Ph.D. course

Probabilistic well-posedness of dispersive PDEs

Tadahiro Oh

Fall 2025 - Spring 2026

Table of contents: (numbering continued from the course from 2024-2025)

- Lecture 1: Introduction. 3
  Chap 4: Stochastic dispersive PDEs.
- Lecture 2: **Sec 4.1**: Stochastic convolutions 21
- Lecture 3: Stochastic convolutions on $\mathbb{R}^d$ 35
  Sec 4.2: 1-d stochastic NLS and stochastic NLW
- Lecture 4: 1-d stochastic NLS on $\mathbb{T}$: Fourier restriction norm method 50
  global well-posedness: Ito approach
- Lecture 5: global well-posedness: Ito approach for 1-d SNLS 68
  pathwise approach for 1-d defocusing cubic SNLW
  Sec 4.3: Invariant Gibbs measure for SdNLW:
- Lecture 6: Invariant Gibbs measure for SdNLW 80
  truncated dynamics, OU process, invariance by generator computation
- Lecture 7: **Chap 5**: 2-d stochastic NLW 98
  Sec 5.1: NLW with random initial data
- Lecture 8: NLW with random initial data, continued 114
  Construction of the Wick powers
- Lecture 9: Garcia-Rodemich-Rumsey inequality 134
  Sec 5.2: Stochastic NLW with additive space-time white noise

Lec 1 : Oct. 23, 2025 (THU)

Dispersive PDEs:

- **Nonlinear Schrödinger eqn (NLS):**

$$\begin{cases} i\partial_t u + \Delta u \mp |u|^{p-1}u = 0 \\ u|_{t=0} = u_0 \in H^s(M) \end{cases}$$

$M = \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ or $\mathbb{R}^d$

- **Nonlinear wave eqn (NLW):**

$$\begin{cases} \partial_t^2 u - \Delta u \pm |u|^{p-1}u = 0 \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \in \mathcal{H}^s(M) := H^s(M) \times H^{s-1}(M) \end{cases}$$

- $\|f\|_{H^s(\mathbb{T}^d)} = \|\langle\nabla\rangle^s f\|_{L^2_x} = \|\langle n\rangle^s \hat{f}(n)\|_{\ell^2_n}$

  Bessel potential of order $-s$

  $= \left(\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s} |\hat{f}(n)|^2\right)^{1/2}$

$\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$

$|\nabla| = \sqrt{-\Delta}$

$\langle\nabla\rangle = \sqrt{1-\Delta}$

Goal: Study these eqns with rough random initial data and/or stochastic forcing.

- initiated by Bourgain '94 -
  Tzvetkov, Burq-Tzv, Oh mid '00.
  more recently: Gubinelli-Koch-Oh, Bringmann, Deng-Nahmod-Yue.
- of physical importance.
- at the intersection of harmonic analytic approach to dispersive PDEs & stoch. analysis.

See also the course note from 2021 (← random data / additive noise), 2022 (↑ multiplicative noise)

· NLS & NLW are Hamiltonian PDEs.

Hamiltonian for NLS: $H(u) = \frac{1}{2}\int |\nabla u|^2 dx \pm \frac{1}{p+1}\int |u|^{p+1} dx$

- for NLW: $E(\vec{u}) = E(u, \partial_t u) = H(u) + \frac{1}{2}\int (\partial_t u)^2 dx$

· NLS: L^2 = symplectic space (symplectic form) with $\omega(f,g) = \mathrm{Im}\int f\,\overline{g}\,dx$.

$J = i$.

(Gâteaux deriv at u in the direction of Φ)

$$\omega(\Phi, J\nabla H) \overset{\text{def}}{=} dH|_u(\Phi) \overset{\text{def}}{=} \frac{d}{d\varepsilon} H(u+\varepsilon\Phi)\Big|_{\varepsilon=0}$$

· linear case: $H(u) = \frac{1}{2}\int |\nabla u|^2 dx \Rightarrow dH_u(\Phi) = \mathrm{Re}\langle \nabla u, \nabla \Phi\rangle$

$$\overset{\text{IBP}}{=} \mathrm{Re}\langle \Phi, -\Delta u\rangle$$

$$= \mathrm{Im}\langle \Phi, i\Delta u\rangle$$

$$= \omega(\Phi, i\Delta u)$$

$\Rightarrow \partial_t u = J\nabla H = i\Delta u$

- can repeat a similar computation for the full Hamiltonian to write NLS as $\partial_t u = J\nabla H$.

Formally, $\partial_t u = i\frac{\partial H}{\partial \bar{u}}$

· NLW: Symplectic space $= \dot{\mathcal{H}}^{1/2} = \dot{H}^{1/2} \times \dot{H}^{-1/2}$

$$\langle \vec{f}, \vec{g} \rangle_{\dot{\mathcal{H}}^{1/2}} = \int (|\nabla| f_1) g_1 \, dx = \int (|\nabla|^{-1} f_2) g_2 \, dx$$

· almost symplectic structure: $J = \begin{pmatrix} 0 & -|\nabla|^{-1} \\ |\nabla| & 0 \end{pmatrix}$.

$J : \dot{\mathcal{H}}^{1/2} \to \dot{\mathcal{H}}^{1/2}$ and $J^2 = -Id$.

· symplectic form $\omega(\vec{f}, \vec{g}) = \langle \vec{f}, J\vec{g} \rangle_{\dot{\mathcal{H}}^{1/2}}$

Then, (NLW) $\iff \partial_t \begin{pmatrix} u \\ v \end{pmatrix} = J\nabla E$, $\quad v = \partial_t u$

where $\omega(\vec{\Phi}, J\nabla H) = dE|_{(u,v)}(\vec{\Phi}) = \frac{d}{d\varepsilon} E(\vec{u} + \varepsilon\vec{\Phi})\Big|_{\varepsilon=0}$.

$\vec{u} = (u, v)$

Formally, $\partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \mp \begin{pmatrix} 0 \\ |u|^{p-1}u \end{pmatrix}$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{\partial E(u,v)}{\partial (u,v)}$$

· Given a Hamiltonian system on $\mathbb{R}^{2N}$

$$\partial_t q_j = \frac{\partial H}{\partial p_j}, \quad \partial_t p_j = -\frac{\partial H}{\partial q_j}, \quad j = 1, \cdots, N$$

$$H(q,p) = H(q_1, \cdots, q_N, p_1, \cdots, p_N)$$

① Liouville thm : $div\, X = \sum_{j=1}^{N} \left(\frac{\partial}{\partial q_j} \frac{\partial H}{\partial p_j} + \frac{\partial}{\partial p_j} \left(-\frac{\partial H}{\partial q_j} \right) \right) = 0$

$\Rightarrow dq\,dp = \prod_{j=1}^{N} dq_j\, dp_j$ is invariant under the flow.

② Hamiltonian is conserved under the flow.

$$\partial_t H(q(t), p(t)) = \sum_{j=1}^{N} \left(\frac{\partial H}{\partial q_j} \partial_t q_j + \frac{\partial H}{\partial p_j} \partial_t p_j \right) = 0$$

$\partial_t q_j = \frac{\partial H}{\partial p_j}$, $\partial_t p_j = -\frac{\partial H}{\partial q_j}$

$\Rightarrow$ Gibbs measure ρ:

$$d\rho = Z^{-1} \underbrace{e^{-\beta H(q,p)}}_{②} \underbrace{dq\,dp}_{①}$$

$\beta > 0$ = inverse temperature

is invariant under the dynamics.

- Bourgain '94, '96 : study of invariant Gibbs measures for NLS on $\mathbb{T}^d$

co-dim'l Hamiltonian system

Friedlander '85, McKean '95

Lebowitz-Rose-Speer '88 (see the notes from last year)

NLS: $i\partial_t u + \underline{(1-\Delta)}\, u \mp |u|^{p-1}u = 0$

↑ to avoid an issue at the freq 0.

$$H(u) = \frac{1}{2}\|u\|_{H^1}^2 \pm \frac{1}{p+1}\int |u|^{p+1}dx.$$

· NO ∞-dim'l Lebesgue meas du !!

$$\Rightarrow \quad d\rho = Z^{-1} e^{-H(u)}\, du$$

$$= Z^{-1} e^{\mp \frac{1}{p+1}\int |u|^{p+1}dx}\, e^{-\frac{1}{2}\|u\|_{H^1}^2}\, du$$

↑ Gaussian meas on $\mathcal{D}'(\mathbb{T}^d)$

on $H^s(\mathbb{T}^d)$, $s < 1-\frac{d}{2}$

$W^{s,r}(\mathbb{T}^d)$, $1 \le r \le \infty$.

· − sign defocusing case: $d=1$, trivial
$d=2$, see the notes from 2024/2025

$d=3$, $p=4$: $\overline{\Phi}^4_3$-meas

· + sign: focusing case

$d=1$, LRS '88. Bourgain '94, Oh-Sosoe-Tolomeo '22

$d \ge 2$, Oh-Okamoto-Tolomeo '24, '25, Oh-Seong-Tolomeo '24

- Gaussian measure : $d\mu_s = Z^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du$

  - truncated version : $d\hat{u}(n)$ = Lebes meas on $\mathbb{C}$.

$$d\mu_{s,N} = Z_N^{-1} \, e^{-\frac{1}{2}\sum_{|n|\le N} \langle n\rangle^{2s} |\hat{u}(n)|^2} \prod_{|n|\le N} d\hat{u}(n)$$

$$= Z_N^{-1} \prod_{|n|\le N} e^{-\frac{1}{2}\langle n\rangle^{2s}|\hat{u}(n)|^2} d\hat{u}(n)$$

$$= |g_n|^2 \qquad \sim_n dg_n$$

Q: $N \to \infty$? u_N under $\mu_{s,N}$ (i.e. Law $(u_N) = \mu_{s,N}$ on $\mathbb{C}^{2N+1}$)

$$\Rightarrow \boxed{u_N = \sum_{|n|\le N} \frac{g_n}{\langle n\rangle^s} e^{in\cdot x}}$$

$\{g_n\}_{|n|\le N}$ = family of indep. standard complex-valued Gaussian r.v.'s

- $\mathbb{E}\left[\|u_N\|_{H^\sigma}^2\right] = \mathbb{E}\left[\sum_{|n|\le N} \frac{|g_n|^2}{\langle n\rangle^{2(s-\sigma)}}\right]$

$$\sim \sum_{|n|\le N} \frac{1}{\langle n\rangle^{2(s-\sigma)}} \lesssim 1, \text{ unif. in } N\in\mathbb{N}$$

if $2(s-\sigma) > d \iff \boxed{\sigma < s - \frac{d}{2}}$

- $u_N \to u$, a.s. in $H^\sigma(\mathbb{T}^d)$, $\sigma < s - \frac{d}{2}$

$\Rightarrow \underline{\mu_s = \mathrm{Law}(u)}$ $\quad u \in H^s(\mathbb{T}^d)$

- (H^s, H^σ, μ_s) = abstract Wiener space.

↑ Cameron-Martin space

$\underline{W^{\sigma,r}\ ?}$

$\underline{r<\infty}$: $\quad \| \, \| u_N \|_{W^{\sigma,r}} \|_{L^p(\Omega)} \le \| \, \| \overbrace{\langle \nabla \rangle^\sigma u_N(x)}^{\sum_{|n|\le N} \frac{g_n}{\langle n \rangle^{s-\sigma}} e^{in\cdot x}} \|_{L^p(\Omega)} \|_{L^r_x}$

$p \ge r$ finite

$\left(\|f\|_{W^{\sigma,r}} = \|\langle \nabla \rangle^\sigma f\|_{L^r} \right.$

$$\lesssim p^{1/2} \| \, \| \sum_{|n|\le N} g_n \frac{e^{in\cdot x}}{\langle n \rangle^{s-\sigma}} \|_{L^2(\Omega)} \|_{L^r_x}$$

throw away

$$\| \, \mathbb{1}_{|n|\le N} \frac{e^{in\cdot x}}{\langle n \rangle^{s-\sigma}} \|_{\ell^2_n}$$

$\sim p^{1/2}$ unif in $N \in \mathbb{N}$. if $\sigma < s - \frac{d}{2}$.

$\underline{r=\infty}$: Use Sobolev and repeat the computation above

$$\|u_N\|_{W^{\sigma,\infty}} \lesssim \|u_N\|_{W^{\sigma+\varepsilon,r}} \qquad \varepsilon r > d$$

(Bényi - Oh '13.

· Gibbs measure for NLS: $d\rho = Z^{-1} e^{\mp \frac{1}{p+1}\int |u|^{p+1} dx} d\mu_1$

$d\mu_1 = Z^{-1} e^{-\frac{1}{2}\|u\|_{H^1}^2} du$

↑ (massive) Gaussian free field on $\mathbb{T}^d$.

