Lec 8 : Dec 08, 2025 (MON)

· NLW on $\mathbb{T}^2$

$k \geq 2$, integer

$$\begin{cases} \partial_t^2 u + (1-\Delta) u + u^k = 0 \\ (u, \partial_t u)|_{t=0} = (u_0^\omega, u_1^\omega) \quad \text{with Law}\,(u_0^\omega, u_1^\omega) = \mu_1 \otimes \mu_0 \end{cases}$$

① freq-truncate the random initial data

$$\begin{cases} \partial_t^2 u_N + (1-\Delta) u_N + u_N^k = 0 \\ (u_N, \partial_t u_N)|_{t=0} = (\pi_N u_0^\omega, \pi_N u_1^\omega) \end{cases}$$

② first order expansion: $u_N = z_N + v_N$.

$z_N = \pi_N z$
$= \pi_N W(t)(u_0^\omega, u_1^\omega)$.

$$\begin{cases} \partial_t^2 v_N + (1-\Delta) v_N + \sum_{\ell=0}^{k} \binom{k}{\ell} v_N^{k-\ell} z_N^\ell = 0 \\ (v_N, \partial_t v_N)|_{t=0} = (0,0). \end{cases}$$

↑ The limit does not exist as $N \to \infty$.

③ Wick renormalization

$$:Z_N^\ell: \;=\; H_\ell(Z_N;\sigma_N) \longrightarrow :Z^\ell: \quad \text{a.s. in } C_T W_x^{-\varepsilon,\infty} \text{ (later)}$$

$$\sigma_N = \mathbb{E}\big[Z_N^2(t,x)\big] = \sum_{n\in\mathbb{N}} \frac{1}{\langle n\rangle^2} \sim \log N \to \infty$$

↑ indep of t and x

for the current problem due to the invariance of $\mu_1 \otimes \mu_0$ under the linear dynamics

$$(NLW_N') \begin{cases} \partial_t^2 v_N + (1-\Delta) v_N + \sum_{\ell=0}^{k} \binom{k}{\ell} v_N^{k-\ell} :Z_N^\ell: = 0 \\ (v_N, \partial_t v_N)|_{t=0} = (0,0) \end{cases}$$

④ Take $N \to \infty$

$$\begin{cases} \partial_t^2 v + (1-\Delta) v + \sum_{\ell=0}^{k} \binom{k}{\ell} v^{k-\ell} :Z^\ell: = 0 \\ (v, \partial_t v)|_{t=0} = (0,0) \end{cases}$$

$= :u^k:$

(5) Undo the first order expansion: $u = z + v$

$$\begin{cases} \partial_t^2 u + (1-\Delta) u + :u^k: = 0 \\ (u, \partial_t u)|_{t=0} = (u_0^\omega, u_1^\omega) \end{cases}$$

renormalized NLW
Wick ordered NLW

- This is just a formal equation

$$k=2: \quad :u^2: = u^2 - \sigma_\infty = u^2 - \infty$$
$$k=3: \quad :u^3: = u^3 - 3\cdot\infty\cdot u$$

$(NLW_N') \Rightarrow u_N = z_N + v_N$

$$H_k(x+y;\sigma) = \sum_{\ell=0}^{k} \binom{k}{\ell} x^{k-\ell} H_\ell(y;\sigma)$$

$$\begin{cases} \partial_t^2 u_N + (1-\Delta) u_N + H_k(u_N; \sigma_N) = 0 \\ (u_N, \partial_t u_N)|_{t=0} = (\pi_N u_0^\omega, \pi_N u_1^\omega) \end{cases}$$

↑ ptwise renormalization

$$\sigma_N = \mathbb{E}[z_N^2(t,x)]$$

$:u^k:$ in the limit.

