Lec 5: Nov 20, 2025 (THU)

Last time, we started to discuss

$\Phi \in HS(L^2; L^2)$, trans inv.

Thm 4.2.b: 1-d cubic SNLS on $\mathbb{T}$ is globally well-posed in $L^2(\mathbb{T})$.

$$M(u) = \int |u|^2 dx = \sum_{n\in\mathbb{Z}} (p_n^2 + q_n^2)$$

$p_n = \mathrm{Re}\,\hat{u}(n)$

$q_n = \mathrm{Im}\,\hat{u}(n)$

SNLS: $i\partial_t u + \partial_x^2 u + |u|^2 u = \Phi\xi$.

$$\Rightarrow \quad d\hat{u}(n) = \left(-i n^2 \hat{u}(n) + i \mathcal{F}_x(|u|^2 u)(n)\right) dt - i \Phi_n \, dB_n$$

- $dp_n = \left(n^2 q_n - \mathrm{Im}\,\mathcal{F}_x(|u|^2u)(n)\right) dt + \underbrace{\mathrm{Im}\,(\Phi_n dB_n)}_{= \mathrm{Im}\Phi_n \, d(\mathrm{Re}B_n) + \mathrm{Re}\Phi_n \, d(\mathrm{Im}B_n)}$

- $dq_n = \left(-n^2 p_n + \mathrm{Re}\,\mathcal{F}_x(|u|^2u)(n)\right) dt$

$$\underbrace{-\mathrm{Re}\,(\Phi_n dB_n)}_{= -\mathrm{Re}\Phi_n \, d(\mathrm{Re}B_n) + \mathrm{Im}\Phi_n \, d(\mathrm{Im}B_n)}$$

Apply Ito's lemma to $M(u) = \sum_{n\in\mathbb{Z}} (p_n^2 + q_n^2)$.

$\frac{\partial}{\partial q_n}\frac{\partial}{\partial p_n} p_n^2 = 0$, $\frac{\partial}{\partial p_n}\frac{\partial}{\partial q_n} q_n^2 = 0$ ⟹ NO $dp_n dq_n$ (2)

$$dM(u) \overset{Ito}{=} \underbrace{2\sum (p_n\, dp_n + q_n\, dq_n)} + \underbrace{\sum \left((dp_n)^2 + (dq_n)^2\right) dt}_{= 2\|\Phi\|^2_{HS(L^2;L^2)}\, dt}$$

$$= 2\sum \Big[p_n \big(\mathrm{Im}\,\Phi_n\, d(\mathrm{Re}\,B_n) + \mathrm{Re}\,\Phi_n\, d(\mathrm{Im}\,B_n) \big) + q_n \big(-\mathrm{Re}\,\Phi_n\, d(\mathrm{Re}\,B_n) + \mathrm{Im}\,\Phi_n\, d(\mathrm{Im}\,B_n) \big) \Big]$$

Apply BDG ineq.

- $\sum_n (dp_n)^2 = \sum_n \left((\mathrm{Im}\,\Phi_n)^2 + (\mathrm{Re}\,\Phi_n)^2 \right) = \|\Phi\|^2_{HS(L^2;L^2)}$
- $\sum_n (dq_n)^2 = \sum_n \left((\mathrm{Re}\,\Phi_n)^2 + (\mathrm{Im}\,\Phi_n)^2 \right) = \|\Phi\|^2_{HS(L^2;L^2)}$

$d(\mathrm{Re}\,B_n)\, d(\mathrm{Im}\,B_n) = 0$ (indep.)

Note: $dp_n\, dq_n = \mathrm{Im}\,\Phi_n \cdot (-\mathrm{Re}\,\Phi_n) + \mathrm{Re}\,\Phi_n \cdot \mathrm{Im}\,\Phi_n = 0$.

⟸ We do not need it here.