· We first need / want to study (local) well-posedness of NLS with the GFF initial data

$$u_0 = \sum_{n\in\mathbb{Z}^d} \frac{g_n}{\langle n\rangle} e^{in\cdot x} \overset{\text{a.s.}}{\in} H^s(\mathbb{T}^d) \setminus H^{1-\frac{d}{2}}(\mathbb{T}^d)$$

$$s < 1 - \frac{d}{2}$$

$(< 0 \text{ for } d \geq 2)$

From page ⑨,

$\mathbb{E}\big[\|u\|_{H^{1-\frac{d}{2}}}^2\big] = \infty \Rightarrow P(u \notin H^{1-\frac{d}{2}}) > 0$

$\Rightarrow$ Kolmogorov's 0-1 law $\Rightarrow$ a.s. not in $H^{1-\frac{d}{2}}$

Rmk: Law $(u) = \mu_s$, i.e. $\boxed{u = \sum_{n \in \mathbb{Z}^d} \frac{g_n}{\langle n \rangle^s} e^{in\cdot x}} \in H^{s-\frac{d}{2}-\varepsilon}$ a.s.

$f \in H^s(\mathbb{T}^d)$, deterministic

$$\langle f, u \rangle_{H^s} = \sum_{n \in \mathbb{Z}^d} \langle n \rangle^s \widehat{f}(n) \cdot \overline{g_n}$$

$$\Rightarrow \| \langle f, u \rangle_{H^s} \|_{L^2(\Omega)} = \| f \|_{H^s} < \infty$$

- Although $u \notin H^s(\mathbb{T}^d)$ a.s., the inner product $\langle f, u \rangle_{H^s}$ is a.s. finite for each $f \in H^s(\mathbb{T}^d)$
- $\langle f, u \rangle_{H^s}$ is Gaussian for each $f \in H^s(\mathbb{T}^d)$

$\Leftarrow$ This is often used as a definition of a Gaussian probability meas. (In this case, μ_s is a Gaussian probability measure.

Gibbs measure for NLW: $\partial_t^2 u + (1-\Delta)u \pm |u|^{p-1}u = 0$ (13)

$$d\rho = Z^{-1} e^{-E(u,\partial_t u)} du\, d(\partial_t u)$$

$$= Z^{-1} e^{\mp \frac{1}{p+1}\int |u|^{p+1} dx} \underbrace{e^{-\frac{1}{2}\|u\|_{H^1}^2} du \otimes e^{-\frac{1}{2}\|\partial_t u\|_{L^2}^2} d(\partial_t u)}$$

base Gaussian meas on $(u, \partial_t u)$

$\mu_1 \otimes \mu_0 (u, \partial_t u)$

$\Rightarrow$ $\mathrm{Law}(u_0) = \mu_1$, $\boxed{\mathrm{Law}(u_1) = \mu_0}$

$$u_0 = \sum_{n\in\mathbb{Z}^d} \frac{g_n}{\langle n\rangle} e^{in\cdot x} \in H^{1-\frac{d}{2}-}(\mathbb{T}^d) \setminus H^{1-\frac{d}{2}}(\mathbb{T}^d) \quad \text{a.s.}$$

$$u_1 = \sum_{n\in\mathbb{Z}^d} h_n e^{in\cdot x} \in H^{-\frac{d}{2}-}(\mathbb{T}^d) \setminus H^{-\frac{d}{2}}(\mathbb{T}^d)$$

• u and $\partial_t u$ are real-valued.

$\Rightarrow$ u_0 and u_1 are real-valued

$\Leftrightarrow$ $\widehat{u_j}(-n) = \overline{\widehat{u_j}(n)}$, $j=1,2$, $\forall\, n \in \mathbb{Z}^d$

(*) $g_{-n} = \overline{g_n}$ & $h_{-n} = \overline{h_n}$, $\forall\, n \in \mathbb{Z}^d$

Define the index set $I = \left(\bigcup_{k=0}^{d-1} \mathbb{Z}^k \times \mathbb{N} \times \{0\}^{d-k-1} \right) \cup \{\vec{0}\}$

$$'' = \mathbb{Z}^d / 2''.$$

$\{(g_n, h_n)\}_{n \in I}$ = family of indep. std. $\mathbb{C}$-valued Gaussian r.v.'s

and extend onto $\mathbb{Z}^d$ by (*)

· μ_0 = (spatial) white noise on $\mathbb{T}^d$.

$d\mu_0 = Z^{-1} e^{-\frac{1}{2}\|u\|_{L^2}^2} du$

· $\text{Law}(u) = \mu_0$

$$\Rightarrow \quad u = \sum_{n \in \mathbb{Z}^d} g_n e^{in\cdot x} \in H^{-\frac{d}{2}-}(\mathbb{T}^d)$$

$$W^{-\frac{d}{2}-,\infty}(\mathbb{T}^d)$$

$$C^{-\frac{d}{2}-}(\mathbb{T}^d)$$

· White noise u:

formal: $\mathbb{E}[u(x)\overline{u(y)}] = \delta(x-y)$

$$\left(= \mathbb{E}\Big[\sum_n g_n e^{in\cdot x} \sum_k \bar{g}_k e^{-ik\cdot y}\Big] = \sum_n e^{in\cdot(x-y)} \right.$$

$$\mathbb{E}[g_n^{\ell_1} \bar{g}_k^{\ell_2}] = \ell_1!\, 1_{n=k} 1_{\ell_1=\ell_2}$$

rigorous: $\mathbb{E}[u(f_1)\overline{u(f_2)}] = \langle f_1, f_2\rangle_{L^2} \quad \forall f_1, f_2 \in L^2$

$$\left(= \mathbb{E} \sum_n \hat{f}_1(n)\overline{g_n} \cdot \sum_k \overline{\hat{f}_2(k)}\, g_k = \sum_n \hat{f}_1(n)\overline{\hat{f}_2(n)} \right.$$

Chap 4: Stochastic dispersive PDEs

Stochastic NLS (SNLS): $i\partial_t u + \Delta u + \mathcal{N}(u) = \Phi\xi$

Stochastic NLW (SNLW): $\partial_t^2 u + (1-\Delta)u + \mathcal{N}(u) = \Phi\xi$.

- ξ = space-time white noise on $\mathbb{R}_t \times \mathbb{T}_x^d$
- Φ = Hilbert-Schmidt op from L^2 to H^s.

formal: $\mathbb{E}[\xi(t_1,x_1)\overline{\xi(t_2,x_2)}] = \delta(t_1-t_2)\delta(x_1-x_2)$.

rigorous: $\mathbb{E}[\xi(f_1)\overline{\xi(f_2)}] = \langle f_1, f_2\rangle_{L^2}$

$\forall\, f_1, f_2 \in L^2(\mathbb{R}\times\mathbb{T}^d)$

· Duhamel formulation for SNLS: $S(t) = e^{it\Delta}$

$$u(t) = S(t)u_0 - i\int_0^t S(t-t')\,N(u)(t')\,dt' - i\int_0^t S(t-t')\,\Xi(dt')$$

suppose $\Phi = Id$

$=: \Psi$ stochastic convolution

· Define $B_n(t) = \langle \Xi, \mathbb{1}_{[0,t]} \cdot e^{in\cdot x} \rangle_{\mathcal{X}'-\mathcal{X}}$

$\mathbb{1}_{[0,t]} \cdot e^{in\cdot x} \in L^2$, $\mathbb{E}[(B_n(t))^2] = t$

B_n is a BM. ⇐
- $B_n(0) = 0$
- $B_n(t)$ = mean 0 Gaussian r.v.
- indep increment on disjoint intervals.

· $\{B_n\}_{n\in\mathbb{Z}^d}$ = family of Indep BM's.

(real-valued case: $B_{-n} = \overline{B_n}$)

L^2- cylindrical Wiener process

On $\mathbb{T}^d$

$$W(t) = \sum_{n \in \mathbb{Z}^d} B_n(t)\, e^{in\cdot x} \in W^{-\frac{d}{2}-,\infty}(\mathbb{T}^d) \quad \text{a.s.}$$

$\Leftarrow$ dilated spatial white noise

$$\Psi(t) = \int_0^t S(t-t')\, \Xi(dt') \qquad \Phi = Id$$

$$= \int_0^t S(t-t')\, dW(t')$$

$$= \sum_{n \in \mathbb{Z}} e^{in\cdot x} \underbrace{\int_0^t e^{-i(t-t')|n|^2}\, dB_n(t')}_{= \text{Wiener integral (next time)}}$$

Lec 2: Oct 30, 2025 (THU)

$$\Psi(t) = \int_0^t S(t-t')\,\xi(dt')$$

$$= \int_0^t S(t-t')\,dW(t')$$

$$= \sum_{n\in\mathbb{Z}^d} e^{in\cdot x} \underbrace{\int_0^t e^{-i(t-t')|n|^2}\,dB_n(t')}_{\text{Wiener integral}}$$

· **Wiener integral:** Given $f \in L^2([a,b])$ deterministic

$$I(f) = \int_a^b f(t)\,dB(t) \qquad B = \text{B.M.}$$

· constructed as a limit of Riemann sums in $L^2(\Omega)$

see Kuo, chap 2.

· mean 0 Gaussian r.v. $\quad \mathbb{E}[I(f)] = 0$

· $\mathrm{Var}(I(f)) = \mathbb{E}[|I(f)|^2] = \|f\|^2_{L^2([a,b])}$

$I : L^2([a,b]) \longrightarrow L^2(\Omega)$ is an isometry. (2)

- $\mathbb{E}[I(f)\,\overline{I(g)}] = \langle f, g\rangle_{L^2}$

$$\Rightarrow \quad \Psi(t) = \sum_{n\in\mathbb{Z}^d} e^{in\cdot x} \underbrace{\int_0^t e^{-i(t-t')|n|^2}\, dB_n(t')}_{\sim N_{\mathbb{C}}(0,t)}$$

⇐ dilated spatial white noise for each fixed t.

$$\in H^{-\frac{d}{2}-}(\mathbb{T}^d) \quad \text{a.s.} \quad (\text{for each fixed } t)$$

Note: For $f \in C^1$, we can define $I(f)$ as a Paley-Wiener-Zygmund integral. (by IBP):

$$I(f) = \int_a^b f\, dB = f(b)B(b) - \int_a^b f'(t)\, B(t)\, dt$$

Recall $B(0) = 0$.

Sec 4.1: Stochastic convolutions

· Stoch. convolution for SNLS:

$$\begin{cases} i\partial_t \Psi + \Delta \Psi = (i)\Phi \xi \\ \Psi|_{t=0} = 0. \end{cases}$$

(↑ so that we do not need to keep i in Ψ.)

$$\Psi(t) = \int_0^t S(t-t')\Phi\, dW(t')$$

$$= \sum_{n\in\mathbb{Z}^d} \int_0^t S(t-t')\Phi(e_n)\, dB_n(t')$$

($n \in \mathbb{N}$ on $\mathbb{R}^d$)

$$\begin{cases} e_n = e^{in\cdot x} \text{ on } \mathbb{T}^d. \\ \{e_n\}_{n\in\mathbb{N}} = \text{O.N.B. of } L^2(\mathbb{R}^d) \end{cases}$$

· If Φ is translation invariant (spatial homogeneous)

i.e. Fourier multiplier operator (Grafakos vol. 1. Chap 2)

$$\begin{cases} \tau_y f(x) = f(x-y) \\ \Rightarrow \Phi(\tau_y f) = \tau_y \Phi(f). \end{cases}$$

On $\mathbb{T}^d$

$$\Psi(t) = \sum_{n\in\mathbb{Z}^d} \phi_n e_n \int_0^t e^{-i(t-t')|n|^2} dB_n(t)$$

· **Lemma 4.1.1: Kolmogorov continuity criterion**

$\{X_t\}$ with values in a metric space

Suppose $\mathbb{E}\left[d(X_{t_1}, X_{t_2})^p \right] \le C_0 |t_1 - t_2|^{1+\alpha}$

for some $p \ge 1$, $\alpha > 0$.

Then, $$P\left(\sup_{t_1 \neq t_2} \frac{d(X_{t_1}, X_{t_2})}{|t_1 - t_2|^{\alpha/p} - \varepsilon} \ge \lambda \right) \le \frac{C_1}{\lambda^p}, \quad \forall\, 0 < \varepsilon < \frac{\alpha}{p}$$

$\Rightarrow$ X_t is a.s. $\left(\frac{\alpha}{p} - \varepsilon\right)$- Hölder continuous.

ex: B.M. $\quad \mathbb{E}\left[|\underbrace{B(t_2) - B(t_1)}_{\text{Gaussian}}|^2 \right] = |t_2 - t_1|$

$\Rightarrow \mathbb{E}\left[|B(t_2) - B(t_1)|^p \right] \sim p^{p/2} |t_2 - t_1|^{p/2}$, $\quad p < \infty$

($\frac{p}{2} = 1 + \overbrace{\frac{p}{2} - 1}^{\alpha}$)

$\Rightarrow \frac{\alpha}{p} = \frac{\frac{p}{2} - 1}{p} = \frac{1}{2} - \frac{1}{p} \to \frac{1}{2}-$ as $p \to \infty$ $\Rightarrow$ BM is $(\frac{1}{2}-)$- Hölder conti a.s.

