Lec 7 : Dec 04, 2025 (THU)

- Φ^{k+1}_d - measure on $\mathbb{T}^d$: $d\rho = Z^{-1} e^{-\frac{1}{k+1}\int u^{k+1} dx} d\mu_1$, $k \in 2\mathbb{N}+1$.
  - $d\mu_1 = Z^{-1} e^{-\frac{1}{2}\|u\|^2_{H^1}} du$ = GFF on $\mathbb{T}^d$.

- Stochastic quantization: Parisi - Wu '81.
  - study stoch. dynamics which preserves a target measure

$\Rightarrow$ stochastic quantization eqn (SQE):

$$\partial_t u + (1-\Delta) u + u^k = \sqrt{2}\,\xi \quad \Leftarrow \text{SNLH.}$$

$d=2$: Da Prato - Debussche '03

$d=3$ & $k=3$: Hairer '14

also called

- dynamical Φ^{k+1}_d - model (parabolic Φ^{k+1}_d - model)

(Also, stoch Cahn-Hilliard eqn preserves the Φ^{k+1}_d - meas: Röckner - Yang - R. Zhu AAP '21.

· Canonical stoch. quantization: Ryang-Saito-Shigemoto '85.

Canonical SQE (= Hamiltonian SQE).

Langvin eqn

$$\begin{cases} \partial_t u = \dfrac{\partial E}{\partial v} \\ \partial_t v = -\dfrac{\partial E}{\partial u} - v + \sqrt{2}\xi \end{cases}$$

→ measure

Φ^{k+1}_d-meas on u } → Gibbs meas.

⊗ μ_0 on v

↑ white noise

ex: stoch damped NLW = hyperbolic Φ^{k+1}_d-model.

$$\partial_t^2 u + \partial_t u + (1-\Delta)u + u^k = \sqrt{2}\xi.$$

$d=2$: GKOT '22

$d=3$ & $k=3$: BDNY '24.

· In the complex-valued setting, NLS: $i\partial_t u + (1-\Delta)u + |u|^{k-1}u = 0$

preserves the Φ^{k+1}_d-measure (= Gibbs meas for NLS) ↑ "dispersive Φ^{k+1}_d-model"

$d=2$: Bourgain '96

SPDE version? ↓ heat

$$i\partial_t u + (1-i)(1-\Delta)u + |u|^{k-1}u = \sqrt{2}\xi$$

stoch complex Ginzburg-Landau eqn ⇐ analysis for SNLH applies

Chap 5 : 2-d stochastic NLW

Sec 5.1 : NLW with random initial data

$$\text{(NLW)} \quad \begin{cases} \partial_t^2 u + (1-\Delta) u + u^k = 0 \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases} \qquad \text{on } \mathbb{T}^2.$$

Main goal: Prove a.s. LWP w.r.t. the Gibbs measure ρ

↑ almost sure local existence of a uniq soln

- probabilistic conti dependence:
  Burq-Tzvetkov '14, Pocovnicu '17.

On $\mathbb{T}^2$, $d\rho = Z^{-1} e^{-\frac{1}{k+1}\int u^{k+1} dx} d\mu_1 \otimes d\mu_0 (u, \partial_t u)$.

↑ view it as a weight
(integrable w.r.t. $\mu_1 \otimes \mu_0$)

Need renormalization
(See Lec 18-19 from 2024)

· $\rho \ll \mu_1 \otimes \mu_0$ (In fact, $\rho \sim \mu_1 \otimes \mu_0$

$\rightsquigarrow$ LWP a.s. w.r.t. ρ $\Longleftrightarrow$ LWP a.s. w.r.t. $\mu_1 \otimes \mu_0$.

$$\begin{cases} u_0^\omega = \sum_{n \in \mathbb{Z}^2} \frac{g_n(\omega)}{\langle n \rangle} e^{in\cdot x} \\ u_1^\omega = \sum_{n \in \mathbb{Z}^2} h_n(\omega) e^{in\cdot x} \end{cases}$$

$\{g_n, h_n\}$, family of $\mathbb{C}$-valued std. Gaussian r.v.'s conditioned that

$g_{-n} = \overline{g_n}$

$h_{-n} = \overline{h_n}$

$\Rightarrow$ u_0^ω, u_1^ω are real-valued.