Burkholder – Davis – Gundy inequality: X, (local) martingale

course notes from 2022
Lec 10–11.
Book: Karatzas–Shreve
Thm 3.28 on p. 166

$1 \le p < \infty$

$$\mathbb{E}\Big[\sup_{0\le t\le T} |X_t|^p\Big] \sim \mathbb{E}\big[\langle X_t\rangle_{[0,T]}^{p/2}\big]$$

↑ quadratic variation

For an Itô process $dX = f\,dt + g\,dB$,

$$\langle X_t\rangle_{[0,T]} = \int_0^T g^2\,dt$$

ex: $2\sum_n p_n \operatorname{Im}\Phi_n\, d(\operatorname{Re} B_n)$

$$\Rightarrow \text{quadratic variation} = 4\int_0^T \sum_n p_n^2(t)\,(\operatorname{Im}\Phi_n)^2\,dt$$

$\sum a_n b_n \le \sum a_n \sum b_n$ for $a_n, b_n \ge 0$

$M(u(0)) + \int_0^t dM(u(t'))$ $\qquad \le \|u(t)\|_{L^2}^2 \,\|\Phi\|_{HS(L^2;L^2)}^2$

$$\Rightarrow \mathbb{E}\Big[\sup_{0\le t\le T} M(u(t))\Big] \le \mathbb{E}\big[M(u(0))\big] + \mathbb{E}\Big[\sup_{0\le t\le T}\int_0^t dM(u(t'))\Big]$$

(*)
$$\le \mathbb{E}\big[M(u(0))\big] + CT^{1/2}\,\mathbb{E}\Big[\sup_{0\le t\le T} M(u(t))^{1/2}\cdot\|\Phi\|_{HS(L^2;L^2)}\Big]$$

$$\underset{\text{BDG}}{\sim} \mathbb{E}\big[\langle M(u)\rangle_{[0,T]}^{1/2}\big]$$

$$\cdot\quad T^{1/2}\,\mathbb{E}\Big[\sup_{0\le t\le T} M^{1/2}(u(t))\cdot\|\Phi\|_{HS(L^2;L^2)}\Big]$$

Cauchy's ineq.

$$ab \le \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2$$

$$\lesssim \varepsilon\,\mathbb{E}\Big[\sup_{0\le t\le T} M(u(t))\Big] + \frac{1}{\varepsilon}\,T\,\|\Phi\|^2_{HS(L^2;L^2)}.$$

↑ hide in LHS of ⊛

$$\Rightarrow\quad \mathbb{E}\Big[\sup_{0\le t\le T} M(u(t))\Big] \lesssim \mathbb{E}\big[M(u(0))\big] + T\,\|\Phi\|^2_{HS(L^2;L^2)}$$

$\Rightarrow$ The L^2-norm of $u(t)$ is a.s. finite on $[0,T]$. $\forall\, T \gg 1$

Thus, by iterating the LWP argument (pathwise), we obtain GWP of SNLS in $L^2(\mathbb{T})$.

SNLS: $i\partial_t u + \partial_x^2 u + |u|^2 u = \Phi\xi$

$$\text{SNLS}_N \begin{cases} i\partial_t u_N + \partial_x^2 u_N + \pi_N\left(|u_N|^2 u_N\right) = \pi_N \Phi\xi \\ u_N|_{t=0} = \pi_N u_0 \end{cases}$$

π_N = freq proj onto $\{|n|\le N\}$

- On the Fourier side, SNLS_N is a finite dim'l system of SDEs $(p_n, q_n, |n|\le N)$.

$\Leftarrow$ We can apply Ito's lemma to SNLS_N for each fixed N

- Given $T \gg 1$, in applying BDG ineq, we do not know (a priori) if $u_N(\omega)$ exists on $[0,T]$. $\Rightarrow$ Use a stopping time argument
  - Fix a target time $T \gg 1$. Let τ_N be a stopping time

de Bouard - Debussche '03
Oh - Okamoto '20.

$0 < \tau_N \le \min\left(T, T^N_{\max}\right)$, $T^N_{\max} = T^N_{\max}(\omega)$ = maximal time of existence (forward) for u_N.