· We say that $\Phi : H_1 \to H_2$ is Hilbert-Schmidt

if $\|\Phi\|_{HS(X;Y)} = \Big(\sum_{n\in\mathbb{N}} \|\Phi(e_n)\|_Y^2\Big)^{1/2} < \infty$

· $\{e_n\}_{n\in\mathbb{N}}$ = O.N.B. of X $\quad$ H_1, H_2, Hilbert space

· If Φ is translation invariant, i.e.

$\Phi(e_n) = \phi_n e_n$ (on $\mathbb{T}^d$)

A special case of γ-radonifying operators in the Banach space setting.
(see Appendix of Forlano-Oh-Wang '20)

then

$$\|\Phi\|_{HS(L^2(\mathbb{T}^d);\, H^s(\mathbb{T}^d))} \simeq \Big(\sum_{n\in\mathbb{Z}^d} \|\phi_n e_n\|_{H^s}^2\Big)^{1/2} = \|\langle n\rangle^s \phi_n\|_{l^2_n} = \|\check{\phi}\|_{H^s}$$

Prop 4.1.2: $\Phi \in HS(L^2(\mathbb{T}^d); H^s(\mathbb{T}^d))$, trans. inv.

Then, $\Psi(t) = \int_0^t S(t-t')\Phi\, dW(t') \in C_t^{\frac{\alpha}{2}-} W_x^{s-\alpha, r}(\mathbb{T}^d)$

a.s. $1 \le r \le \infty$, $\alpha > 0$.

only on $\mathbb{T}^d$.

Also, $\Psi \in C_t H_x^s$, a.s.

Pf: $0 < t_1 \le t_2$

$$\mathbb{E}\left[\Psi(t_1, x_1)\,\overline{\Psi(t_2, x_2)}\right]$$

$$= \sum_{n\in\mathbb{Z}^d} |\phi_n|^2 e_n(x_1 - x_2) \int_0^{t_1} e^{-i(t_1 - t_2)|n|^2}\, dt'$$

indep

Wiener isom.

$$\Leftarrow \text{Apply } \langle \nabla_{x_1} \rangle^s \text{ and } \langle \nabla_{x_2} \rangle^s$$

$$\sum_{n\in\mathbb{Z}^d} \phi_n e_n \int_0^t e^{-i(t-t')|n|^2}\, dB_n(t')$$

independent for $n \in \mathbb{Z}^d$

$$\int_0^{t_1} = \int_0^{t_2} + \int_{t_1}^{t_2}$$

$$t_1\, e^{-i(t_1 - t_2)|n|^2}$$

$$\mathbb{E}\left[\int_0^{t_1} f\, dB_n \overline{\int_0^{t_2} g\, dB_n}\right] = \int_0^{t_1} f\, \bar{g}\, dt.$$

↑ indep of $\int_0^{t_1} f\, dB_n$

$$\mathbb{E}\left[\langle\nabla_{x_1}\rangle^s \Psi(t_1,x_1)\,\overline{\langle\nabla_{x_2}\rangle^s \Psi(t_2,x_2)}\right]$$

$$= \sum_{n\in\mathbb{Z}^d} |\Phi_n|^2 \langle n\rangle^{2s} \underbrace{e_n(x_1-x_2)}\cdot t_1 \underbrace{e^{-i(t_1-t_2)|n|^2}}$$

$\to 1$

- Now, set $x_1 = x_2$ and $t_1 = t_2$

$$\|\underbrace{\langle\nabla\rangle^s \Psi(t,x)}\|_{L^p(\Omega)} \sim p^{1/2} \|\langle\nabla\rangle^s \Psi(t,x)\|_{L^2(\Omega)}$$

Gaussian

$$= p^{1/2} t^{1/2} \|\Phi\|_{HS(L^2;H^s)}$$

$r<\infty$, $r\le p<\infty$

$$\left\| \|\Psi(t)\|_{W^{s,r}_x} \right\|_{L^p(\Omega)} = \left\| \|\langle\nabla\rangle^s \Psi(t)\|_{L^r_x} \right\|_{L^p(\Omega)}$$

$$\overset{\text{Mink}}{\le} \left\| \|\langle\nabla\rangle^s \Psi(t,x)\|_{L^p(\Omega)} \right\|_{\underbrace{L^r(\mathbb{T}^d)}}$$

$$\sim p^{1/2} t^{1/2} \|\Phi\|_{HS(L^2;H^s)}$$

throw away since $|\mathbb{T}^d| \sim 1$.

$$\Rightarrow \quad \Psi(t) \in \begin{cases} W^{s,r}(\mathbb{T}^d), & \text{a.s.} \quad 1 \le r < \infty \\ W^{s-,\infty}(\mathbb{T}^d), & \text{a.s.} \quad r = \infty \end{cases} \quad \text{(by Sobolev}$$

$$\text{Also, } \quad P\left(\|\Psi(t)\|_{W^{s,r}} > \lambda \right) \le C\, e^{-c\,\lambda^2/t\,\|\Phi\|_{HS}^2}, \quad r < \infty$$

Next, we study the temporal regularity.

· Given $h \in \mathbb{R}$, let

$$\delta_h \Psi(t,x) = \Psi(t+h,x) - \Psi(t,x)$$

$$\Rightarrow \mathbb{E}\left[\delta_h \Psi(t,x) \overline{\delta_h \Psi(t,y)} \right]$$

$$= \mathbb{E}\left[\Psi(t+h,x) \overline{\Psi(t+h,y)} \right] - \mathbb{E}\left[\Psi(t,x) \overline{\Psi(t+h,y)} \right]$$

$$- \mathbb{E}\left[\Psi(t+h,x) \overline{\Psi(t,y)} \right] + \mathbb{E}\left[\Psi(t,x) \overline{\Psi(t,y)} \right]$$

$$= \sum_{n \in \mathbb{Z}^d} |\phi_n|^2 e_n(x-y) \underbrace{\left\{ (t+h) - t e^{ih|n|^2} - t e^{-ih|n|^2} + t \right\}}_{=: F_n(t,h)} \qquad h > 0$$

$$|F_n(t,h)| \le |h| + t\,|1-e^{ih|n|^2}| + t\,|1-e^{-ih|n|^2}| \qquad (9)$$

$$\overset{\text{MVT}}{\lesssim} |h| + t \underbrace{\min(1, h|n|^2)}_{(h|n|^2)^\alpha} \quad \forall\ 0 \le \alpha \le 1$$

$$\Rightarrow \|\langle\nabla\rangle^s \delta_h \Psi(t,x)\|_{L^p(\Omega)}$$

$$\sim p^{1/2} \|\langle\nabla\rangle^s \delta_h \Psi(t,x)\|_{L^2(\Omega)} \qquad |t| \le T.$$

$$\lesssim p^{1/2} T^{1/2} |h|^{\alpha/2} \|\Phi\|_{HS(L^2; H^{s+\alpha})}$$

$$\Rightarrow \Big\| \|\delta_h \Psi(t)\|_{W^{s,r}_x} \Big\|_{L^p(\Omega)} \le C_r p^{1/2} T^{1/2} |h|^{\alpha/2} \|\Phi\|_{HS(L^2; H^{s+\alpha})}$$

$r < \infty$, $p < \infty$

Lem 4.1.1

$$\Rightarrow \Psi \in C_t^{\frac{\alpha}{2}-\frac{1}{p}-} W_x^{s-\alpha, r} \quad \text{a.s.}$$

if $\Phi \in HS(L^2; H^s)$ (by replacing $s+\alpha$ by s).

take $p \gg 1$

$\Rightarrow$ $\Psi \in C_t^{\frac{\alpha}{2}-\varepsilon} W_x^{s-\alpha, r}$ a.s. if $\Phi \in HS(L^2; H^s)$

$r \leq \infty, \quad \alpha > 0$

$\underline{r=2}$: $\widetilde{\Psi}(t) = S(-t)\Psi(t) = \int_0^t S(-t')\Phi \, dW(t')$

$\Rightarrow \mathbb{E}\left[\widetilde{\Psi}(t_1, x_1)\overline{\widetilde{\Psi}(t_2, x_2)}\right] = \sum_{n \in \mathbb{Z}} |\Phi_n|^2 e_n(x_1 - x_2) \cdot t_1$ $\quad 0 < t_1 \leq t_2$

By repeating the computation,

$$\left\| \left\| \delta_h \widetilde{\Psi}(t) \right\|_{H_x^s} \right\|_{L^p(\Omega)} \leq C_T p^{1/2} \|\Phi\|_{HS(L^2; H^s)} \underline{|h|^{1/2}}$$

Lem 4.1.1 $\Rightarrow$ $\widetilde{\Psi} \in C_t^{\frac{1}{2}-} H_x^s$, a.s.

$S(t)f \in C_t H_x^s$, $f \in H^s$

$\Rightarrow$ $\Psi \in C_t^{\frac{1}{2}-} H_x^s$, a.s. since $S(t)$ is unitary on H^s

$\square$ (

$$\Psi(t+h) - \Psi(t) = S(t+h)\left[\widetilde{\Psi}(t+h) - \widetilde{\Psi}(t)\right] + \underbrace{\left(S(t+h) - S(t)\right)}_{\to 0 \text{ as } h \to 0.} \widetilde{\Psi}(t)$$

Note: Later in the course, I will present two more proofs of Prop 4.2.

---

ex: $\Phi = \mathrm{Id} \Rightarrow \Phi \in HS(L^2; H^s)$, $s < -\frac{d}{2}$

$\Rightarrow \Psi \in C_t^{\frac{1}{2}-} H_x^{-\frac{d}{2}-}$, a.s., $s < -\frac{d}{2}$.

regularity of spatial white noise.

· Wave case:

$$\begin{cases} \partial_t^2 \Psi + (1-\Delta)\Psi = \xi, & \Phi = \mathrm{Id} \\ (\Psi, \partial_t \Psi)|_{t=0} = (0,0) \end{cases}$$

$$\Rightarrow \Psi_{\text{wave}}(t) = \int_0^t \frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle} \, dW(t')$$

$$\in C_t H_x^s, \text{ a.s.} \qquad \boxed{s < 1 - \frac{d}{2}}$$

$$C_t W_x^{s,r} \qquad 1 \le r \le \infty$$

$\mathbb{R}^d$ case:

Prop 4.1.3: $\Phi \in HS(L^2; H^s)$. (No translation invariance is assumed.)

$$\Rightarrow \Psi(t) = \int_0^t S(t-t')\Phi\, dW(t') \in C_t H^s_x(\mathbb{R}^1), \text{ a.s.}$$

$$= \sum_{n\in\mathbb{N}} \int_0^t S(t-t')\Phi(e_n)\, dB_n(t')$$

$\{e_n\}_{n\in\mathbb{N}}$ = O.N.B. of $L^2(\mathbb{R}^d)$

Lem 4.1.4: $T > 0$, $0 < \alpha < 1$, $q > \frac{1}{\alpha}$ (Da Prato-Zabczyk, Prop 5.9; Da Prato, "Kolm. eqn for SPDEs".)

If $f \in L^q_T H^s_x$, then

$$F(t) = \int_0^t S(t-t')(t-t')^{\alpha-1} f(t')\, dt' \in C_T H^s_x.$$

and $\|F\|_{L^p_T H^s_x} \lesssim_{q,T} \|f\|_{L^q_T H^s_x}$

Pf: Factorization method

(*) $$\int_r^t (t-t')^{\alpha-1}(t'-r)^{-\alpha}\,dt' = \frac{\pi}{\sin \pi\alpha}, \quad \begin{matrix} 0<\alpha<1 \\ 0 \le r \le t' \le t. \end{matrix}$$

- Fix $0 < \alpha < \frac{1}{2}$.

$$\Psi(t) = \int_0^t S(t-r)\,\Phi\,dW(r)$$

$$\overset{(*)}{=} \frac{\sin \pi\alpha}{\pi} \int_0^t \left(\int_r^t (t-t')^{\alpha-1}(t'-r)^{-\alpha}\,dt' \right) S(t-r)\,\Phi\,dW(r)$$

stoch Fubini

$$= \frac{\sin \pi\alpha}{\pi} \int_0^t S(t-t')\,(t-t')^{\alpha-1} \times \underbrace{\int_0^{t'} S(t'-r)(t'-r)^{-\alpha}\,\Phi\,dW(r)}_{= f(t')}\,dt'$$

D.P.-Zab.
Oh-Wang-Zine '22
Appendix

In view of Lem 4.5, it suffices to show

(**) $$\mathbb{E}\left[\int_0^T \|f(t')\|_{H^s}^q \, dt'\right] \le C(T, q, \|\Phi\|_{HS}) < \infty$$

$$q > \frac{1}{\alpha}$$

$$f(t') = \int_0^{t'} S(t'-r)(t'-r)^{-\alpha} \Phi \, dW(r)$$

$$= \sum_{n\in\mathbb{N}} \int_0^{t'} S(t'-r)(t'-r)^{-\alpha} \Phi(e_n) \, dB_n(t')$$

$$\mathbb{E}[|f(t',x)|^2] = \sum_{n\in\mathbb{N}} \int_0^{t'} |S(t'-r)(t'-r)^{-\alpha} \Phi(e_n)(x)|^2 \, dr$$

$$\Rightarrow \| \|f(t')\|_{H^s} \|_{L^p(\Omega)} \lesssim p^{1/2} \underbrace{\left(\int_0^{t'} (t'-r)^{-2\alpha} dr\right)^{1/2}}_{\lesssim 1 \text{ for } \alpha < 1/2} \underbrace{\left(\sum_{n\in\mathbb{N}} \|\Phi(e_n)\|_{H^s}^2\right)^{1/2}}_{= \|\Phi\|_{HS(L^2;H^s)}}$$

$\Rightarrow$ (**)

□

Lec 3: Nov 03, 2025 (MON)

Prop 4.1.5: $\Phi \in HS(L^2(\mathbb{R}^d); H^s(\mathbb{R}^d))$, $s \in \mathbb{R}$

$1 \le q < \infty$, $2 \le r < \infty$ and $r \le \frac{2d}{d-2}$, $d \ge 3$

($2^* = \frac{2d}{d-2}$, $H^1(\mathbb{R}^d) \subset L^{\frac{2d}{d-2}}(\mathbb{R}^d)$)

Then, $\Psi \in \underbrace{L^q_T W^{s,r}_x(\mathbb{R}^d)}_{\uparrow L^q([-T,T]; W^{s,r}(\mathbb{R}^d))}$, a.s.

for any finite $T > 0$.

de Bouard-Debussche '03
Oh-Pocovnicu-Wang '20

- Wave case: Brun-Li-Liu, JDE '24.