· What if we do not renormalize?

renormalized truncated NLW ($k=3$):

$$\begin{cases} \partial_t^2 u_N + (1-\Delta) u_N + u_N^3 - 3 \underline{\sigma_N} u_N = 0 \\ (u_N, \partial_t u_N)|_{t=0} = (\pi_N u_0^\omega, \pi_N u_1^\omega) \end{cases} \qquad \underline{\sigma_N} \to \infty$$

· If we instead consider

No renormalization

$$\begin{cases} \partial_t^2 u_N + (1-\Delta) u_N + u_N^3 = 0 \\ (u_N, \partial_t u_N)|_{t=0} = (\pi_N u_0^\omega, \pi_N u_1^\omega) \end{cases}$$

← LWP for each $N \in \mathbb{N}$

BUT $u_N \to 0$.

triviality

(Oh-Okamoto-Robert, SPA '20 for SdNLW

⇒ Renormalization is needed to see any non-trivial limiting behavior.

· see also weak universality in GKO '18

Prop 5.1.4: $l \in \mathbb{N}$, $\varepsilon > 0$, $T > 0$.

$:z_N^l:$ converges to a limit, denoted by $:z^l:$

- a.s. in $C_T W_x^{-\varepsilon,\infty}(\mathbb{T}^2)$
- in $L^p(\Omega; C_T W_x^{-\varepsilon,\infty})$, $p < \infty$.

For now, we assume Prop 5.1.4 and prove a.s LWP of

(NLW_ν)
$$\begin{cases} \partial_t^2 v + (1-\Delta) v = -\sum_{l=0}^{k} \binom{k}{l} v^{k-l} \underbrace{:z^l:}_{\in C_T W_x^{-\varepsilon,\infty} \text{ a.s.}} \\ (v, \partial_t v)\big|_{t=0} = \underbrace{(v_0, v_1)}_{\text{deterministic initial data}} \end{cases}$$

↑ pretend they are deterministic.

Consider

$$\circledast \quad \begin{cases} \partial_t^2 v + (1-\Delta) v = -\sum_{\ell=0}^{k} \binom{k}{\ell} v^{k-\ell} \mathbb{Z}_\ell & \mathbb{Z}_0 \equiv 1 \\ (v, \partial_t v)\big|_{t=0} = (v_0, v_1) \in \mathcal{H}^s(\mathbb{T}^2) & \text{for some } s > 0. \end{cases}$$

deterministic

Enhanced data set : $\Xi = (v_0, v_1, \mathbb{Z}_1, \cdots, \mathbb{Z}_k)$.

$$\mathcal{X}^s = \mathcal{H}^s(\mathbb{T}^2) \times \left(C([0,1]; W^{-\varepsilon,\infty}(\mathbb{T}^2)) \right)^{\otimes k}$$

Prop 5.1.5: Given $\varepsilon > 0$, let $s = 1 - \varepsilon$.

Then, $\circledast$ is LWP. More precisely, $\exists\, T = T(\|\Xi\|_{\mathcal{X}^s}) > 0$

and a unique soln $(v, \partial_t v) \in C_T \mathcal{H}^s_x$ to $\circledast$.

Moreover, the map : $\Xi \in \mathcal{X}^s \longmapsto (v, \partial_t v) \in C_T \mathcal{H}^s_x$ is continuous.

As a corollary, we obtain

Thm 5.1.6: NLW on $\mathbb{T}^2$ with Law $((u(0), \partial_t u(0))) = \mu_1 \otimes \mu_0$

is a.s. LWP in the sense that the v-eqn (NLW_v) is a.s. LWP.
(with $(v_0, v_1) = (0,0)$)

Moreover, if v_N satisfies

$$\begin{cases} \partial_t^2 v_N + (1-\Delta) v_N = -\sum_{\ell=0}^{k} \binom{k}{\ell} v_N^{k-\ell} :z_N^\ell: \\ (v_N, \partial_t v_N)|_{t=0} = (0,0), \end{cases}$$

$:z_N^\ell:$ — Prop 5.1.4 → $:z^\ell:$

Prop 5.1.4 & Prop 5.1.5 →

then $(v_N, \partial_t v_N) \to (v, \partial_t v)$, a.s. in $C_T \mathcal{H}^{1-\varepsilon}_x$

$T = T_\omega$ ← random local existence time, a.s. positive

$\Rightarrow (u_N, \partial_t u_N) \mapsto (u, \partial_t u)$, a.s. in $C_T \mathcal{H}^{-\varepsilon}_x$.