Then, apply BDG ineq with $\sup_{0\le t\le \tau_N}$ (instead of $\sup_{0\le t\le T}$

$$\Rightarrow \mathbb{E}\Big[\sup_{0\le t\le \tau_N} M(u_N(t))\Big] \lesssim \mathbb{E}[M(u(0))] + T\|\Phi\|^2_{HS(L^2;L^2)}.$$

$u(0)$: not u_N — uniformly in $N\in\mathbb{N}$.

$\Rightarrow$ $u_N(\omega)$ exists up to time $\tau_N(\omega)$

τ: any stopping time

$\Rightarrow$ u_N exists a.s. on $[0,T]$.

- By a slight modification of the LWP argument,

$$\|u-u_N\|_{X^{0,\frac12-\varepsilon}\ \text{local in time}\ \cap\, C_T L^2_x} \to 0 \quad \text{a.s., as } N\to\infty.$$

$\Rightarrow$ existence of u on $[0,T]$, a.s.

$\Rightarrow$ GWP of u in $L^2(\mathbb{T})$. $\square$

· 1-d SNLW: $\partial_t^2 u + (1-\partial_x^2) u \underbrace{+}_{\text{defocusing}} u^3 = \xi$

LWP in $\mathcal{H}^s(\mathbb{T})$, $s \geq \frac{1}{2} - \frac{1}{2k}$ (Thm 4.2.2 from Lec 3)

Thm 4.2.7: 1-d defocusing cubic SNLW with additive space-time white noise is globally well-posed in $\mathcal{H}^1(\mathbb{T})$.

Pf: pathwise approach (= adaption of a deterministic approach by viewing SNLW as a perturbed NLW

$$\Psi = \int_0^t \frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle} \xi(dt') \in C_t W^{s,\infty}(\mathbb{T}), \text{ a.s. } s < \frac{1}{2}$$

First order expansion: $u = \Psi + v$

$$\Rightarrow \partial_t^2 v + (1-\partial_x^2) v + \underbrace{(v+\Psi)^3}_{= v^3 + \underbrace{3v^2\Psi + 3v\Psi^2 + \Psi^3}_{\text{perturbation}}} = 0$$

$$(v, \partial_t v)\big|_{t=0} = (u_0, u_1) \in \mathcal{H}^1$$

$$v(t) = W(t)(u_0, u_1) - \int_0^t \frac{\sin((t-t')\langle\nabla\rangle}{\langle\nabla\rangle} (v+\Psi)^3(t)\, dt' \qquad (8)$$

$$\Rightarrow \|\vec{v}\|_{C_T\mathcal{N}^1_x} \lesssim \|(u_0,u_1)\|_{\mathcal{N}^1} + T \underbrace{\|(v+\Psi)^3\|_{C_T L^2_x}}_{\text{Hölder}}$$

$\vec{v} = (v, \partial_t v)$

$$\underset{\text{Young}}{\lesssim} \|v\|^3_{C_T L^6_x} + \underbrace{\|\Psi\|^3_{C_T L^6_x}}_{\leq C_{\omega} < \infty}$$

$$\underset{\text{Sob}}{\lesssim} \|v\|^3_{C_T \mathcal{N}^1_x}$$

$\Rightarrow$ LWP for $\vec{v} \in C_T\mathcal{N}^1_x$.

Goal: GWP for $\vec{v}$ in $C(\mathbb{R}_+; \mathcal{N}^1)$.

$\Leftarrow$ Suffices to show

$$\sup_{0\leq t\leq T} \|\vec{v}(t)\|_{\mathcal{N}^1_x} \leq C(\omega; T) < \infty \text{ , a.s. for each given } T \gg 1.$$

(Then, we can iterate the LWP argument for $\vec{v}$.)