• Strichartz estimates on $\mathbb{R}^d$

See Sec 1.4 from 2024.

We say that (q,r) is admissible if

$$2 \le q, r \le \infty, \quad (q,r,d) \neq (2,\infty,2),$$

$$\boxed{\frac{2}{q} + \frac{d}{r} = \frac{d}{2}}.$$

For $d \ge 3$, $r \le \frac{2d}{d-2}$

① (homog): If (q,r) is admissible, then

$$\| S(t) f \|_{L_t^q(\mathbb{R}; L_x^r(\mathbb{R}^d))} \lesssim \| f \|_{L_x^2}$$

② (dual): $$\| \int_{\mathbb{R}} S(-t) F(t) dt \|_{L_x^2} \lesssim \| F \|_{L_t^{q'} L_x^{r'}}$$

③ (nonhomog): $$\| \int_0^t S(t-t') F(t') dt' \|_{L_t^q L_x^r} \lesssim \| F \|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

(q,r), $(\tilde{q},\tilde{r})$, admissible.

Pf of Prop 4.1.5: $\Psi(t,x) = \int_0^t S(t-t') \Phi \, dW(t')$

$$= \sum_{n \in \mathbb{N}} \underbrace{\int_0^t S(t-t') \Phi(e_n)(x) \, dB_n(t')}_{= \text{Wiener integral}}$$

$$\Rightarrow \mathbb{E}\left[|\langle \nabla \rangle^s \Psi(t,x)|^2 \right]$$

$$= \sum_{n \in \mathbb{N}} \int_0^t |S(t-t') \langle \nabla \rangle^s \Phi(e_n)(x)|^2 \, dt'$$

$$\Rightarrow \underline{\| \langle \nabla \rangle^s \Psi(t,x) \|_{L^p(\Omega)}} \sim p^{1/2} \| \langle \nabla \rangle^s \Psi(t,x) \|_{L^2(\Omega)}$$

$$= p^{1/2} \left\| \left(\int_0^t |S(\underbrace{t-t'}_{= \tau}) \langle \nabla \rangle^s \Phi(e_n)(x)|^2 \, dt' \right)^{1/2} \right\|_{\ell^2_n}$$

(*)

$$= \underline{p^{1/2} \| S(\tau) \langle \nabla \rangle^s \Phi(e_n)(x) \|_{\ell^2_n L^2_\tau([0,t])}}$$

$q, r < \infty$. Take $p \geq q \vee r \; (= \max(q,r))$

$$\Big\| \, \| \Psi \|_{L_T^q W_x^{s,r}} \, \Big\|_{L^p(\Omega)} = \Big\| \, \| \langle \nabla \rangle^s \Psi \|_{L_T^q L_x^r} \, \Big\|_{L^p(\Omega)}$$

$$\overset{\text{Mink}}{\leq} \Big\| \, \| \langle \nabla \rangle^s \Psi(t,x) \|_{L^p(\Omega)} \, \Big\|_{L_T^q L_x^r}$$

$$\overset{(*)}{\lesssim} p^{1/2} T^{1/q} \Big\| \, \| S(\tau) \langle \nabla \rangle^s \Phi(e_n)(x) \|_{\ell_n^2 L_\tau^2([0,T])} \, \Big\|_{L_x^r}$$

$$\overset{\text{Mink}}{\leq} p^{1/2} T^{1/q} \Big\| \, \| S(\tau) \langle \nabla \rangle^s \Phi(e_n) \|_{L_T^2 L_x^r} \, \Big\|_{\ell_n^2}$$

Choose $\tilde{q} \geq 2$ s.t. $(\tilde{q}, r)$ is admissible.

$$\overset{\text{Hölder in } \tau}{\lesssim} p^{1/2} T^{\theta} \Big\| \, \| S(\tau) \langle \nabla \rangle^s \Phi(e_n) \|_{L_T^{\tilde{q}} L_x^r} \, \Big\|_{\ell_n^2}$$

$$\overset{\text{Str.}}{\lesssim} p^{1/2} T^{\theta} \Big\| \, \| \Phi(e_n) \|_{H^s} \, \Big\|_{\ell_n^2} = \underset{\text{ww}}{p^{1/2}} T^{\theta} \| \Phi \|_{HS(L^2; H^s)}. \qquad \square$$

Rmk: ① By Chebyshev, we have (see the proof of Lem 1.8.1 in Lec 8 from 2024.)

$$P\left(\|\Psi\|_{L_T^q W_x^{s,r}} > \lambda \right) \le C e^{-c \lambda^2 / T^{2/\theta} \|\Phi\|^2_{HS(L^2;H^s)}}.$$

② On $\mathbb{T}^d$, if Φ is not trans. invariant, we also need to use the Strichartz estimates on $\mathbb{T}^d$ (much weaker than those on $\mathbb{R}^d$, often with a derivative).

Sec 4.2: 1-d SNLS & SNLW

SNLS: $i\partial_t u + \Delta u + |u|^{k-1}u = \Phi\xi$, $\quad k \in 2\mathbb{N}+1$

SNLW: $\partial_t^2 u + (1-\Delta)u + u^k = \Phi\xi$, $\quad k \geq 2$, integer.

- **SNLS on $\mathbb{T}^d$ or $\mathbb{R}^d$**: algebra case $s > \frac{d}{2}$

$$\|fg\|_{H^s} \lesssim \|f\|_{H^s}\|g\|_{H^s}, \quad s > \frac{d}{2}$$

$$\Gamma_{u_0,\Phi}(u) = S(t)u_0 + i\int_0^t S(t-t')|u|^{k-1}u(t')\,dt' - i\Psi$$

WTS: $\exists!\ u \in \overline{B_R} \subset C_T H_x^s$ s.t. $u = \Gamma(u)$

$$\|\Gamma(u)\|_{C_T H^s_x} \le \|u_0\|_{H^s} + \left\| \int_0^t S(t-t')|u|^{k-1}u(t')\,dt' \right\|_{C_T H^s_x} \quad (7)$$

Mink
$$\le \sup_{|t|\le T} \int_0^t \|S(t-t')|u|^{k-1}u(t')\|_{H^s_x}\,dt'$$

$$\lesssim \|u(t')\|^k_{H^s_x}$$

$$\le \|u\|^k_{C_T H^s_x}$$

$$+ \|\Psi\|_{C_T H^s_x} \quad \Leftarrow \text{finite a.s. (Prop 4.1.2 \& 4.1.3)}$$

$\Rightarrow$ $0 < T \le 1$.

$$\|\Gamma[u]\|_{C_T H^s_x} \le \|u_0\|_{H^s} + C_1 T \|u\|^k_{C_T H^s_x} + \|\Psi\|_{C_1 H^s_x}$$

$$\le \frac{R}{2} + C_1 T R^k \le R, \quad \forall u \in \overline{B}_R \subset C_T H^s_x \qquad = C([0,1]; H^s_x)$$

by setting $R = 2\left(\|u_0\|_{H^s} + \|\Psi\|_{C_1 H^s_x}\right)$

$$T \le \frac{1}{2C_1 R^{k-1}}.$$

R is random, a.s. finite $\Rightarrow$ $T > 0$, a.s.

Similarly, we have

$$\| \Gamma(u) - \Gamma(v) \|_{C_T H^s_x} \le C_2 \left(\|u\|^{k-1}_{C_T H^s_x} + \|v\|^{k-1}_{C_T H^s_x} \right) \|u - v\|_{C_T H^s_x}$$

← No u_0 or Ψ.

$$\le \frac{1}{2} \|u - v\|_{C_T H^s_x}, \quad \forall\, u, v \in \overline{B_R}.$$

by setting $T \le \frac{1}{4 C_2 R^{k-1}}$.

$\Rightarrow$ $\Gamma_{u_0, \Psi}$ is a contration in $\overline{B_R} \subset C_T H^s_x$

$\Rightarrow$ By the Banach fixed pt thm, $\exists!\ u \in \overline{B_R}$ s.t. $u = \Gamma(u)$.

i.e. $$u(t) = S(t) u_0 + i \int_0^t S(t - t') |u|^k u(t')\, dt' - i \Psi(t)$$

on $[0, T_\omega]$, where T_ω is a.s. positive.

Thm 4.2.1: $s > \frac{d}{2}$, $\Phi \in HS(L^2; H^s)$.
Then, SNLS is locally well-posed in $H^s(M)$, $M = \mathbb{T}^d$ or $\mathbb{R}^d$.

Rmk: · The proof gives uniqueness of the solution only in $\overline{B_R} \subset C_T H^s_x$

⇐ By a standard continuity argument, we can upgrade uniqueness to hold in the entire class $C_T H^s_x$ (unconditional uniqueness).

· The question of uncond. uniqueness for stoch. disp. PDEs in the low regularity setting is mostly open

Oh - Sosoe - Wang '25

1-d SNLW on $\mathbb{T}$: $\partial_t^2 u + (1-\Delta) u + u^k = \xi$, $\Phi = Id$.

$$S(t) = \frac{\sin(t\langle\nabla\rangle)}{\langle\nabla\rangle}$$

Duhamel formulation (= mild formulation):

$$u(t) = \partial_t S(t) u_0 + S(t) u_1 - \int_0^t S(t-t') u^k(t') dt' + \Psi(t)$$

where $\Psi(t) = \Psi_{wave}(t) = \int_0^t S(t-t') dW(t')$

$$= \int_0^t \frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle} dW(t') \in C_t H^{\frac{1}{2}-}_x(\mathbb{T})$$

a.s.

$C_t W^{\frac{1}{2}-,\infty}_x(\mathbb{T})$.

Thm 4.22: $s \geq \frac{1}{2} - \frac{1}{2k}$.

Then, 1-d SNLW on $\mathbb{T}$ with an additive space-time white noise is locally well-posed in $\mathcal{H}^s(\mathbb{T}) := H^s(\mathbb{T}) \times H^{s-1}(\mathbb{T})$

$\ni (u_0, u_1)$.

$\vec{u} = (u, \partial_t u)$.

$$\Gamma(\vec{u}) = \underbrace{\partial_t S(t) u_0 + S(t) u_1}_{= \cos(t\langle\nabla\rangle) u_0 + \frac{\sin(t\langle\nabla\rangle)}{\langle\nabla\rangle} u_1} - \int_0^t S(t-t') u^k(t') dt' + \Psi(t)$$

Set $\vec{\Gamma}(\vec{u}) = (\Gamma(\vec{u}), \partial_t \Gamma(\vec{u}))$.

$$\Rightarrow \|\vec{\Gamma}(\vec{u})\|_{C_T \mathcal{H}^s_x} \leq \|(u_0, u_1)\|_{\mathcal{H}^s} + \int_0^T \underbrace{\|u^k(t')\|_{H^{s-1}_x}}_{\| u^k(t')\|_{L^2_x} = \|u(t')\|^k_{L^{2k}_x} \overset{\text{Sob}}{\lesssim} \|u(t)\|^k_{H^s_x}} dt' + \underbrace{\|\vec{\Psi}\|_{C_1 \mathcal{H}^s_x}}_{< \infty, \text{ a.s.}}$$

$0 < T \leq 1$

$\frac{1}{2} - \frac{1}{2k} \leq s < \frac{1}{2}$

$s \geq \frac{1}{2} - \frac{1}{2k}$

$$\leq \underbrace{\left(\|(u_0, u_1)\|_{\mathcal{H}^s} + \|\vec{\Psi}\|_{C_1 \mathcal{H}^s_x} \right)}_{=: \frac{R}{2}} + \underbrace{C_1 T \|\vec{u}\|^k_{C_T \mathcal{H}^s_x}}_{\leq \frac{R}{2}, \ \forall \ \vec{u} \in \overline{B_R} \subset C_T \mathcal{H}^s_x}$$

$$\leq R.$$

for $T \leq \frac{1}{2 C_1 R^{k-1}}$

A similar computation yields

$$\|\vec{\Gamma}(\vec{u}) - \vec{\Gamma}(\vec{v})\|_{C_T\mathcal{H}^s_x} \le C_2 T \left(\|\vec{u}\|^{k-1}_{C_T\mathcal{H}^s_x} + \|\vec{v}\|^{k-1}_{C_T\mathcal{H}^s_x} \right) \|\vec{u} - \vec{v}\|_{C_T\mathcal{H}^s_x}$$

$$\le \frac{1}{2} \|\vec{u} - \vec{v}\|_{C_T\mathcal{H}^s_x}$$

$\forall\ \vec{u}, \vec{v} \subset \overline{B_R}$, by choosing $T \le \frac{1}{4C_2 R^{k-1}}$.