· **Proof of Prop 5.1.5:**

$$\Gamma(v)(t) = W(t)(v_0, v_1) - \sum_{l=0}^{k} \binom{k}{l} \int_0^t \underbrace{\frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle}}_{S(t-t')} \left(v^{k-l}(t') Z_l(t')\right) dt'.$$

$0 < T \le 1$

$Z_0 = 1.$

· $\underline{l=0}$: $\left\| \int_0^t S(t-t') v^k(t') dt' \right\|_{C_T H^{1-\varepsilon}_x}$ (S = $S(t-t')$)

$$\lesssim T \|v^k\|_{C_T H^{-\varepsilon}_x}$$

$\frac{\varepsilon}{2} = \frac{1+\varepsilon}{2} - \frac{1}{2}$ $\qquad$ $\frac{s}{d} = \frac{1}{p} - \frac{1}{q}$

Sob Hölder $\lesssim T \|v\|_{C_T L^{\frac{2k}{1+\varepsilon}}_x}$

$\frac{1}{2} - \frac{1+\varepsilon}{2k} = \frac{k-1-\varepsilon}{2k} \le \frac{1-\varepsilon}{2}$

$k \ge 2.$

$$\lesssim T \|v\|^k_{C_T H^{1-\varepsilon}_x}$$

· $\underline{l=k}$:

$$\left\| \int_0^t S(t-t') Z_k(t') dt' \right\|_{C_T H^{1-\varepsilon}_x} \lesssim T \|Z_k\|_{C_T H^{-\varepsilon}_x} \le T \|\Xi\|_{X^s}.$$

Lem 5.1.7: $s \geq 0$, $1 < p \leq \infty$, $1 < q, r < \infty$

$q, r' \geq p'$ (for our purpose, $p = \infty \Rightarrow p' = 1$ — trivially satisfied)

Benyi-Oh-Zhao PAMS '24 (GKO '18) Also on $\mathbb{R}^d$

s.t. $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{r} + \frac{s}{d}$.

Then,
$$\| \langle \nabla \rangle^{-s} (fg) \|_{L^r(\mathbb{T}^d)} \lesssim \| \langle \nabla \rangle^{-s} f \|_{L^p(\mathbb{T}^d)} \| \langle \nabla \rangle^{s} g \|_{L^q(\mathbb{T}^d)}$$

Moral: The product fg "can be defined" if the sum of regularities is non-negative.

- $\ell = 1, \dots, k-1$:
$$\| \int_0^t S(t-t') (v^{k-\ell}(t') Z_\ell(t')) dt' \|_{C_T H^{1-\varepsilon}_x} \lesssim T \| v^{k-\ell} Z_\ell \|_{C_T H^{-\varepsilon}_x}$$

Lem 5.1.7 ($q = r = 2$, $p = \infty$)
$$\lesssim T \| Z_\ell \|_{C_T W^{-\varepsilon,\infty}_x} \| \langle \nabla \rangle^{\varepsilon} (v^{k-\ell}) \|_{C_T L^2_x}$$

$\frac{1}{2} - \frac{1}{2(k-1)} \leq \frac{1-\varepsilon}{2}$

frac. Leibniz
$$\lesssim \| v \|^{k-\ell}_{C_T W^{\varepsilon, 2(k-\ell)}_x} \lesssim \text{Sob} \lesssim \| v \|^{k-\ell}_{C_T H^{1-\varepsilon}_x}$$

$$\Rightarrow \|\vec{\Gamma}(\vec{v})\|_{C_T\mathcal{H}^{1-\varepsilon}_x} \lesssim \|(v_0,v_1)\|_{\mathcal{H}^{1-\varepsilon}} \leq \|\Xi\|_{X^{1-\varepsilon}}. \qquad s = 1-\varepsilon$$

$$+ T\left(\|\vec{v}\|^k_{C_T\mathcal{H}^{1-\varepsilon}_x} + \sum_{\ell=1}^{k} \|\vec{v}\|^{k-\ell}_{C_T\mathcal{H}^{1-\varepsilon}_x} \|\Xi\|_{X^s} \right)$$

- A similar computation yields the difference estimate.

