t') dt' - i \mathbb{1}_{[0,1]}(t) \cdot \Psi(t)$$

$0 < T \leq 1$.

Set $\Gamma(u) = \text{RHS}$.

- $\eta(t) = \eta\left(\frac{t}{T}\right) = \mathbb{1}_{[0,1]}(t) = 1$ on $[0, T]$.

- We only consider the case $s = 0$.

$$\|\Gamma(u)\|_{X^{0,\frac{1}{2}-\varepsilon}} \overset{(3)}{\leq} c_1 \|u_0\|_{L^2} + \underbrace{\left\| \eta\left(\frac{t}{T}\right) \int_0^t \cdots dt' \right\|_{X^{0,\frac{1}{2}+\varepsilon}}}_{\lesssim T^{\varepsilon} \| |u|^2 u \|_{X^{0,-\frac{1}{2}+2\varepsilon}} \ (4)} + \underbrace{\| \mathbb{1}_{[0,1]}(t) \cdot \Psi \|_{X^{0,\frac{1}{2}-\varepsilon}}}_{\text{Prop 4.24} \ \leq C_\omega < \infty, \text{ a.s.}}$$

$$\| \eta_b(\tfrac{t}{T}) \int_0^t S(t-t') |u|^2 u(t') \, dt' \|_{X^{0,\frac{1}{2}+\varepsilon}}$$

④

$$\lesssim T^{\varepsilon} \| |u|^2 u \|_{X^{0,-\frac{1}{2}+2\varepsilon}}$$

$(X^{s,b})' = X^{-s,-b}$ see page 17

duality

$$= T^{\varepsilon} \sup_{\|v\|_{X^{0,\frac{1}{2}-2\varepsilon}}=1} \underbrace{\left| \int u \, \bar{u} \, u \, \bar{v} \, dx dt \right|}$$

$$\leq \|u\|^3_{L^4_{t,x}} \|v\|_{L^4_{t,x}}$$

L^4- Strichartz

$$\lesssim \|u\|^3_{X^{0,\frac{3}{8}}} \underbrace{\|v\|_{X^{0,\frac{3}{8}}}}$$

⑥

$$\leq \|v\|_{X^{0,\frac{1}{2}-2\varepsilon}} = 1$$

$$\lesssim T^{\varepsilon} \|u\|^3_{X^{0,\frac{3}{8}}}$$

$$\Rightarrow \|\Gamma(u)\|_{X^{0,\frac{1}{2}-\varepsilon}} \le \underbrace{\left(C_1 \|u_0\|_{L^2} + \|\mathbb{1}_{[0,1]}\cdot \Phi\|_{X^{0,\frac{1}{2}-\varepsilon}}\right)}_{=:\frac{1}{2}R \text{ ← random}} + C_2 T^{\varepsilon} \|u\|^3_{X^{0,\frac{3}{8}}}$$

$$\le \frac{1}{2}R + C_2 T^{\varepsilon} R^3 \le R \quad \text{if } T = T(R) > 0 \text{ is suff. small}$$

$R < \infty$, a.s. ⇒ $T > 0$, a.s.

$\forall\ u \in \overline{B_R} \subset X^{0,\frac{1}{2}-\varepsilon}$.

A similar computation yields the diffence estime.

$$\Rightarrow \exists!\ u \in \overline{B_R} \subset X^{0,\frac{1}{2}-\varepsilon} \text{ s.t. } u = \Gamma(u)$$

$\Rightarrow u$ is a soln to SNLS with $u|_{t=0} = u_0$ on the time interval $[0, T_\omega]$.

↑ a.s. positive.

· Need to show $u \in C_T L^2_x$

$$u(t) = S(t)u_0 + i\underbrace{\eta\left(\frac{t}{T}\right)\int_0^t S(t-t')|u|^2 u(t')\,dt'}_{\in X^{0,\frac{1}{2}+\varepsilon} \subset C_t L^2_x} - i\Psi$$

$S(t)u_0 \in C_t L^2_x$ on $[0,T]$; $\Psi \in C_t L^2_x$

$X^{0,\frac{1}{2}+\varepsilon}$: $\frac{1}{2}+\varepsilon > \frac{1}{2}$ ②

$\Rightarrow u \in C_T L^2_x$, a.s.