$\Rightarrow \vec{\Gamma}$ is a contraction in $\overline{B_R} \subset C_T\mathcal{H}^s_x$, $T = T_\omega > 0$, a.s.

□

Sobolev's ineq: $1 < p \le q < \infty$. $s \ge \frac{1}{p} - \frac{1}{q}$

$$\|f\|_{W^{s,q}} \lesssim \|f\|_{L^p}$$

· 1-d cubic SNLS on $\mathbb{R}$:

$$i\partial_t u + \partial_x^2 u + |u|^2 u = \Phi\xi$$

Thm 4.2.3: $s \geq 0$. $\Phi \in HS(L^2; H^s)$

Then, 1-d cubic SNLS on $\mathbb{R}$ is locally well-posed in $H^s(\mathbb{R})$.

Pf: We only consider $s = 0$.

· For $s > 0$, we can easily modify the proof, using the fractional Leibniz rule. (see Bényi-Oh-Zhao '25).

$$\Gamma[u] = S(t)u_0 + i\int_0^t S(t-t')|u|^2 u(t')\,dt' - i\Psi(t).$$

We need to use the Strichartz estimates.

- $(q,r) = (\infty, 2)$, $(8,4)$ is admissible

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad d = 1.$$

Set $X(T) = C_T L^2_x \cap L^8_T L^4_x$.

$$\|\Gamma(u)\|_{X(T)} \overset{\text{Strichartz}}{\lesssim} \|u_0\|_{L^2} + \|\,|u|^2 u\,\|_{L^{\frac{8}{7}}_T L^{4/3}_x} + \|\Psi\|_{X(1)}$$

$$\lesssim T^{1/2} \|u\|_{L^8_T L^4_x}$$

$x: \frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$

$t: \frac{7}{8} = \frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$

$0 < T \le 1$.

$\|\Psi\|_{X(1)} < \infty$, a.s. (Prop 4.1.3 & 4.1.5)

$$\lesssim \left(\|u_0\|_{L^2} + \|\Psi\|_{X(1)} \right) + T^{1/2} \|u\|^3_{X(1)}$$

- Together with a difference estimate, we conclude LWP.

□

Rmk: Uniqueness holds only in $\overline{B_R} \subset C_T L^2_x \cap L^8_T L^4_x$.

$\Leftarrow$ conditional uniqueness.

- Next time, 1-d cubic NLS on $\mathbb{T}$.
  - Fourier restriction norm method ($X^{s,b}$-space).
- Lec 4 on Nov. 13 (THU).

Lec 4 : Nov. 13, 2025 (THU)

- 1-d cubic SNLS on $\mathbb{T}$

$$\begin{cases} i\partial_t u + \partial_x^2 u + |u|^2 u = \Phi\xi \\ u|_{t=0} = u_0 \in H^s(\mathbb{T}) \end{cases}, \qquad \Phi \in HS(L^2; H^s)$$

trans. inv.

$\Phi(e_n) = \phi_n e_n$

- Main difficulty: lack of (good) Strichartz estimates
  - Strichartz estimates on $\mathbb{T}^d$: $\|S(t)f\|_{L_t^q L_x^r} \lesssim \|f\|_{L_x^2}$?

$d=1$: $f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{inx}$

$$\Rightarrow S(t)f(x) = \sum_{n\in\mathbb{Z}} e^{-it n^2} \hat{f}(n) e^{inx}$$ ← periodic in time

  - Strichartz estimates on $\mathbb{T}^d$: only locally in time.
  - not as good as those on $\mathbb{R}^d$: need $q=r$.
  - Zygmund '74, Bourgain '93, '13 ← analytic number theory (HL circle method
    Bourgain-Demeter '15, Killip-Vişan '16.

For KdV : $\partial_t u + \partial_x^3 u + \partial_x (u^2) = 0$

'80 ~ Kenig-Ponce-Vega, oscillatory integral approach (on $\mathbb{R}$)

KdV: LWP in $H^s(\mathbb{R})$, $s > \frac{3}{4}$ (not on $\mathbb{T}$)

· Fourier restriction norm method : Bourgain '93

$X^{s,b}$ - spaces

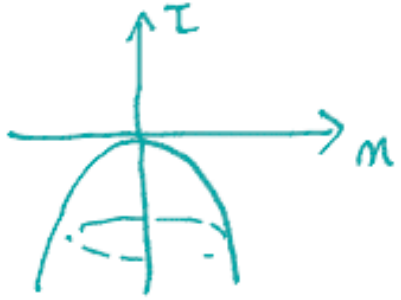
(wave : Klainerman-Machedon '93 hyperbolic Sobolev space

$$\|u\|_{X^{s,b}} = \| \langle n \rangle^s \langle \tau + |n|^2 \rangle^b \hat{u}(\tau, n) \|_{l_n^2 L_\tau^2 (\mathbb{Z}^d \times \mathbb{R})}$$

space-time F.T.

Linear Sch. eqn : $i\partial_t u + \Delta u = 0 \Rightarrow -(\tau + |n|^2)\hat{u}(\tau, n) = 0$

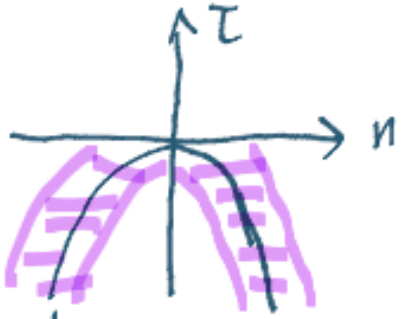
measure supp on the paraboloid $\tau + |n|^2 = 0$



· The weight $\underline{\langle \tau + |n|^2 \rangle^b}$ $(b > 0)$ "penalizes" functions whose Fourier support is away from $\tau + |n|^2 = 0$

$\Rightarrow$ In running a contraction argument in $\overline{B}_R \subset X^{s,b}$, we do NOT see functions whose Fourier supp is away from $\tau + |n|^2 = 0$.

i.e. $u \in \overline{B}_R \subset X^{s,b}$ $\rightsquigarrow$ u is close to being a linear soln



· suitable for perturbative study:

deterministic NLS

$$u(t) = S(t)u_0 + \underbrace{i\int_0^t S(t-t')|u|^2 u(t')\,dt'}_{\text{"perturbation"}}$$

· The Fourier restriction norm method has been effective in the low regularity study of nonlinear dispersive PDEs.

ex.: Bourgain '93

· 1-d cubic NLS on $\mathbb{T}$: LWP in $L^2(\mathbb{T})$. ← improved from $s > 1/2$

· KdV on $M = \mathbb{R}$ or $\mathbb{T}$: LWP in $L^2(M)$ ← $s > \frac{3}{4}$ on $\mathbb{R}$; $s > \frac{3}{2}$ on $\mathbb{T}$

(Kenig-Ponce-Vega: LWP in $H^{-\frac{3}{4}+}(\mathbb{R})$

in $H^{-\frac{1}{2}}(\mathbb{T})$ Colliander-Keel-Staffilani-Takaoka-Tao '03

· Basic properties: see the course notes from 2024

① interaction representation: $v(t) = S(-t)u(t)$

$$\Rightarrow \|u\|_{X^{s,b}} = \|\langle n\rangle^s \langle \tau + |n|^2\rangle^b \hat{u}(\tau,n)\|_{l^2_n L^2_\tau}$$

$$= \|\langle n\rangle^s \langle \tau\rangle^b \hat{v}(\tau,n)\|_{l^2_n L^2_\tau} = \|\langle \nabla\rangle^s \langle \partial_t\rangle^b v\|_{L^2_{t,x}}$$

② $b > 1/2$: $X^{s,b} \subset C_t H^s_x$

↖ essentially — Riemann-Lebesgue lemma

$H^b_t \subset \mathcal{F}^{-1}_t(L^1) \subset C_0$ (for $b > \frac{1}{2}$)

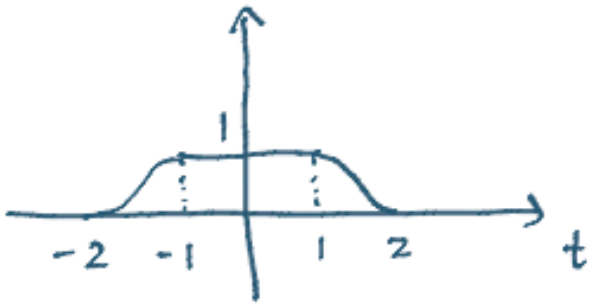
③ $\forall s, b \in \mathbb{R}$

$$\| \eta(t) S(t) f \|_{X^{s,b}} \lesssim \| f \|_{H^s}$$

↑ smooth cutoff function in time

$\eta \equiv 1$ on $[-1,1]$

$\operatorname{supp} \eta \subset [-2,2]$



· Pf: $\mathcal{F}_x(S(t)f)(n) = e^{-it|n|^2} \hat{f}(n) \Rightarrow \widehat{\eta(t) S(t) f}(\tau, n) = \hat{\eta}(\tau + |n|^2) \hat{f}(n)$

$$\Rightarrow \| \eta(t) S(t) f \|_{X^{s,b}} = \| \langle n \rangle^s \langle \tau + |n|^2 \rangle^b \hat{\eta}(\tau + |n|^2) \hat{f}(n) \|_{l^2_n L^2_\tau}$$

$$= \| \eta \|_{H^b_t} \| f \|_{H^s_x}$$

Q: $S(t)f \in X^{s,b}$? (without a cutoff in time.)

Ans: **NO!!**

$$\widehat{S(t)f}(\tau, n) = \delta(\tau + |n|^2)\,\hat{f}(n)$$

(Dirac delta)

---

④ $$\left\| \eta(t) \int_0^t S(t-t') F(t')\,dt' \right\|_{X^{s,b}} \lesssim \|F\|_{X^{s,b-1}}$$

$\forall\, s \in \mathbb{R}$

but $b > 1/2$

$0 < T \le 1$ $$\left\| \eta\left(\frac{t}{T}\right) \int_0^t S(t-t') F(t')\,dt' \right\|_{X^{s,b}} \lesssim T^{\varepsilon} \|F\|_{X^{s,b-1+\varepsilon}}$$

⑤ Transference: If $\| S(t) f \|_{L_t^q L_x^r} \lesssim \| f \|_{H_x^s}$,

then $\| u \|_{L_t^q L_x^r} \lesssim \| u \|_{X^{s,b}}$, $b > 1/2$

⑥ periodic L^4-Strichartz estimate on $\mathbb{T}$.