□

Rmk: By using the Strichartz estimate, we can lower the regularity of initial data.

ex: $k=3$. $(v_0,v_1) \in \mathcal{H}^{\frac{1}{4}+\varepsilon}$ as in the deterministic case

see GKO '18.

Thm 5.1.8: The defocusing renormalized NLW on $\mathbb{T}^2$ ($k \in 2\mathbb{N}+1$) is a.s. GWP w.r.t. the Gibbsian initial data, and the Gibbs meas is invariant under the resulting NLW dynamics.

Truncated NLW:
$$\begin{cases} \partial_t^2 u_N + (1-\Delta) u_N + \pi_N \left(:(\pi_N u_N)^k: \right) = 0, \quad k \in 2\mathbb{N}+1 \\ \text{Law}\left((u_N(0), \partial_t u_N(0))\right) = \rho_N \end{cases}$$

$$= \left(\rho_N^{low} \otimes (\pi_N^{\perp})_{\#} \mu_1 \right) \otimes \mu_0$$

- nonlinear dynamics for $|n| \le N$ $\Leftarrow$ Thm 5.1.6 applies (after small modification)
- lin. dynamics for $|n| \ge N$

$\Rightarrow$ $(u_N, \partial_t u_N) \longrightarrow (u, \partial_t u)$ locally in time.

Then, apply Bourgain's inv. meas. argument. See the notes from 2024

One key diff: we study v_N, v eqns.

$N \in \mathbb{N}$. · key prop: $\forall\, T, \varepsilon > 0$, $\exists\, \Omega_N = \Omega_N(T, \varepsilon)$ $\subset \mathcal{H}^{-\varepsilon}(\mathbb{T}^2)$

① $\rho_N(\Omega_N^c) < \varepsilon$

② $\| (v_N, \partial_t v_N) \|_{C_T \mathcal{H}^{1-\varepsilon}_x} \lesssim \left(\log \frac{1}{\varepsilon}\right)^{k/2}$ ← unif in N

See Oh-Robert-Tzvetkov, AHL '23 (Sec 5)

Rmk: $k = 3$: We can prove pathwise GWP (GKOT, '22).

SNLH: pathwise GWP for any $k \in 2\mathbb{N}+1$ (Mourrat-Weber, AOP '17) Also, on $\mathbb{R}^2$

NLW (or Sd NLW) on $\mathbb{R}^2$:

Oh-Tolomeo-Wang-Zheng, CMP '25+

· Back to the construction of Wick powers $:z^\ell:$ (Prop 5.1.4 on page 5.

· hypercontractivity & Wiener chaos estimate

Fix $d \in \mathbb{N}$. $H = L^2(\mathbb{R}^d; \mu_d)$ $d\mu_d = \frac{1}{(2\pi)^{d/2}} e^{-|x|^2/2} dx$

$x = (x_1, \cdots, x_d)$

homog Wiener chaos of order k.

$$\mathcal{H}_k = \overline{\left\{ \prod_{j=1}^{d} H_{k_j}(x_j) \;:\; k = k_1 + \cdots + k_d \right\}}^{\|\cdot\|_H}$$

or $L^p(\mathbb{R}^d, \mu_d)$ $p < \infty$

↑ When $d = \infty$, $k_j = 0$ except for finitely many j's.

Ito - Wiener decomposition

$$H = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$$

· Ornstein-Uhlenbeck operator $L = \Delta - x \cdot \nabla$

See the note "OU process, ..." on my website.

$F \in \mathcal{H}_k$ is an eigenfunc of L with eigenvalue $-k$.

· Lem 5.1.9: hypercontractivity of the OU semigroup e^{tL}

$\infty > p \geq q > 1$. Then, for $t \geq \frac{1}{2} \log\left(\frac{p-1}{q-1}\right)$, we have

$$\| e^{tL} f \|_{L^p(\mathbb{R}^d, \mu_d)} \leq \| f \|_{L^q(\mathbb{R}^d, \mu_d)}$$

indep of $d \in \mathbb{N} \Rightarrow$ holds true for $d = \infty$.