Energy (= Hamiltonian) for the (deterministic) def. cubic NLW:

$$E(\vec{v}) = \frac{1}{2}\int |\langle\nabla\rangle v|^2\,dx + \frac{1}{2}\int (\partial_t v)^2\,dx + \frac{1}{4}\int v^4\,dx.$$

- conserved under NLW.
- $\|\vec{v}(t)\|_{\mathcal{H}^1_x}^2 \lesssim E(\vec{v}(t))$ $\Rightarrow$ suffices to control $E(\vec{v}(t))$ for a soln $\vec{v}$ to the perturbed NLW:

$$\partial_t^2 v + (1-\partial_x^2)v + v^3 + 3v^2\Psi + 3v\Psi^2 + \Psi^3 = 0.$$

- $\partial_t E(\vec{v}(t)) = \int \partial_t v\,(\partial_t^2 v + (1-\partial_x^2)v + v^3)\,dx$

$= -3v^2\Psi - 3v\Psi^2 - \Psi^3$

No v^3 term.

$$\overset{\text{C-S}}{\underset{\text{Young}}{\lesssim}} \underbrace{\left(\int (\partial_t v)^2\,dx\right)^{1/2}}_{\lesssim (E(\vec{v}(t)))^{1/2}} \left(\|\Psi\|_{C_T L^\infty_x}^2 \underbrace{\int v^4\,dx}_{\lesssim E(\vec{v}(t))} + \|\Psi\|_{C_T L^6_x}^6 \right)^{1/2}$$

$$\lesssim \left(1 + \|\Psi\|_{C_T L^\infty_x}^2\right) E(\vec{v}(t)) + \|\Psi\|_{C_T L^6_x}^6$$

By Gronwall, we obtain

$$\sup_{0\le t\le T} \|\vec{\Psi}(t;\omega)\|_{\mathcal{H}^1_x} \le C(\omega,T) < \infty$$

$\uparrow$ a.s.

□

- See Brun-Li-Liu, JDE '24 for higher dim'l case and the references.

Sec 4.3: Invariant Gibbs measure for SdNLW.

Stochastic damped NLW on $\mathbb{T}^d$: (defocusing: $k \in 2\mathbb{N}+1$)

$$\text{(SdNLW)} \quad \partial_t^2 u + \partial_t u + (1-\Delta)u + u^k = \sqrt{2}\,\xi$$

Let $v = \partial_t u$.

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ \Delta-1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ -u^k \end{pmatrix}}_{\text{deterministic NLW}} + \underbrace{\begin{pmatrix} 0 \\ -v+\sqrt{2}\,\xi \end{pmatrix}}_{\text{Ornstein-Uhlenbeck process for } v.}$$

· deterministic NLW: $\partial_t^2 u + (1-\Delta)u + u^k = 0$

· Hamiltonian PDE : $E(\vec{u})$ is conserved

↓

the volume $du\,d(\partial_t u)$ on the phase space is invariant

↑ does not exist.

⇒ We expect that the Gibbs meas ρ, formally given by

$$d\rho(u,v) = Z^{-1} e^{-E(u,v)} du\,dv \qquad (v = \partial_t u)$$

is invariant under the NLW dynamics

Φ_d^{k+1} - measure

$$d\rho(u,v) = Z^{-1} \boxed{e^{-\frac{1}{k+1}\int u^{k+1}dx - \frac{1}{2}\|u\|_{H^1}^2} du} \otimes e^{-\frac{1}{2}\|v\|_{L^2}^2} dv$$

↑ massive Gaussian free field μ_1, see Lec 1.

↑ (spatial) white noise μ_0

- Construction of the Φ^{k+1}_d-measure: constructive QFT. (Euclidean)
  - see the notes from 2024.

  $d=1,2$: $k \in 2\mathbb{N}+1$
  $d=3$: $k=3$
  $d \geq 4$: not possible.

- Invariant Gibbs meas for NLW
  - $d=2$: Oh-Thomann '20 ($k \in 2\mathbb{N}+1$)
  - $d=3$: Bringmann-Deng-Nahmod-Yue '24. ($k=3$)