· Zygmund '74: $\| \eta(t) S(t) f \|_{L^4_{t,x}} \lesssim \| f \|_{L^2}$

$\xRightarrow{\text{transf}}$ $\| \eta(t) u \|_{L^4_{t,x}} \lesssim \| u \|_{X^{0,b}}$, $b > 1/2$

· Bourgain '93: $\| u \|_{L^4_{t,x}} \lesssim \| u \|_{X^{0,\frac{3}{8}}}$

See Lec 12 from 2024.

$$\|\Phi u\|_{X^{s,b}} \lesssim \|u\|_{X^{s,b}} \qquad \forall s,b \in \mathbb{R}$$

Φ, smooth cutoff in time

$$\|\mathbb{1}^{(t)}_{[t_1,t_2]} u\|_{X^{s,b}} \lesssim \|u\|_{X^{s,b}} \qquad -\tfrac{1}{2} < b < \tfrac{1}{2}$$

$$\left(\Leftarrow \|\mathbb{1}^{(t)}_{[t_1,t_2]} f(t)\|_{H^b_t} \lesssim \|f\|_{H^b_t} \right.$$

Prop 4.2.4: $\Psi(t) = \int_0^t S(t-t')\Phi\, dW(t')$, $\Phi \in HS(L^2;H^s)$ trans. inv.

on $\mathbb{T}^d$

$$\left\| \|\mathbb{1}_{[0,T]}\Psi\|_{X^{s,b}} \right\|_{L^p(\Omega)} \lesssim_T p^{1/2} \|\Phi\|_{HS(L^2;H^s)}$$

$$\left(\overset{\text{Chebyshev}}{\Rightarrow} P\left(\|\mathbb{1}_{[0,T]}\Psi\|_{X^{s,b}} > \lambda \right) \leq C e^{-c_T \frac{\lambda^2}{\|\Phi\|^2_{HS(L^2;H^s)}}} \right.$$

Pf: Let $\widetilde{\Psi}(t) = S(-t)\Psi(t)$.

$$\widehat{\widetilde{\Psi}}(t) = \int_0^t S(-t')\Phi\, dW(t')$$

$$= \sum_{n\in\mathbb{Z}^d} \Phi_n e_n \int_0^t e^{it'|n|^2} dB_n(t')$$

F.T. in x $\Rightarrow$

$$\widehat{\widetilde{\Psi}}(t,n) = \Phi_n \int_0^t e^{it'|n|^2} dB_n(t')$$

F.T. int $\Rightarrow$

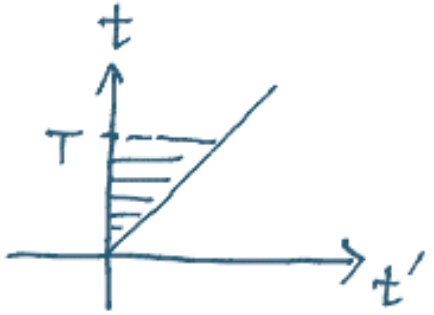
$$\widehat{\mathbb{1}_{[0,T]}\cdot\widetilde{\Psi}}(\tau,n) = \Phi_n \int_{\mathbb{R}} e^{-it\tau}\, \mathbb{1}_{[0,T]}(t) \int_0^t e^{it'|n|^2} dB_n(t')\, dt.$$

Stoch. Fubini

$$= \Phi_n \int_0^T e^{it'|n|^2} \int_{t'}^T e^{-it\tau} dt\, dB_n(t')$$

Φ_n: Wiener integral

$\int_{t'}^T e^{-it\tau} dt$: $\lesssim_T \min\left(1, \frac{1}{|\tau|}\right) \sim \frac{1}{\langle\tau\rangle}$



$$\Rightarrow \quad \| \widehat{\mathbb{1}_{[0,T]}(t) \cdot \Phi}(\tau, n) \|_{L^p(\Omega)} \sim p^{1/2} \| \cdots \|_{L^2(\Omega)}$$

(*)

$$\lesssim p^{1/2} \left(T^{3/2} \vee T^{1/2} \right) \frac{|\Phi_n|}{\langle \tau \rangle}$$

$$\overset{p \geq 2}{\Rightarrow} \left\| \| \mathbb{1}_{[0,T]} \cdot \Phi \|_{X^{s,b}} \right\|_{L^p(\Omega)} = \left\| \| \langle n \rangle^s \langle \tau \rangle^b \widetilde{\mathbb{1}_{[0,T]} \cdot \Phi}(\tau, n) \|_{l^2_n L^2_\tau} \right\|_{L^p(\Omega)}$$

$$\overset{\text{Mink}}{\leq} \left\| \langle n \rangle^s \langle \tau \rangle^b \| \widehat{\mathbb{1}_{[0,T]} \cdot \Phi}(\tau, n) \|_{L^p(\Omega)} \right\|_{l^2_n L^2_\tau}$$

(*)

$$\lesssim C_T p^{1/2} \underbrace{\| \langle n \rangle^s \Phi_n \|_{l^2_n}}_{= \| \Phi \|_{HS(L^2; H^s)}} \underbrace{\| \langle \tau \rangle^{b-1} \|_{L^2_\tau}}_{< \infty \text{ iff } b < 1/2}$$

□

Thm 4.2.5: Let $s \geq 0$.

Then, 1-d cubic SNLS is locally well-posed in $H^s(\mathbb{T})$.

Pf: Duhamel formulation (= mild formulation)

$$u(t) = \eta(t) S(t) u_0 + i\, \eta\left(\frac{t}{T}\right) \int_0^t S(t-t') |u|^2 u(t') dt' - i\, \mathbb{1}_{[0,1]}(t) \cdot \Psi(t)$$

$0 < T \leq 1$.

Set $\Gamma(u) = \text{RHS}$.

- $\eta(t) = \eta\left(\frac{t}{T}\right) = \mathbb{1}_{[0,1]}(t) = 1$ on $[0, T]$.

- We only consider the case $s = 0$.

$$\|\Gamma(u)\|_{X^{0,\frac{1}{2}-\varepsilon}} \underset{③}{\leq} C_1 \|u_0\|_{L^2} + \underbrace{\left\| \eta\left(\frac{t}{T}\right) \int_0^t \cdots dt' \right\|_{X^{0,\frac{1}{2}+\varepsilon}}}_{\underset{④}{\lesssim} T^{\varepsilon} \| |u|^2 u \|_{X^{0,-\frac{1}{2}+2\varepsilon}}} + \underbrace{\| \mathbb{1}_{[0,1]}(t) \cdot \Psi \|_{X^{0,\frac{1}{2}-\varepsilon}}}_{\text{Prop 4.2.4} \quad \leq C_\omega < \infty, \text{ a.s.}}$$

$$\| \eta_b\left(\tfrac{t}{T}\right) \int_0^t S(t-t') |u|^2 u(t') \, dt' \|_{X^{0,\frac{1}{2}+\varepsilon}}$$

④

$$\lesssim T^{\varepsilon} \| |u|^2 u \|_{X^{0,-\frac{1}{2}+2\varepsilon}}$$

$$(X^{s,b})' = X^{-s,-b}$$

see page 17

duality

$$= T^{\varepsilon} \sup_{\|v\|_{X^{0,\frac{1}{2}-2\varepsilon}}=1} \underbrace{\left| \int u \bar{u} u \bar{v} \, dx dt \right|}$$

$$\leq \|u\|^3_{L^4_{t,x}} \|v\|_{L^4_{t,x}}$$

L^4- Strichartz

$$\lesssim \|u\|^3_{X^{0,\frac{3}{8}}} \underbrace{\|v\|_{X^{0,\frac{3}{8}}}}$$

⑥

$$\leq \|v\|_{X^{0,\frac{1}{2}-2\varepsilon}} = 1$$

$$\lesssim T^{\varepsilon} \|u\|^3_{X^{0,\frac{3}{8}}}$$

$$\Rightarrow \|\Gamma(u)\|_{X^{0,\frac{1}{2}-\varepsilon}} \le \underbrace{\left(C_1 \|u_0\|_{L^2} + \|\mathbb{1}_{[0,1]} \cdot \Phi\|_{X^{0,\frac{1}{2}-\varepsilon}}\right)}_{=: \frac{1}{2}R \leftarrow \text{random}} + C_2 T^{\varepsilon} \|u\|^3_{X^{0,\frac{3}{8}}}$$

$R < \infty$, a.s. $\Rightarrow T > 0$, a.s.

$$\le \frac{1}{2}R + C_2 T^{\varepsilon} R^3 \le R \quad \text{if } T = T(R) > 0 \text{ is suff. small}$$

$\forall\ u \in \overline{B_R} \subset X^{0,\frac{1}{2}-\varepsilon}$.

A similar computation yields the diffence estime.

$$\Rightarrow \exists!\ u \in \overline{B_R} \subset X^{0,\frac{1}{2}-\varepsilon} \text{ s.t. } u = \Gamma(u)$$

$\Rightarrow u$ is a soln to SNLS with $u|_{t=0} = u_0$ on the time interval $[0, T_\omega]$.

$\uparrow$ a.s. positive.

· Need to show $u \in C_T L^2_x$

$$u(t) = S(t)u_0 + i\underbrace{\eta\left(\frac{t}{T}\right)\int_0^t S(t-t')|u|^2 u(t')\,dt'}_{\in X^{0,\frac{1}{2}+\varepsilon} \subset C_t L^2_x} - i\Psi$$

$S(t)u_0 \in C_t L^2_x$ on $[0,T]$; $\Psi \in C_t L^2_x$

$X^{0,\frac{1}{2}+\varepsilon}$: $\frac{1}{2}+\varepsilon > \frac{1}{2}$ ②

$\Rightarrow u \in C_T L^2_x$, a.s.

□

$\Psi \in HS(L^2; L^2)$, trans inv.

Thm 4.2.6: 1-d cubic SNLS on $\mathbb{T}$ is globally well-posed in $L^2(\mathbb{T})$.

Ito calculus approach

Pf: $i\partial_t u + \partial_x^2 u + |u|^2 u = \Phi\xi$

· $\underline{\Phi \equiv 0}$. i.e. deterministic cubic NLS.

Mass $M(u) = \int |u|^2 dx$ is conserved under NLS

$$\partial_t M(u) = 2\,\mathrm{Re} \int \partial_t u \cdot \bar{u}\, dx$$

$$= 2\mathrm{Re}\, i \underbrace{\int \partial_x^2 u \cdot \bar{u}\, dx}_{\overset{\text{IBP}}{=} \int |\partial_x u|^2 dx} + 2\mathrm{Re}\, i \underbrace{\int |u|^2 u \cdot \bar{u}\, dx}_{= \int |u|^4 dx}$$

$$= 0.$$

⇒ By iteratively applying the LWP argument,
we obtain GWP of deterministic NLS in $L^2(\mathbb{T})$.

For SNLS, we use Ito's lemma on

$$M(u) = \int |u|^2 dx = \sum_{n\in\mathbb{Z}} (p_n^2 + q_n^2)$$

$p_n = \mathrm{Re}\,\hat{u}(n)$

$q_n = \mathrm{Im}\,\hat{u}(n)$

· Ito's lemma: $dX = f\,dt + g\,dB$ $\quad B = BM$

Consider $F(X_t)$

Then, $dF(X) = \partial_x F(X)\,dX + \frac{1}{2}\partial_x^2 F(X)(dX)^2$

$= \partial_x F(X)(f\,dt + g\,dB)$ $\quad (dt)^2 = 0$

$+ \frac{1}{2}\partial_x^2 F(X)\, g^2\,dt$ $\quad dt\,dB = dB\,dt = 0$

$(dB)^2 = dt$

· $F(X_{t_2}) - F(X_{t_1}) = \int_{t_1}^{t_2} \partial_x F(X_t) f_t + \frac{1}{2}\partial_x^2 F(X_t) g_t^2\,dt$

$+ \int_{t_1}^{t_2} \partial_x F(X_t)\, g_t\, dB_t$

Appendix : On the duality of the $X^{s,b}$ - spaces

- $u \in X^{s,b}$

$$\int u \bar{v}\, dx dt = \int S(-t)u(t) \cdot \overline{S(-t)v(t)}\, dx dt$$

$U(t) = S(-t)u(t)$

$V(t) = S(-t)v(t)$

$$\leq \| \langle \nabla \rangle^{s} \langle \partial_t \rangle^{b} U \|_{L^2_{t,x}} \, \| \langle \nabla \rangle^{-s} \langle \partial_t \rangle^{-b} V \|_{L^2_{t,x}}$$

$$= \| u \|_{X^{s,b}} \, \| v \|_{X^{-s,-b}}$$

$$\Rightarrow \quad \| u \|_{X^{s,b}} = \sup_{\| v \|_{X^{-s,-b}} = 1} \left| \int u \bar{v}\, dx dt \right|$$

It is in this sense that we wrote $(X^{s,b})' = X^{-s,-b}$ on page 12

Note that the usual duality pairing comes without the conjugate sign.

$$\int u v\, dx dt$$

Under the standard duality pairing $\int uv\,dxdt$,

↑ NO conjugate

we have $$\left(X^{s,b}_{\tau+|n|^2}\right)' = X^{-s,-b}_{\tau-|n|^2}$$

↑ really comes from $\tau = -|-n|^2$

(⇐ follows from page 17 and $\|\overline{v}\|_{X^{s,b}_{\tau+|n|^2}} = \|v\|_{X^{s,b}_{\tau-|-n|^2}}$

Lec 5: Nov 20, 2025 (THU)

Last time, we started to discuss

$\Phi \in HS(L^2; L^2)$, trans inv.

Thm 4.2.b: 1-d cubic SNLS on $\mathbb{T}$ is globally well-posed in $L^2(\mathbb{T})$.

$$M(u) = \int |u|^2 dx = \sum_{n\in\mathbb{Z}} (p_n^2 + q_n^2)$$

$p_n = \mathrm{Re}\,\hat{u}(n)$

$q_n = \mathrm{Im}\,\hat{u}(n)$

SNLS: $i\partial_t u + \partial_x^2 u + |u|^2 u = \Phi\xi$.

$$\Rightarrow \quad d\hat{u}(n) = \left(-i n^2 \hat{u}(n) + i \mathcal{F}_x(|u|^2 u)(n)\right) dt - i \Phi_n \, dB_n$$

- $dp_n = \left(n^2 q_n - \mathrm{Im}\,\mathcal{F}_x(|u|^2u)(n)\right) dt + \underbrace{\mathrm{Im}\,(\Phi_n dB_n)}_{= \mathrm{Im}\Phi_n \, d(\mathrm{Re}B_n) + \mathrm{Re}\Phi_n \, d(\mathrm{Im}B_n)}$

- $dq_n = \left(-n^2 p_n + \mathrm{Re}\,\mathcal{F}_x(|u|^2u)(n)\right) dt$

$$\underbrace{-\mathrm{Re}\,(\Phi_n dB_n)}_{= -\mathrm{Re}\Phi_n \, d(\mathrm{Re}B_n) + \mathrm{Im}\Phi_n \, d(\mathrm{Im}B_n)}.$$

Apply Ito's lemma to $M(u) = \sum_{n\in\mathbb{Z}} (p_n^2 + q_n^2)$.

