Lec 6 : Nov 27, 2025 (THU)

· Stochastic damped NLW on $\mathbb{T}^d$

(SdNLW) $\partial_t^2 u + \partial_t u + (1-\Delta)u + u^k = \sqrt{2}\xi$

defocusing: $k \in 2\mathbb{N}+1$

(1-d: we can consider $|u|^{p-1}u$, $p>2$

With $v = \partial_t u$, we can write SdNLW

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta - 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ -u^k \end{pmatrix} + \begin{pmatrix} 0 \\ -v + \sqrt{2}\xi \end{pmatrix}$$

deterministic NLW OU process for v.

· Hamiltonian (= energy) for deterministic NLW:

$$E(u, \partial_t u) = \frac{1}{2}\int |\langle \nabla \rangle u|^2 + \frac{1}{2}\int (\partial_t u)^2 + \frac{1}{k+1}\int u^{k+1}\,dx$$

$\int u^{k+1}\,dx = 0$ b/c $k+1$, even

$+$ mw ↑ def.

$$= \frac{1}{2}\|u\|_{H^1}^2 + \frac{1}{2}\|v\|_{L^2}^2$$

· Gibbs measure ρ :

$$d\rho = Z^{-1} e^{-E(u,v)} du dv \qquad // \; d\mu_1 = \text{massive GFF}$$

$$= Z^{-1} e^{-\frac{1}{k+1}\int u^{k+1} dx} \underbrace{e^{-\frac{1}{2}\|u\|_{H^1}^2} du}$$

$$\otimes \underbrace{e^{-\frac{1}{2}\|v\|_{L^2}^2} dv}_{= d\mu_0 = \text{(spatial) white noise.}}$$

fractional Gaussian field: $d\mu_s = Z_s^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du$ (see Lec 1

- $u(x) = \sum_{n\in\mathbb{Z}^d} \frac{g_n}{\langle n\rangle^s} e^{in\cdot x} \in W^{s-\frac{d}{2}-,\infty}(\mathbb{T}^d) \setminus W^{s-\frac{d}{2},\infty}(\mathbb{T}^d)$ a.s.

$$\Rightarrow d\rho(u,v) = Z^{-1} \underbrace{e^{-\frac{1}{k+1}\int u^{k+1} dx}}_{\sim\sim\sim\sim\sim\sim\sim\sim} \underbrace{d\mu_1 \otimes d\mu_0(u,v)}_{\text{on } W^{1-\frac{d}{2}-,\infty}(\mathbb{T}^d) \times W^{-\frac{d}{2}-,\infty}(\mathbb{T}^d)}$$

- $d=1$: $e^{-\frac{1}{k+1}\int u^{k+1}dx} \in (0,1]$, a.s.

  $\uparrow$

  $u \in L^\infty$, a.s.

  - The density is integrable w.r.t. $d\mu_1 \otimes d\mu_0 (u,v)$

  $\Rightarrow$ ρ exists as a prob. meas on $W^{\frac{1}{2}-,\infty}(\mathbb{T}) \times W^{-\frac{1}{2}-,\infty}(\mathbb{T})$

- $d=2$: u = distribution of negative regularity $(1-\frac{d}{2}-<0)$.

  $\Rightarrow$ $e^{-\frac{1}{k+1}\int u^{k+1}dx} = e^{-\infty} = 0$, a.s.

  - Need to renormalize the interaction potential
    - see the notes from 2024.

Φ_d^{k+1} - meas $\otimes$ μ_0

Nelson '65 : hypercontractivity of the OU semigp & Nelson's estimate

$\Rightarrow$ ρ exists for any $k \in 2\mathbb{N}+1$.

· $d=3$: only $k=3$, i.e. Φ^4_3 - meas. (70's ~

· $d \geq 4$: not possible. ((marginal) triviality : Aizenman, A- Duminil-Copin '21)

· deterministic NLW : $\partial_t^2 u + (1-\Delta) u + u^k = 0$

· Gibbs meas ρ is known to be invariant.

· If $\Phi(t) : (u(0), \partial_t u(0)) \mapsto (u(t), \partial_t u(t))$, then we have

$$(\Phi(t))_* \rho(A) = \rho(\Phi(-t) A) = \rho(A)$$

pushforward

for any measuable set A.