$\mathrm{Law}(u_0^\omega, u_1^\omega) = \mu_1 \otimes \mu_0$

$(u_0^\omega, u_1^\omega) \underset{\text{a.s.}}{\in} \mathcal{H}^{0-}(\mathbb{T}^2) \setminus \mathcal{H}^0(\mathbb{T}^2)$

($\mathcal{H}^s = H^s \times H^{s-1}$

Consider the cubic NLW on $\mathbb{T}^2$:

$$\partial_t^2 u + (1-\Delta) u + u^3 = 0$$

- Critical regularity: $s_{scaling} = \frac{d}{2} - \frac{2}{k-1} = 0$ $\quad d=2,\ k=3$

$$s_{conf} = \frac{d+1}{4} - \frac{1}{k-1} = \frac{1}{4}$$

s_{conf}: critical regularity w.r.t. conformal invariance (Lorentzian invariance)

$\Rightarrow$ $s_{crit} = \max(s_{scaling}, s_{conf}, 0) = \frac{1}{4}$ when $d=2,\ k=3$

- cubic NLW on $\mathbb{T}^2$ is LWP in $\mathcal{H}^s(\mathbb{T}^2)$ for $s > \frac{1}{4}$
  - Strichartz estimate — Lindblad-Sogge '95, GKO '18
- focusing cubic NLW: $\partial_t^2 u + (1-\Delta) u - u^3 = 0$
  - ill-posed for $s < \frac{1}{4}$ — Tao's book
  - defocusing case? OPEN!!

· Probabilistic LWP for rougher data?

· $\mathrm{Law}(u_0^\omega, u_1^\omega) = \mu_\alpha \otimes \mu_{\alpha-1}$.

$\alpha - \frac{d}{2} -$
$=$
$\alpha - 1 -$

$$\begin{cases} u_0^\omega = \sum_{n\in\mathbb{Z}^2} \frac{g_n}{\langle n\rangle^\alpha} e^{in\cdot x} \\ u_1^\omega = \sum_{n\in\mathbb{Z}^2} \frac{h_n}{\langle n\rangle^{\alpha-1}} e^{in\cdot x} \end{cases} \Rightarrow (u_0^\omega, u_1^\omega) \underset{a.s.}{\in} \mathcal{H}^{\alpha-1-}(\mathbb{T}^2)$$

$\Rightarrow$ Take $\alpha > 1$ s.t. $(u_0^\omega, u_1^\omega) \underset{a.s.}{\in} \mathcal{H}^0(\mathbb{T}^2)$

· Can view u_0^ω & u_1^ω as randomization of deterministic functions:

$$\begin{cases} u_0 = \sum_{n\in\mathbb{Z}^2} \frac{\perp}{\langle n\rangle^\alpha} e^{in\cdot x} \\ u_1 = \sum_{n\in\mathbb{Z}^2} \frac{\perp}{\langle n\rangle^{\alpha-1}} e^{in\cdot x} \end{cases} \Rightarrow (u_0, u_1) \in \mathcal{H}^{\alpha-1-}(\mathbb{T}^2).$$

· Start with a pair of (determ) functions:

$$u_0 = \sum_{n\in\mathbb{Z}^2} a_n e^{in\cdot x}, \qquad u_1 = \sum_{n\in\mathbb{Z}^2} b_n e^{in\cdot x}$$

and consider their randomization: Burq - Tzvetkov '08.

$$u_0^\omega = \sum_{n\in\mathbb{Z}^2} g_n a_n e^{in\cdot x}, \qquad u_1^\omega = \sum_{n\in\mathbb{Z}^2} h_n b_n e^{in\cdot x}$$

u_0^ω: $u_0 *$ white noise

· If $(u_0, u_1) \in \mathcal{H}^s \setminus \mathcal{H}^{s+\varepsilon}$ (for some ε),

then $(u_0^\omega, u_1^\omega) \underset{a.s.}{\in} \mathcal{H}^s \setminus \mathcal{H}^{s+\varepsilon}$ No smoothing under randomization.

but $(u_0^\omega, u_1^\omega) \underset{a.s.}{\in} W^{s,r} \times W^{s-1,r}$ $\forall\, r < \infty$ gain of integrability.