$\frac{\partial}{\partial q_n}\frac{\partial}{\partial p_n} p_n^2 = 0$, $\frac{\partial}{\partial p_n}\frac{\partial}{\partial q_n} q_n^2 = 0$ ⟹ NO $dp_n dq_n$ (2)

$$dM(u) \overset{\text{Ito}}{=} \underbrace{2\sum (p_n\, dp_n + q_n\, dq_n)} + \underbrace{\sum \left((dp_n)^2 + (dq_n)^2\right) dt}_{= 2\|\Phi\|^2_{HS(L^2;L^2)}\, dt}$$

$$= 2\sum \Big[p_n \big(\mathrm{Im}\,\Phi_n\, d(\mathrm{Re}\,B_n) + \mathrm{Re}\,\Phi_n\, d(\mathrm{Im}\,B_n) \big) + q_n \big(-\mathrm{Re}\,\Phi_n\, d(\mathrm{Re}\,B_n) + \mathrm{Im}\,\Phi_n\, d(\mathrm{Im}\,B_n) \big) \Big]$$

Apply BDG ineq.

- $\sum_n (dp_n)^2 = \sum_n \left((\mathrm{Im}\,\Phi_n)^2 + (\mathrm{Re}\,\Phi_n)^2 \right) = \|\Phi\|^2_{HS(L^2;L^2)}$
- $\sum_n (dq_n)^2 = \sum_n \left((\mathrm{Re}\,\Phi_n)^2 + (\mathrm{Im}\,\Phi_n)^2 \right) = \|\Phi\|^2_{HS(L^2;L^2)}$

$d(\mathrm{Re}\,B_n)\, d(\mathrm{Im}\,B_n) = 0$ (indep.)

Note: $dp_n\, dq_n = \mathrm{Im}\,\Phi_n \cdot (-\mathrm{Re}\,\Phi_n) + \mathrm{Re}\,\Phi_n \cdot \mathrm{Im}\,\Phi_n = 0$.

⟸ We do not need it here.

- **Burkholder – Davis – Gundy inequality**: X, (local) martingale

$1 \le p < \infty$

$$\mathbb{E}\Big[\sup_{0\le t\le T} |X_t|^p\Big] \sim \mathbb{E}\big[\langle X_t\rangle_{[0,T]}^{p/2}\big]$$

↑ quadratic variation

course notes from 2022
Lec 10-11.
Book: Karatzas-Shreve
Thm 3.28 on p. 166

For an Ito process $dX = f\,dt + g\,dB$,

$$\langle X_t\rangle_{[0,T]} = \int_0^T g^2\,dt$$

ex: $2\sum_n p_n \operatorname{Im}\Phi_n\, d(\operatorname{Re} B_n)$

$$\Rightarrow \text{quadratic variation} = 4\int_0^T \sum_n p_n^2(t)\,(\operatorname{Im}\Phi_n)^2\,dt$$

$\sum a_n b_n \le \sum a_n \sum b_n$ for $a_n, b_n \ge 0$

$M(u(0)) + \int_0^t dM(u(t'))$ $\qquad \le \|u(t)\|_{L^2}^2 \,\|\Phi\|_{HS(L^2;L^2)}^2$

$$\Rightarrow \mathbb{E}\Big[\sup_{0\le t\le T} M(u(t))\Big] \le \mathbb{E}\big[M(u(0))\big] + \mathbb{E}\Big[\sup_{0\le t\le T}\int_0^t dM(u(t'))\Big]$$

(*)
$$\le \mathbb{E}\big[M(u(0))\big] + CT^{1/2}\,\mathbb{E}\Big[\sup_{0\le t\le T} M(u(t))^{1/2}\cdot\|\Phi\|_{HS(L^2;L^2)}\Big]$$

$\overset{\text{BDG}}{\sim} \mathbb{E}\big[\langle M(u)\rangle_{[0,T]}^{1/2}\big]$

$$\cdot\quad T^{1/2}\,\mathbb{E}\Big[\sup_{0\le t\le T} M^{1/2}(u(t))\cdot\|\Phi\|_{HS(L^2;L^2)}\Big]$$

Cauchy's ineq.

$$ab \le \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2$$

$$\lesssim \varepsilon\,\mathbb{E}\Big[\sup_{0\le t\le T} M(u(t))\Big] + \frac{1}{\varepsilon}\,T\,\|\Phi\|^2_{HS(L^2;L^2)}.$$

↑ hide in LHS of ⊛

$$\Rightarrow\quad \mathbb{E}\Big[\sup_{0\le t\le T} M(u(t))\Big] \lesssim \mathbb{E}\big[M(u(0))\big] + T\,\|\Phi\|^2_{HS(L^2;L^2)}$$

$\Rightarrow$ The L^2-norm of $u(t)$ is a.s. finite on $[0,T]$. $\forall\, T \gg 1$

Thus, by iterating the LWP argument (pathwise), we obtain GWP of SNLS in $L^2(\mathbb{T})$.

$$\text{SNLS}: \ i\partial_t u + \partial_x^2 u + |u|^2 u = \Phi\xi$$

$$\text{SNLS}_N \begin{cases} i\partial_t u_N + \partial_x^2 u_N + \pi_N\left(|u_N|^2 u_N\right) = \pi_N \Phi\xi \\ u_N|_{t=0} = \pi_N u_0 \end{cases}$$

π_N = freq proj onto $\{|n| \le N\}$

- On the Fourier side, SNLS_N is a finite dim'l system of SDEs $(p_n, q_n, |n| \le N)$.

$\Leftarrow$ We can apply Ito's lemma to SNLS_N for each fixed N

- Given $T \gg 1$, in applying BDG ineq, we do not know (a priori) if $u_N(\omega)$ exists on $[0,T]$. $\Rightarrow$ Use a stopping time argument
  - Fix a target time $T \gg 1$. Let τ_N be a stopping time

de Bouard - Debussche '03
Oh - Okamoto '20.

$0 < \tau_N \le \min\left(T, T^N_{\max}\right)$, $T^N_{\max} = T^N_{\max}(\omega)$ = maximal time of existence (forward) for u_N.

Then, apply BDG ineq with $\sup_{0\le t\le \tau_N}$ (instead of $\sup_{0\le t\le T}$

$$\Rightarrow \mathbb{E}\Big[\sup_{0\le t\le \tau_N} M(u_N(t))\Big] \lesssim \mathbb{E}[M(u(0))] + T\,\|\Phi\|^2_{HS(L^2;L^2)}.$$

not u_N uniformly in $N\in\mathbb{N}$.

$\Rightarrow$ $u_N(\omega)$ exists up to time $\tau_N(\omega)$

any stopping time

$\Rightarrow$ u_N exists a.s. on $[0,T]$.

- By a slight modification of the LWP argument,

$$\|u-u_N\|_{X^{0,\frac{1}{2}-\varepsilon}_{\text{local in time}}\cap C_T L^2_x} \to 0 \quad \text{a.s., as } N\to\infty.$$

$\Rightarrow$ existence of u on $[0,T]$, a.s.

$\Rightarrow$ GWP of u in $L^2(\mathbb{T})$. $\boxtimes$

· 1-d SNLW: $\partial_t^2 u + (1-\partial_x^2) u \underset{\text{defocusing}}{+} u^3 = \xi$

LWP in $\mathcal{H}^s(\mathbb{T})$, $s \geq \frac{1}{2} - \frac{1}{2k}$ (Thm 4.2.2 from Lec 3)

Thm 4.2.7: 1-d defocusing cubic SNLW with additive space-time white noise is globally well-posed in $\mathcal{H}^1(\mathbb{T})$.

Pf: pathwise approach (= adaption of a deterministic approach by viewing SNLW as a perturbed NLW

$$\Psi = \int_0^t \frac{\sin\big((t-t')\langle\nabla\rangle\big)}{\langle\nabla\rangle} \xi(dt') \in C_t W^{s,\infty}(\mathbb{T}), \text{ a.s. } s < \frac{1}{2}$$

First order expansion: $u = \Psi + v$

$$\Rightarrow \partial_t^2 v + (1-\partial_x^2) v + (v+\Psi)^3 = 0$$

$$(v, \partial_t v)\big|_{t=0} = (u_0, u_1) \in \mathcal{H}^1$$

$$= v^3 + \underbrace{3v^2\Psi + 3v\Psi^2 + \Psi^3}_{\text{perturbation}}$$

$$v(t) = W(t)(u_0, u_1) - \int_0^t \frac{\sin((t-t')\langle\nabla\rangle}{\langle\nabla\rangle} (v+\Psi)^3(t)\, dt' \qquad (8)$$

$$\Rightarrow \|\vec{v}\|_{C_T\mathcal{N}^1_x} \lesssim \|(u_0,u_1)\|_{\mathcal{N}^1} + T \underbrace{\|(v+\Psi)^3\|_{C_T L^2_x}}_{\text{Hölder}}$$

$\vec{v} = (v, \partial_t v)$

$$\underset{\text{Young}}{\lesssim} \|v\|^3_{C_T L^6_x} + \underbrace{\|\Psi\|^3_{C_T L^6_x}}_{\leq C_\omega < \infty}$$

$$\underset{\text{Sob}}{\lesssim} \|v\|^3_{C_T \mathcal{N}^1_x}$$

$\Rightarrow$ LWP for $\vec{v} \in C_T\mathcal{N}^1_x$.

Goal: GWP for $\vec{v}$ in $C(\mathbb{R}_+; \mathcal{N}^1)$.

$\Leftarrow$ Suffices to show

$$\sup_{0\leq t\leq T} \|\vec{v}(t)\|_{\mathcal{N}^1_x} \leq C(\omega; T) < \infty \text{ , a.s. for each given } T \gg 1.$$

(Then, we can iterate the LWP argument for $\vec{v}$.)

Energy (= Hamiltonian) for the (deterministic) def. cubic NLW:

$$E(\vec{v}) = \frac{1}{2}\int |\langle\nabla\rangle v|^2\,dx + \frac{1}{2}\int (\partial_t v)^2\,dx + \frac{1}{4}\int v^4\,dx.$$

- conserved under NLW.
- $\|\vec{v}(t)\|_{\mathcal{H}^1_x}^2 \lesssim E(\vec{v}(t))$ $\Rightarrow$ suffices to control $E(\vec{v}(t))$ for a soln $\vec{v}$ to the perturbed NLW:

$$\partial_t^2 v + (1-\partial_x^2)v + v^3 + 3v^2\Psi + 3v\Psi^2 + \Psi^3 = 0.$$

- $\partial_t E(\vec{v}(t)) = \int \partial_t v\,(\partial_t^2 v + (1-\partial_x^2)v + v^3)\,dx$

$= -3v^2\Psi - 3v\Psi^2 - \Psi^3$

No v^3 term.

$$\overset{\text{C-S}}{\lesssim} \underset{\text{Young}}{\lesssim} \underbrace{\left(\int (\partial_t v)^2\,dx\right)^{1/2}}_{\lesssim (E(\vec{v}(t)))^{1/2}} \left(\|\Psi\|_{C_T L^\infty_x}^2 \underbrace{\int v^4\,dx}_{\lesssim E(\vec{v}(t))} + \|\Psi\|_{C_T L^6_x}^6 \right)^{1/2}$$

$$\lesssim \left(1 + \|\Psi\|_{C_T L^\infty_x}^2\right) E(\vec{v}(t)) + \|\Psi\|_{C_T L^6_x}^6$$

By Gronwall, we obtain

$$\sup_{0\le t\le T} \|\vec{\Psi}(t;\omega)\|_{\mathcal{H}^1_x} \le C(\omega,T) < \infty$$

$\uparrow$ a.s.

□

- See Brun-Li-Liu, JDE '24 for higher dim'l case and the references.

Sec 4.3: Invariant Gibbs measure for SdNLW.

Stochastic damped NLW on $\mathbb{T}^d$: (defocusing: $k \in 2\mathbb{N}+1$)

$$\text{(SdNLW)} \quad \partial_t^2 u + \partial_t u + (1-\Delta)u + u^k = \sqrt{2}\,\xi$$

Let $v = \partial_t u$.

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ \Delta-1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ -u^k \end{pmatrix}}_{\text{deterministic NLW}} + \underbrace{\begin{pmatrix} 0 \\ -v+\sqrt{2}\,\xi \end{pmatrix}}_{\text{Ornstein-Uhlenbeck process for } v.}$$

· deterministic NLW: $\partial_t^2 u + (1-\Delta)u + u^k = 0$

· Hamiltonian PDE : $E(\vec{u})$ is conserved

↓

the volume $du\,d(\partial_t u)$ on the phase space is invariant

↑ does not exist.