· SdNLW has the OU part: $\partial_t v = -v + \sqrt{2}\xi$.

$$d\hat{v}(n) = -\hat{v}(n)\, dt + \sqrt{2}\, dB_n, \quad n \in \mathbb{Z}^d$$

· Ornstein-Uhlenbeck process (real-valued)

$$\begin{cases} dX = -aX\,dt + b\,dB, & a>0,\ b\in\mathbb{R} \\ X(0) = x_0 \end{cases}$$

$$\Rightarrow \boxed{X(t) = e^{-at}x_0 + b\int_0^t e^{-a(t-t')}\,dB(t')}$$

· Suppose $x_0 \sim N\left(0, \frac{b^2}{2a}\right)$, indep of B.

Then, $X(t)$ is a mean 0 Gaussian r.v. with variance

$$\mathbb{E}[X^2(t)] = e^{-2at}\mathbb{E}[x_0^2] + b^2\int_0^t e^{-2a(t-t')}\,dt'$$

$$= e^{-2at}\frac{b^2}{2a} + \frac{b^2}{2a}\left(1 - e^{-2at}\right)$$

$$= \frac{b^2}{2a}.$$

$\Rightarrow N\left(0, \frac{b^2}{2a}\right)$ is invariant under OU.

- $d\hat{v}(n) = -\hat{v}(n)\,dt + \sqrt{2}\,dB_n$

  $\hat{v}(0,n) = g_n \sim \mathcal{N}_{\mathbb{C}}(0,1)$

$\Rightarrow$ $\mathcal{N}_{\mathbb{C}}(0,1)$ is invariant, $n \in \mathbb{Z}^d$

$\Rightarrow$ μ_0 (and thus ρ) is invariant under the OU part:

$$\partial_t v = -v + \sqrt{2}\,\xi$$

- Recall:

$$\text{Sd NLW} = \underbrace{(\text{determ.})\ \text{NLW}}_{\text{both preserve the Gibbs meas } \rho.} + \underbrace{\text{OU for } v}_{\text{both preserve the Gibbs meas } \rho.}$$

Hence, we "expect" that ρ is invariant under Sd NLW.

- can make this rigorous via regularization (= freq truncation).

· freq-truncated SdNLW:

$$(SdNLW_N) \quad \partial_t^2 u_N + \partial_t u_N + (1-\Delta) u_N + \pi_N (\pi_N u_N)^k = \sqrt{2}\xi$$

π_N: freq proj onto $\{|n| \le N\}$ — NO π_N

· truncated Gibbs meas ρ_N:

$$d\rho_N = Z_N^{-1} \underbrace{e^{-\frac{1}{k+1}\int (\pi_N u_N)^{k+1} dx} d\mu_1}_{\text{density}} \otimes d\mu_0(u_N, \underbrace{\partial_t u_N}_{\| \; v_N})$$

$$e^{-\frac{1}{k+1}\int (\pi_N u_N)^{k+1} dx} \, d\mu_{1,N} \otimes d\mu_{1,N}^{\perp}$$

where $\mu_{1,N} = (\pi_N)_* \mu_1$ and $\mu_{1,N}^{\perp} = (\pi_N^{\perp})_* \mu_1$

$\pi_N^{\perp} = Id - \pi_N$

Split $SdNLW_N$ into two parts:

$$\text{(low)} \quad \partial_t^2 u_N^{low} + \partial_t u_N^{low} + (1-\Delta) u_N^{low} + \pi_N\big((u_N^{low})^k\big) = \sqrt{2}\,\pi_N \xi$$

$$\text{(high)} \quad \partial_t^2 u_N^{high} + \partial_t u_N^{high} + (1-\Delta) u_N^{high} = \sqrt{2}\,\pi_N^{\perp} \xi$$

- (high) is linear (but ∞-dim'l)

  $\Rightarrow$ $\mu_{1,N}^{\perp} \otimes \mu_{0,N}^{\perp}$ is invariant under (high)

- (low) $\simeq$ NLW_N: $\partial_t^2 u_N^{low} + (1-\Delta) u_N^{low} + \pi_N\big((u_N^{low})^k\big) = 0$