□

$\Psi \in HS(L^2; L^2)$, trans inv.

Thm 4.2.6: 1-d cubic SNLS on $\mathbb{T}$ is globally well-posed in $L^2(\mathbb{T})$.

Ito calculus approach

Pf: $i\partial_t u + \partial_x^2 u + |u|^2 u = \Phi\xi$

· $\underline{\Phi \equiv 0}$. i.e. deterministic cubic NLS.

Mass $M(u) = \int |u|^2 dx$ is conserved under NLS

$$\partial_t M(u) = 2\,\mathrm{Re} \int \partial_t u \cdot \bar{u}\, dx$$

$$= 2\mathrm{Re}\, i \underbrace{\int \partial_x^2 u \cdot \bar{u}\, dx}_{\overset{\text{IBP}}{=} \int |\partial_x u|^2 dx} + 2\mathrm{Re}\, i \underbrace{\int |u|^2 u \cdot \bar{u}\, dx}_{= \int |u|^4 dx}$$

$$= 0.$$

$\Rightarrow$ By iteratively applying the LWP argument,
we obtain GWP of deterministic NLS in $L^2(\mathbb{T})$.

For SNLS, we use Ito's lemma on

$$M(u) = \int |u|^2 dx = \sum_{n\in\mathbb{Z}} (p_n^2 + q_n^2)$$

$p_n = \mathrm{Re}\,\hat{u}(n)$

$q_n = \mathrm{Im}\,\hat{u}(n)$

- Ito's lemma: $dX = f\,dt + g\,dB$ $\quad B = BM$

Consider $F(X_t)$

Then, $dF(X) = \partial_x F(X)\,dX + \frac{1}{2}\partial_x^2 F(X)(dX)^2$

$= \partial_x F(X)(f\,dt + g\,dB)$ $\quad (dt)^2 = 0$

$+ \frac{1}{2}\partial_x^2 F(X)\, g^2\,dt$ $\quad dt\,dB = dB\,dt = 0$

$(dB)^2 = dt$

- $F(X_{t_2}) - F(X_{t_1}) = \int_{t_1}^{t_2} \partial_x F(X_t) f_t + \frac{1}{2}\partial_x^2 F(X_t) g_t^2\,dt$

$+ \int_{t_1}^{t_2} \partial_x F(X_t)\, g_t\, dB_t$

Appendix : On the duality of the $X^{s,b}$ - spaces

- $u \in X^{s,b}$

$$\int u \bar{v}\, dx dt = \int S(-t)u(t) \cdot \overline{S(-t)v(t)}\, dx dt$$

$U(t) = S(-t)u(t)$

$V(t) = S(-t)v(t)$

$$\leq \| \langle \nabla \rangle^{s} \langle \partial_t \rangle^{b} U \|_{L^2_{t,x}} \, \| \langle \nabla \rangle^{-s} \langle \partial_t \rangle^{-b} V \|_{L^2_{t,x}}$$

$$= \| u \|_{X^{s,b}} \, \| v \|_{X^{-s,-b}}$$

$$\Rightarrow \quad \| u \|_{X^{s,b}} = \sup_{\| v \|_{X^{-s,-b}} = 1} \left| \int u \bar{v}\, dx dt \right|$$

It is in this sense that we wrote $(X^{s,b})' = X^{-s,-b}$ on page 12

Note that the usual duality pairing comes without the conjugate sign.

$$\int u v\, dx dt$$

Under the standard duality pairing $\int uv\,dxdt$,

↑ NO conjugate

we have $$\left(X^{s,b}_{\tau+|n|^2}\right)' = X^{-s,-b}_{\tau-|n|^2}$$

↑ really comes from $\tau = -|-n|^2$

(⇐ follows from page 17 and $\|\overline{v}\|_{X^{s,b}_{\tau+|n|^2}} = \|v\|_{X^{s,b}_{\tau-|-n|^2}}$