· Linear soln : $W(t)(u_0, u_1) = \partial_t S(t) u_0 + S(t) u_1$

$$S(t) = \frac{\sin(t\langle\nabla\rangle)}{\langle\nabla\rangle}$$

· **Lem 5.1.1.:** $(u_0, u_1) \in \mathcal{H}^s$. Then,

$T < \infty$, $q, r < \infty$

$$P\left(\| W(t)(u_0^\omega, u_1^\omega) \|_{L_T^q W_x^{s,r}} > \lambda \right) \le C \exp\left(- c \frac{\lambda^2}{T^{\frac{2}{q}} \|(u_0,u_1)\|_{\mathcal{H}^s}^2} \right) \quad \forall\ 0 < \lambda < \infty$$

$$\Rightarrow \quad W(t)(u_0^\omega, u_1^\omega) \underset{a.s.}{\in} L_T^q W_x^{s,r}$$

Pf:

$p \ge q \vee r$

$$\| \| W(t)(u_0^\omega, u_1^\omega) \|_{L_T^q W_x^{s,r}} \|_{L^p(\Omega)} \overset{\text{Mink}}{\le} \| \| \langle\nabla\rangle^s W(t)(u_0^\omega, u_1^\omega) \|_{L^p(\Omega)} \|_{L_T^q L_x^r}$$

$$\le p^{1/2} \| \| \langle\nabla\rangle^s W(t)(u_0^\omega, u_1^\omega) \|_{L^2(\Omega)} \|_{L_T^q L_x^r}$$

$$\sim p^{1/2} T^{\frac{1}{q}} \| (u_0, u_1) \|_{\mathcal{H}^s}$$

□

Rmk: By Sobolev, $W(t)(u_0^\omega, u_1^\omega) \in L_T^q W_x^{s-\varepsilon,\infty}$, $\forall \varepsilon > 0$.

Thm 5.1.2: $(u_0, u_1) \in \mathcal{H}^0(\mathbb{T}^2)$.

Then, the cubic NLW is a.s. LWP w.r.t. the randomized initial data

$$(u_0^\omega, u_1^\omega) = \Big(\sum_{n\in\mathbb{Z}^2} g_n a_n e^{in\cdot x}, \sum_{n\in\mathbb{Z}^2} h_n b_n e^{in\cdot x} \Big).$$

Rmk: · a.s. LWP for Law $(u_0^\omega, u_1^\omega) = \mu_\alpha \otimes \mu_{\alpha-1}$, $\alpha > 1$

· Gibbs meas : $\alpha = 1$.

Pf: First order expansion : $u = \underbrace{W(t)(u_0^\omega, u_1^\omega)}_{= z} + v$.

Then, v satisfies

$$\text{(NLW')} \quad \begin{cases} \partial_t^2 v + (1-\Delta)v + (v+z)^3 = 0. \\ (v, \partial_t v)|_{t=0} = (0,0). \end{cases}$$

· $\Gamma(v) = \int_0^t \frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle} (v+z)^3(t')\, dt'$.

Set $\vec{\Gamma}(v, \partial_t v) = (\vec{\Gamma}(v), \partial_t \vec{\Gamma}(v))$

$$\Rightarrow \| \vec{\Gamma}(v, \partial_t v) \|_{C_T \mathcal{H}^1_x} \leq T^{1/2} \| (v+z)^3 \|_{L^2_{T,x}}$$

$$(v+z)^3 = v^3 + 3v^2 z + 3v z^2 + z^3.$$

$0 < T \leq 1$

- $\| v \|_{C_T L^6_x} \overset{\text{Sob}}{\lesssim} \| v \|_{C_T H^1_x} \leq \| \vec{v} \|_{C_T \mathcal{H}^1_x}$, $\quad \vec{v} = (v, \partial_t v)$
- $\| z \|_{L^6_{T,x}} \leq \| z \|_{L^6([0,1]; L^6_x)} \leq C_\omega < \infty$ a.s.

$$\overset{\text{Young}}{\leq} C \| \vec{v} \|^3_{C_T \mathcal{H}^1_x} + C'_\omega$$

· A diff. estimate follows from a similar computation.

$\Rightarrow \exists!$ soln $(v, \partial_t v)$ to (NLW′) in $C_T \mathcal{H}^1_x$. $\quad T_\omega = T\left(\| z^\omega \|_{L^6([0,1]; L^6_x)} \right) > 0$

□

Thm 5.1.3: Let $s > 0$. $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^2)$.

Then, the cubic NLW is a.s. GWP with the same randomized initial data as in Thm 5.1.2.

Pf: The same proof as in the 1-d case (Thm 4.2.7. from Lec 5) holds. Namely, compute $\partial_t E(\vec{v})$ and apply Gronwall.

Need $\|z\|_{L^\infty_T L^b_x} \underset{a.s.}{<} \infty$ ← See also Prop 1.8.5 from Lec 8, 2024

$\partial_t^2 z = -\langle \nabla \rangle^2 z.$

↑ Sobolev in time. Then, convert a loss of time derivative to that of spatial derivative.