  + OU for $\pi_N v$: $\partial_t v_N^{low} = - v_N^{low} + \sqrt{2}\,\pi_N \xi$

  finite dim'l Hamiltonian system of ODEs (on the Fourier side

$\Rightarrow$ $d\rho_N^{low} = Z_N^{-1}\, e^{-\frac{1}{k+1}\int (\pi_N u_N^{low})^{k+1} dx}\, d\mu_{1,N} \otimes d\mu_{0,N}(u_N^{low}, v_N^{low})$

is invariant under (low) (We'll come back to this point

$\Leftarrow$ Lie-Trotter product formula.

$$e^{t(A+B)} = \lim_{n\to\infty} \left(e^{\frac{t}{n}A}\, e^{\frac{t}{n}B}\right)^n$$

$\Rightarrow$ $\mathrm{SdNLW}_N = (\mathrm{low}) + (\mathrm{high})$

$\Rightarrow$ Truncated Gibbs meas ρ_N is invariant under SdNLW_N.

· 1-d SdNLW: GWP in $\mathcal{H}^{\frac{1}{2}-}(\mathbb{T}) \supset \mathrm{supp}\,\rho$

$\Rightarrow$ Invariance of ρ under SdNLW follows from

① invariance of ρ_N under SdNLW_N

② $(u_N, v_N) \longrightarrow (u, v)$ PDE argument

see the notes from 2024

$\forall F \in C_b$ $\int \mathbb{E}\left[F(\Phi(t)\vec{\Phi})\right] d\rho = \int F(\vec{\Phi})\, d\rho$.

expectation w.r.t. noise

· $\Phi(t): \vec{\Phi} \longrightarrow \vec{u}(t)$, soln map

· $d=2$: No pathwise GWP in a space containing $\operatorname{supp}\rho$.
(except for $k=3$: GKOT '22).

$\Leftarrow$ Bourgain's invariant measure argument see the notes from 2024.
Bourgain '94, '96. Oh-Robert-Tzvetkov '23

· Back to the invariance of ρ_N^{low} under (low).

· Generator $\mathcal{L}$ of a Markov semigroup:

$$dX = f\,dt + g\,dB, \quad X(0) = x_0$$

· Markov semigroup P_t: $P_t F(x_0) = \mathbb{E}[F(X_t)]$

· Generator $\mathcal{L}$ is given by

$$\mathcal{L}F(x_0) = \lim_{t\to 0} \frac{P_t F(x_0) - F(x_0)}{t} = \lim_{t\to 0} \frac{\mathbb{E}[F(X_t)] - F(x_0)}{t}$$

· By Ito's lemma,

$$F(X_t) - F(x_0) = \int_0^t F'(X_s) f(s)\,ds + \underbrace{\int_0^t F'(X_s) g(s)\,dB(s)}_{=0 \text{ under expectation}} + \frac{1}{2}\int_0^t F''(X_s) g^2(s)\,ds$$

FTC

$$\Rightarrow \mathcal{L}F(x_0) = F'(x_0) f(0) + \frac{1}{2} F''(x_0) g^2(0)$$

ex: $X = BM$. $dX = 0 \cdot dt + 1 \cdot dB$.

$$\Rightarrow \mathcal{L} = \frac{1}{2}\partial_x^2$$

· Back to (low):

$$\partial_t^2 u_N^{low} + \partial_t u_N^{low} + (1-\Delta) u_N^{low} + \pi_N\left((\pi_N u_N^{low})^k\right) = \sqrt{2}\,\pi_N \xi$$

$$\Leftrightarrow \quad \begin{cases} \partial_t u_N^{low} = v_N^{low} \\ \partial_t v_N^{low} = (\Delta - 1) u_N^{low} - \pi_N\left((\pi_N u_N^{low})^k\right) - v_N^{low} + \sqrt{2}\,\pi_N \xi \end{cases}$$

· ρ_N^{low} invariant $\Leftrightarrow\!\!\Rightarrow$ $\mathcal{L}_N^* \rho_N^{low} = 0$

See Chap 5 of Da Prato's book "Intro to ∞-dim'l analysis".

$$\circledast \quad \Leftrightarrow\!\!\Rightarrow \quad \int \mathcal{L}_N\left(F(\vec{u})\right) \rho_N^{low}(d\vec{u}) = 0, \qquad \vec{u} = (u, v)$$

In the following, we prove $\circledast$ (by a formal computation).