($\Leftrightarrow$ log Sobolev ineq. (Gross '75

· $q = 2$, $f \in \mathcal{H}_k \Rightarrow e^{tL} f = e^{-tk} f$. Take $t = \frac{1}{2} \log(p-1)$

$$\| f \|_{L^p(\mathbb{R}^d, \mu_d)} \leq e^{tk} \| f \|_{L^2} = (p-1)^{k/2} \| f \|_{L^2}.$$

Let $\mathcal{H}_{\le k} = \bigoplus_{j=0}^{k} \mathcal{H}_j$.

Simon's book. Thm I.22 '74.

Lem 5.1.10: Wiener chaos estimate $\quad 2 \le p < \infty$

$$\|F\|_{L^p(\Omega)} \le (p-1)^{k/2} \|F\|_{L^2(\Omega)}$$

for any $F \in \mathcal{H}_{\le k}$.

Lem 5.1.11: f, g, jointly Gaussian, mean 0, variance σ_f, σ_g.

$$\mathbb{E}\left[H_k(f;\sigma_f)\, H_\ell(g;\sigma_g) \right] = \delta_{k\ell} \cdot k! \left(\mathbb{E}[fg] \right)^k.$$

Proof of Prop 5.1.4

$$Z_N(t,x) = \sum_{|n|\le N} \frac{\cos(t\langle n\rangle)\, g_n + \sin(t\langle n\rangle)\, h_n}{\langle n\rangle} e^{in\cdot x}.$$

- $\mathbb{E}\left[|\langle\nabla\rangle^{-\varepsilon} Z_N(t,x)|^2\right] = \sum_{|n|\le N} \frac{1}{\langle n\rangle^{2+2\varepsilon}} \lesssim 1$, unif in $N \in \mathbb{N}$.

- $\mathbb{E}\left[:Z_N^\ell(t,x): \, :Z_N^\ell(t,y): \right]$

$$\overset{\text{Lem 5.1.11}}{=} \ell! \left(\mathbb{E}\left[Z_N(t,x) Z_N(t,y) \right] \right)^\ell$$

$e_{n_1}(x-y)\cdots e_{n_\ell}(x-y)$

$$= \ell! \sum_{|n_1|,\cdots,|n_\ell|\le N} \prod_{j=1}^{\ell} \frac{1}{\langle n_j\rangle^2} \, e_{n_1+\cdots+n_\ell}(x-y)$$

- Apply $\langle\nabla_x\rangle^{-\varepsilon}$, $\langle\nabla_y\rangle^{-\varepsilon}$, and set $x=y$.

$$\Rightarrow \mathbb{E}\left[|\langle\nabla\rangle^{-\varepsilon} :Z_N^\ell(t,x):|^2 \right] \lesssim \sum_{|n_1|,\cdots,|n_\ell|\le N} \frac{1}{\langle n_1+\cdots+n_\ell\rangle^{2\varepsilon}} \prod_{j=1}^{\ell} \frac{1}{\langle n_j\rangle} \lesssim 1, \text{ unif in } N$$

$r<\infty$

$\Rightarrow \| \; \| : z_N^\ell(t) : \|_{W_x^{-\varepsilon,r}} \|_{L^p(\Omega)}$ Minkowski's integral ineq

$p<\infty$

$\lesssim p^{\ell/2}$, unif in N. (Also for $r=\infty$, by Sobolev)

$\Rightarrow \quad : z_N^\ell(t) :$ is bdd in $L^p(\Omega; W_x^{-\varepsilon,\infty})$ $\quad p<\infty$, $\varepsilon>0$.