$\Rightarrow$ Need $s > 0$.

□

Rmk: Thm 5.1.3 holds true for $s = 0$.

Need a Yudovich-type argument (i.e. double Gronwall)

Burq-Tzvetkov '14

Q: a.s. LWP in negative Sobolev spaces?

· Need renormalization. (Wick renormalization

Consider

$$\begin{cases} \partial_t^2 u_N + (1-\Delta) u_N + \pi_N \left((\pi_N u_N)^k \right) = 0 \\ (u_N, \partial_t u_N)\big|_{t=0} = (\pi_N u_0^\omega, \pi_N u_1^\omega) \\ \text{Law}\,(u_0^\omega, u_1^\omega) = \mu_1 \otimes \mu_0 \end{cases}$$

or u_N^k

$z(t) \in H^{0-}$

· 1st order expansion : $u_N = z_N + v_N$, $z_N = \pi_N z$

$$\Rightarrow \partial_t^2 v_N + (1-\Delta) v_N = - \sum_{\ell=0}^{k} \binom{k}{\ell} v_N^{k-\ell} \underbrace{z_N^\ell}_{\uparrow \Sigma}$$

issue in taking a limit as $N \to \infty$.

Wick power : $:Z_N^\ell(t,x):$ $= H_\ell(Z_N(t,x); \sigma_N)$ ← Hermite polynomial see p.15

$\sigma_N = \mathbb{E}[Z_N^2(t,x)]$ $\qquad Z_N = \sum_{|n|\le N} \frac{\cos(t\langle n\rangle) g_n + \sin(t\langle n\rangle) h_n}{\langle n\rangle} e^{in\cdot x}$

$$= \sum_{|n|\le N} \frac{\cos^2(t\langle n\rangle) + \sin^2(t\langle n\rangle)}{\langle n\rangle^2}$$

$$= \sum_{|n|\le N} \frac{1}{\langle n\rangle^2} \sim \log N \to \infty.$$

Claim: $:Z_N^\ell:$ converges to some limit, denoted by $:Z^\ell:$

- a.s. in $C_T W_x^{-\varepsilon,\infty}$
- in $L^p(\Omega; C_T W_x^{-\varepsilon,\infty})$ $\quad p < \infty$.

· So, we instead consider

$$\partial_t^2 v_N + (1-\Delta) v_N = -\sum_{\ell=0}^{k} \binom{k}{\ell} v_N^{k-\ell} :z_N^\ell:$$

Wick power

$\downarrow N \to \infty$

$$\partial_t^2 v + (1-\Delta) v = -\sum_{\ell=0}^{k} \binom{k}{\ell} v^{k-\ell} :z^\ell:$$

$:z^\ell: \in C_T W_x^{-\varepsilon,\infty}$ a.s.

We can study LWP of this eqn. for $\vec{v} \in C_T \mathcal{H}_x^{1-\varepsilon}$

(and $\vec{v}_N \to \vec{v}$.

Renormalized NLW: $u = z + v$

$$:u^k: = \sum_{\ell=0}^{k} \binom{k}{\ell} v^{k-\ell} :z^\ell:$$

$$\begin{cases} \partial_t^2 u + (1-\Delta) u + :u^k: = 0 \\ \text{Law}(u(0), \partial_t u(0)) = \mu_1 \otimes \mu_0 \end{cases}$$

- Renormalized NLW is a "formal" equation

$k=2$: $:u^2: \;=\; u^2 - \sigma_\infty \;=\; u^2 - \infty$

$k=3$: $:u^3: \;=\; u^3 - 3\sigma_\infty u \;=\; u^3 - 3\cdot\infty\cdot u$

- Hermite polynomial $H_k(x;\sigma)$ with variance parameter $\sigma>0$.
  - defined via the following generating func:

$$e^{tx-\frac{1}{2}\sigma^2 t} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x;\sigma)$$

- $H_0(x;\sigma)=1$, $H_1(x,\sigma)=x$, $H_2(x;\sigma)=x^2-\sigma$, $H_3(x;\sigma)=x^3-3\sigma x$.
- $\left\{ H_k(x;\sigma) \Big/ \sqrt{k!\,\sigma^k} \right\}_{k=0}^{\infty}$ forms an O.N. basis of $L^2\left(\mathbb{R}; \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-x^2}{2\sigma}} dx\right)$.

↑ Can obtain these by apply Gram-Schmidt to $1, x, x^2, \cdots$ in this space

Next lecture : 10 AM UK time on

Dec 8, 2025 (MON)

(last lecture of the year 2025.)