↑ dropping π_N, etc.

· Write $\widehat{u}(n) = a_n + i\, b_n$

$\widehat{v}(n) = p_n + i\, q_n$

$$\begin{cases} \partial_t u = v \\ \partial_t v = (\Delta - 1)\, u - \underbrace{N[u]}_{= u^k} - v + \sqrt{2}\,\xi \end{cases}$$

$$\partial_t a_n = p_n$$
$$\partial_t b_n = q_n$$
$$\partial_t p_n = -\langle n\rangle^2 a_n - \mathrm{Re}\,\widehat{N(\bar a, \bar b)}(n) - p_n + \sqrt{2}\,\partial_t(\mathrm{Re}\, B_n)$$
$$\partial_t q_n = -\langle n\rangle^2 b_n - \mathrm{Im}\,\widehat{N(\bar a, \bar b)}(n) - q_n + \sqrt{2}\,\partial_t(\mathrm{Im}\, B_n).$$

$\bar a = \{a_n\}_{n\in\mathbb{Z}^2}$, etc. (or $|n| \le N$)

Write $L_N = \underbrace{L_N^1}_{\text{NLW part}} + \underbrace{L_N^2}_{\text{OU part}}$.

· OU part: $L_N^2 F(\bar a, \bar b, \bar p, \bar q) = \sum_n \Big(-p_n \partial_{p_n} F + \frac{1}{2}\partial^2_{p_n} F - q_n \partial_{q_n} F + \frac{1}{2}\partial^2_{q_n} F \Big)$

$$\int \mathcal{L}_N^2 \, F(\bar{a}, \bar{b}, \bar{p}, \bar{q}) \, d\rho = 0 \; ?$$

really ρ_N^{low}

$$P(a,b) = \frac{1}{k+1} \int u^{k+1} dx$$

for fixed n //

$-p_n \partial_{p_n} F + \frac{1}{2} \partial_{p_n}^2 F$

$-q_n \partial_{q_n} F + \frac{1}{2} \partial_{q_n}^2 F$

$$e^{-P(a,b)} \; e^{-\frac{1}{2}\sum_n \langle n^2 \rangle a_n^2 + \langle n^2 \rangle b_n^2} \; d\bar{a}\, d\bar{b}$$

$$\times \; e^{-\frac{1}{2}\sum_n p_n^2 + q_n^2} \; d\bar{p}\, d\bar{q}$$

$$\int \frac{1}{2} \partial_{p_n}^2 F \; e^{-\frac{1}{2}\sum_n p_n^2} \, dp_n = -\frac{1}{2} \int \partial_{p_n} F \cdot (-2p_n) \, dp_n$$

$$= \int p_n \partial_{p_n} F \, dp_n$$

do the same for q_n

$$\Rightarrow \int \mathcal{L}_N^2 \, F(\bar{a}, \bar{b}, \bar{p}, \bar{q}) \, d\rho = 0$$

· NLW part:

$$\partial_t a_n = p_n$$
$$\partial_t b_n = q_n$$
$$\partial_t p_n = -\langle n\rangle^2 a_n - \mathrm{Re}\,\widehat{\mathcal{N}(\bar a, \bar b)}(n)$$
$$\partial_t q_n = -\langle n\rangle^2 b_n - \mathrm{Im}\,\widehat{\mathcal{N}(\bar a, \bar b)}(n)$$

chain rule

$$\mathcal{L}_N^1 F(\bar a, \bar b, \bar p, \bar q) = p_n \partial_{a_n} F + q_n \partial_{b_n} F$$
$$+ \left(-\langle n\rangle^2 a_n - \mathrm{Re}\,\widehat{\mathcal{N}(\bar a, \bar b)}(n)\right) \partial_{p_n} F$$
$$+ \left(-\langle n\rangle^2 b_n - \mathrm{Im}\,\widehat{\mathcal{N}(\bar a, \bar b)}(n)\right) \partial_{q_n} F.$$

Q: $\int \mathcal{L}_N^1 F(\bar a, \bar b, \bar p, \bar q)\, d\rho = 0$?