· time regularity: $\delta_h f(t) = f(t+h) - f(t)$.

$$\frac{1}{\ell!} \mathbb{E}\left[\delta_h : z_N^\ell(t,x) : \; \delta_h : z_N^\ell(t,y) : \right]$$

$$\underset{\text{Lem 5.1.11}}{=} \Big(\mathbb{E}\left[z_N(t+h,x)\, z_N(t+h,y) \right] \Big)^\ell - \Big(\mathbb{E}\left[z_N(t,x)\, z_N(t+h,y) \right] \Big)^\ell$$

$$- \Big(\mathbb{E}\left[z_N(t+h,x)\, z_N(t,y) \right] \Big)^\ell + \Big(\mathbb{E}\left[z_N(t,x)\, z_N(t,y) \right] \Big)^\ell$$

$$\mathbb{E}\left[\left| \langle \nabla \rangle^{\xi} \, \delta_h : z_N^{\ell}(t,x) : \right|^2 \right]$$

$$\sim \sum_{|n_1|,\cdots,|n_\ell| \le N} \frac{1}{\langle n_1 + \cdots + n_\ell \rangle^{2\xi}} \prod_{j=1}^{\ell} G_j(t, n_j, h)$$

$$G_1(t,h,n) = \mathbb{E}\left[\delta_h \widehat{z}_N(t_1,n) \cdot \overline{\widehat{z}_N(t_1,n)} \right]$$

$j \neq 1$ $\quad G_j(t,h,n) = \mathbb{E}\left[\widehat{z}_N(t_1,n) \, \overline{\widehat{z}_N(t_2,n)} \right]$

$t_1, t_2 = t$ or $t+h$.

$$\Rightarrow |G_j(t,h,n)| \lesssim \frac{1}{\langle n \rangle^2}, \quad j = 1, \cdots, \ell.$$

$j=1$: By MVT,

$$|G_1(t,h,n)| \lesssim \frac{|h|}{\langle n \rangle}$$

$$|\cos((t+h)\langle n \rangle) - \cos(t\langle n \rangle)| \overset{\text{MVT}}{\lesssim} |h|\langle n \rangle$$

$$\Rightarrow |G_1(t,h,n)| \lesssim \frac{|h|^{\alpha}}{\langle n \rangle^{2-\alpha}}, \quad 0 \le \alpha \le 1.$$

Given $\varepsilon > 0$, choose $2\varepsilon - \alpha > 0$ s.t.

we can sum over $n_1, \cdots, n_\ell$.

$$(**) \Rightarrow \Big\| \, \| \delta_h : z_N^\ell(t) : \|_{W^{-\varepsilon, r}} \Big\|^p_{L^p(\Omega)} \lesssim p^{p \cdot \frac{\ell}{2}} \, |h|^{\frac{\alpha p}{2}}$$

> 1 by choosing $p \gg 1$.

$$\frac{\alpha p}{2} = 1 + \left(\frac{\alpha p}{2} - 1\right)$$

for $r < \infty$ (and for $r = \infty$, by Sobolev.)

$\Rightarrow$ By Kolmogorov's continuity criterion,

$$: z_N^\ell : \; \in C_T W_x^{-\varepsilon, \infty}, \text{ a.s.}$$

in fact $C_T^\beta W_x^{-\varepsilon, \infty}$ $\beta < \varepsilon$.

A similar computation on $: z_M^\ell : - : z_N^\ell :$ for $M \geq N \geq 1$

shows this is Cauchy a.s. in $C_T W_x^{-\varepsilon, \infty}$ (also in $L^p(\Omega; C_T W_x^{-\varepsilon, \infty})$)

$\square$

Kolmogorov's continuity criterion (Exercise 8.2 in "Stoch. Processes")

$$(**) \Rightarrow P\left(\sup_{t_1 \neq t_2} \frac{\| :Z_N^l(t_1): - :Z_N^l(t_2): \|_{W_x^{-\varepsilon,\infty}}}{|t_1 - t_2|^{\frac{\alpha}{2} - \frac{1}{p}}} > \lambda \right) \leq \frac{C}{\lambda^p}$$

for any finite $p \geq 1$.

$\Rightarrow$ $:Z_N^l: \in L^p(\Omega; C_T W_x^{-\varepsilon,\infty})$ with a unif bound in $N \in \mathbb{N}$

- A similar computation yields convergence of $:Z_N^l:$ in $L^p(\Omega; C_T W_x^{-\varepsilon,\infty})$.