$$\int p_n \partial_{a_n} F \, d\rho \overset{\text{IBP}}{=} \int \langle n^2 \rangle a_n p_n F + \underbrace{\partial_{a_n} P(\bar{a},\bar{b}) p_n F}_{①} \, d\rho$$

$$\uparrow$$

$$e^{-P(\bar{a},\bar{b})} \; e^{-\frac{1}{2}\sum_n \langle n^2 \rangle a_n^2} \; d\bar{a}$$

$$\int q_n \partial_{b_n} F \, d\rho \overset{\text{IBP}}{=} \int \langle n \rangle b_n q_n F + \underbrace{\partial_{b_n} P(\bar{a},\bar{b}) q_n F}_{②} \, d\rho$$

$$\int \left(-\langle n^2 \rangle a_n - \widehat{\mathrm{Re}\, N(\bar{a},\bar{b})}(n)\right) \partial_{p_n} F \, d\rho \; e^{-\frac{1}{2}\sum p_n^2} \, d\bar{p}$$

$$\overset{\text{IBP}}{=} -\int \langle n^2 \rangle a_n p_n F \, d\rho - \underbrace{\int \widehat{\mathrm{Re}\, N(\bar{a},\bar{b})}(n) \, p_n F \, d\rho}_{③}$$

$$\int \left(-\langle n^2 \rangle b_n - \widehat{\mathrm{Im}\, N(\bar{a},\bar{b})}(n)\right) \partial_{q_n} F \, d\rho$$

$$\overset{\text{IBP}}{=} -\int \langle n \rangle b_n q_n F \, d\rho - \underbrace{\int \widehat{\mathrm{Im}\, N(\bar{a},\bar{b})}(n) \, q_n F \, d\rho}_{④}$$

$$P(\bar a,\bar b)=\frac{1}{k+1}\int u^{k+1}dx=\frac{1}{k+1}\sum_{n_1+\dots+n_{k+1}=0}\prod_{j=1}^{k+1}\hat u(n_j)$$

$$\hat u(n_j)=a_{n_j}+\mathrm{i}\,b_{n_j}$$

$$\frac{\partial}{\partial a_n}P(\bar a,\bar b)=\widehat{N(\bar a,\bar b)}(-n)$$

prod. rule & chain rule

$$\frac{\partial}{\partial a_{-n}}P(\bar a,\bar b)=\widehat{N(\bar a,\bar b)}(n)$$

$$N(u)=u^k$$

① $$\sum_{n\in\mathbb{Z}^d}\frac{\partial}{\partial a_n}P(\bar a,\bar b)=\sum_{n\in I_*}\left(\frac{\partial}{\partial a_n}P(\bar a,\bar b)+\frac{\partial}{\partial a_{-n}}P(\bar a,\bar b)\right)+\frac{\partial}{\partial a_0}P(\bar a,\bar b)$$

$$=\sum_{n\in I_*}2\,\mathrm{Re}\,\widehat{N(\bar a,\bar b)}(n)+\widehat{u^k}(0)$$

$$\simeq-\sum_n ③$$

$$I_*=\text{“}\mathbb{Z}^d/2\setminus\{\vec 0\}\text{”}=\bigcup_{k=0}^{d-1}\mathbb{Z}^k\times\mathbb{N}\times\{0\}^{d-k-1}$$

By a similar computation, we get

$$\sum_n (2) + \sum_n (4) = 0 \quad \left(\Leftarrow \operatorname{Im} \widehat{u^k}(0) = 0\right)$$

$\Rightarrow$ We showed $\int L_N^1 F(\bar{a}, \bar{b}, \bar{p}, \bar{q})\, d\rho = 0$

$$\Rightarrow \int L_N F(\bar{a}, \bar{b}, \bar{p}, \bar{q})\, d\rho = \sum_{j=1}^{2} \int L_N^j F(\bar{a}, \bar{b}, \bar{p}, \bar{q})\, d\rho$$

$$= 0.$$

see Subsec 5.2, Oh-Robert-Wang '21 (for SNLH)

$\Rightarrow$ We can make the computation above rigorous by inserting π_N and considering the phase space $\bar{a} = \{a_n\}_{|n| \le N}$, etc

$\Rightarrow$ ρ_N^{low} is invariant under (low) i.e. P_t = Markov semigroup for (low).

$$\int P_t F(\vec{\Phi})\, d\rho_N^{low} = \int F(\vec{\Phi})\, d\rho_N^{low}, \quad \forall F \in C_b.$$