⇒ We expect that the Gibbs meas ρ, formally given by

$$d\rho(u,v) = Z^{-1} e^{-E(u,v)} du\,dv \qquad (v = \partial_t u)$$

is invariant under the NLW dynamics

Φ_d^{k+1} - measure

$$d\rho(u,v) = Z^{-1} \boxed{e^{-\frac{1}{k+1}\int u^{k+1}dx - \frac{1}{2}\|u\|_{H^1}^2} du} \otimes e^{-\frac{1}{2}\|v\|_{L^2}^2} dv$$

↑ massive Gaussian free field μ_1, see Lec 1.

↑ (spatial) white noise μ_0

- Construction of the Φ^{k+1}_d-measure: constructive QFT. (Euclidean)
  - see the notes from 2024.

  $d=1,2$: $k \in 2\mathbb{N}+1$
  $d=3$: $k=3$
  $d \geq 4$: not possible.

- Invariant Gibbs meas for NLW
  - $d=2$: Oh-Thomann '20 ($k \in 2\mathbb{N}+1$)
  - $d=3$: Bringmann-Deng-Nahmod-Yue '24. ($k=3$)

Lec 6 : Nov 27, 2025 (THU)

· Stochastic damped NLW on $\mathbb{T}^d$

(SdNLW) $\partial_t^2 u + \partial_t u + (1-\Delta)u + u^k = \sqrt{2}\xi$

defocusing: $k \in 2\mathbb{N}+1$

(1-d: we can consider $|u|^{p-1}u$, $p>2$

With $v = \partial_t u$, we can write SdNLW

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ \Delta - 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ -u^k \end{pmatrix}}_{\text{deterministic NLW}} + \underbrace{\begin{pmatrix} 0 \\ -v + \sqrt{2}\xi \end{pmatrix}}_{\text{OU process for } v.}$$

· Hamiltonian (= energy) for deterministic NLW:

$$E(u, \partial_t u) = \underbrace{\frac{1}{2}\int |\langle \nabla \rangle u|^2 + \frac{1}{2}\int (\partial_t u)^2}_{= \frac{1}{2}\|u\|_{H^1}^2 + \frac{1}{2}\|v\|_{L^2}^2} + \underbrace{\frac{1}{k+1}\int u^{k+1}\,dx}_{\text{def.}}$$

$= 0$ b/c $k+1$, even

· Gibbs measure ρ :

$$d\rho = Z^{-1} e^{-E(u,v)} du dv \qquad // \; d\mu_1 = \text{massive GFF}$$

$$= Z^{-1} e^{-\frac{1}{k+1}\int u^{k+1} dx} \underbrace{e^{-\frac{1}{2}\|u\|_{H^1}^2} du}$$

$$\otimes \underbrace{e^{-\frac{1}{2}\|v\|_{L^2}^2} dv} = d\mu_0 = \text{(spatial) white noise.}$$

fractional Gaussian field: $d\mu_s = Z_s^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du$ (see Lec 1

- $u(x) = \sum_{n\in\mathbb{Z}^d} \frac{g_n}{\langle n\rangle^s} e^{in\cdot x} \in W^{s-\frac{d}{2}-,\infty}(\mathbb{T}^d) \setminus W^{s-\frac{d}{2},\infty}(\mathbb{T}^d)$ a.s.

$$\Rightarrow d\rho(u,v) = Z^{-1} \underbrace{e^{-\frac{1}{k+1}\int u^{k+1} dx}}_{\sim\sim\sim\sim\sim\sim\sim\sim} \underbrace{d\mu_1 \otimes d\mu_0 (u,v)}_{\text{on } W^{1-\frac{d}{2}-,\infty}(\mathbb{T}^d) \times W^{-\frac{d}{2}-,\infty}(\mathbb{T}^d)}$$

- $\underline{d=1}$: $e^{-\frac{1}{k+1}\int u^{k+1}dx} \in (0,1]$, a.s.

  $u \in L^\infty$, a.s.

  - The density is integrable w.r.t. $d\mu_1 \otimes d\mu_0(u,v)$

  $\Rightarrow$ ρ exists as a prob. meas on $W^{\frac{1}{2}-,\infty}(\mathbb{T}) \times W^{-\frac{1}{2}-,\infty}(\mathbb{T})$

- $\underline{d=2}$: u = distribution of negative regularity $(1-\frac{d}{2}-<0)$.

  $\Rightarrow$ $e^{-\frac{1}{k+1}\int u^{k+1}dx} = e^{-\infty} = 0$, a.s.

  - Need to renormalize the interaction potential
    - see the notes from 2024.

Φ^{k+1}_d-meas $\otimes \mu_0$

Nelson '65 : hypercontractivity of the OU semigp & Nelson's estimate

$\Rightarrow$ ρ exists for any $k \in 2\mathbb{N}+1$.

· $\underline{d=3}$: only $k=3$, i.e. Φ^4_3-meas. (70's ~

· $\underline{d \geq 4}$: not possible. ((marginal) triviality: Aizenman, A-Duminil-Copin'21)

· deterministic NLW: $\partial_t^2 u + (1-\Delta) u + u^k = 0$

· Gibbs meas ρ is known to be invariant.

· If $\Phi(t) : (u(0), \partial_t u(0)) \mapsto (u(t), \partial_t u(t))$, then we have

$$(\Phi(t))_* \rho(A) = \rho(\Phi(-t) A) = \rho(A)$$

pushforward

for any measuable set A.

· SdNLW has the OU part: $\partial_t v = -v + \sqrt{2}\xi$.

$$d\hat{v}(n) = -\hat{v}(n)\, dt + \sqrt{2}\, dB_n, \quad n \in \mathbb{Z}^d$$

· Ornstein-Uhlenbeck process (real-valued)

$$\begin{cases} dX = -aX\,dt + b\,dB, & a>0,\ b\in\mathbb{R} \\ X(0) = x_0 \end{cases}$$

$$\Rightarrow \boxed{X(t) = e^{-at}x_0 + b\int_0^t e^{-a(t-t')}\,dB(t')}$$

· Suppose $x_0 \sim N\left(0, \frac{b^2}{2a}\right)$, indep of B.

Then, $X(t)$ is a mean 0 Gaussian r.v. with variance

$$\mathbb{E}[X^2(t)] = e^{-2at}\mathbb{E}[x_0^2] + b^2\int_0^t e^{-2a(t-t')}\,dt'$$

$$= e^{-2at}\frac{b^2}{2a} + \frac{b^2}{2a}\left(1 - e^{-2at}\right)$$

$$= \frac{b^2}{2a}.$$

$\Rightarrow N\left(0, \frac{b^2}{2a}\right)$ is invariant under OU.

- $d\hat{v}(n) = -\hat{v}(n)\,dt + \sqrt{2}\,dB_n$

  $\hat{v}(0,n) = g_n \sim N_{\mathbb{C}}(0,1)$

$\Rightarrow$ $N_{\mathbb{C}}(0,1)$ is invariant, $n \in \mathbb{Z}^d$

$\Rightarrow$ μ_0 (and thus ρ) is invariant under the OU part:

$$\partial_t v = -v + \sqrt{2}\,\xi$$

- Recall:

$$\text{Sd NLW} = \underbrace{(\text{determ.})\ \text{NLW}}_{\text{both preserve the Gibbs meas } \rho.} + \underbrace{\text{OU for } v}_{\text{both preserve the Gibbs meas } \rho.}$$

Hence, we "expect" that ρ is invariant under Sd NLW.

- can make this rigorous via regularization (= freq truncation).

- freq-truncated SdNLW:

$$(\mathrm{SdNLW}_N) \quad \partial_t^2 u_N + \partial_t u_N + (1-\Delta) u_N + \pi_N (\pi_N u_N)^k = \sqrt{2}\,\xi$$

freq proj onto $\{|n| \le N\}$

NO π_N

- truncated Gibbs meas ρ_N:

$$d\rho_N = Z_N^{-1} \underbrace{e^{-\frac{1}{k+1}\int (\pi_N u_N)^{k+1} dx} d\mu_1}_{e^{-\frac{1}{k+1}\int (\pi_N u_N)^{k+1} dx} d\mu_{1,N} \otimes d\mu_{1,N}^{\perp}} \otimes d\mu_0(u_N, \overset{\partial_t u_N}{\overset{\shortparallel}{v_N}})$$

where $\mu_{1,N} = (\pi_N)_* \mu_1$ and $\mu_{1,N}^{\perp} = (\pi_N^{\perp})_* \mu_1$

$\pi_N^{\perp} = \mathrm{Id} - \pi_N$

Split $SdNLW_N$ into two parts:

(low) $\partial_t^2 u_N^{low} + \partial_t u_N^{low} + (1-\Delta) u_N^{low} + \pi_N\big((u_N^{low})^k\big) = \sqrt{2}\,\pi_N \xi$

(high) $\partial_t^2 u_N^{high} + \partial_t u_N^{high} + (1-\Delta) u_N^{high} = \sqrt{2}\,\pi_N^{\perp} \xi$

- (high) is linear (but ∞-dim'l)

  $\Rightarrow$ $\mu_{1,N}^{\perp} \otimes \mu_{0,N}^{\perp}$ is invariant under (high).

- (low) = NLW_N: $\partial_t^2 u_N^{low} + (1-\Delta) u_N^{low} + \pi_N\big((u_N^{low})^k\big) = 0$

  + OU for $\pi_N v$: $\partial_t v_N^{low} = -v_N^{low} + \sqrt{2}\,\pi_N \xi$

  finite dim'l Hamiltonian system of ODEs (on the Fourier side

$\Rightarrow$ $d\rho_N^{low} = Z_N^{-1}\, e^{-\frac{1}{k+1}\int (\pi_N u_N^{low})^{k+1} dx}\, d\mu_{1,N} \otimes d\mu_{0,N}(u_N^{low}, v_N^{low})$

is invariant under (low) (We'll come back to this point

⇐ Lie-Trotter product formula.

$$e^{t(A+B)} = \lim_{n\to\infty} \left(e^{\frac{t}{n}A}\, e^{\frac{t}{n}B}\right)^n$$

$\Rightarrow$ $\mathrm{SdNLW}_N = (\mathrm{low}) + (\mathrm{high})$

$\Rightarrow$ Truncated Gibbs meas ρ_N is invariant under SdNLW_N.

· 1-d SdNLW: GWP in $\mathcal{H}^{\frac{1}{2}-}(\mathbb{T}) \supset \mathrm{supp}\,\rho$

$\Rightarrow$ Invariance of ρ under SdNLW follows from

① invariance of ρ_N under SdNLW_N

② $(u_N, v_N) \longrightarrow (u, v)$ PDE argument

see the notes from 2024

$\forall F \in C_b$ $\int \mathbb{E}\left[F(\Phi(t)\vec{\Phi})\right] d\rho = \int F(\vec{\Phi})\, d\rho$.

expectation w.r.t. noise

· $\Phi(t): \vec{\Phi} \longrightarrow \vec{u}(t)$, soln map

· $d=2$: No pathwise GWP in a space containing $\operatorname{supp}\rho$.
(except for $k=3$: GKOT '22).

$\Leftarrow$ Bourgain's invariant measure argument — see the notes from 2024.
Bourgain '94, '96. Oh-Robert-Tzvetkov '23

· Back to the invariance of ρ_N^{low} under (low).

· Generator $\mathcal{L}$ of a Markov semigroup:

$$dX = f\,dt + g\,dB, \quad X(0) = x_0$$

· Markov semigroup P_t: $P_t F(x_0) = \mathbb{E}[F(X_t)]$

· Generator $\mathcal{L}$ is given by

$$\mathcal{L}F(x_0) = \lim_{t\to 0} \frac{P_t F(x_0) - F(x_0)}{t} = \lim_{t\to 0} \frac{\mathbb{E}[F(X_t)] - F(x_0)}{t}$$

· By Ito's lemma,

$$F(X_t) - F(x_0) = \int_0^t F'(X_s) f(s)\,ds + \underbrace{\int_0^t F'(X_s) g(s)\,dB(s)}_{= 0 \text{ under expectation}} + \frac{1}{2}\int_0^t F''(X_s) g^2(s)\,ds$$

FTC

$$\Rightarrow \mathcal{L}F(x_0) = F'(x_0) f(0) + \frac{1}{2} F''(x_0) g^2(0)$$

ex: $X = BM$. $dX = 0 \cdot dt + 1 \cdot dB$.

$$\Rightarrow \mathcal{L} = \frac{1}{2}\partial_x^2$$

· Back to (low):

$$\partial_t^2 u_N^{low} + \partial_t u_N^{low} + (1-\Delta) u_N^{low} + \pi_N\left((\pi_N u_N^{low})^k\right) = \sqrt{2}\,\pi_N \xi$$

$$\Leftrightarrow \begin{cases} \partial_t u_N^{low} = v_N^{low} \\ \partial_t v_N^{low} = (\Delta - 1) u_N^{low} - \pi_N\left((\pi_N u_N^{low})^k\right) - v_N^{low} + \sqrt{2}\,\pi_N \xi \end{cases}$$

· ρ_N^{low} invariant $\Leftrightarrow\!\!\Rightarrow$ $\mathcal{L}_N^* \rho_N^{low} = 0$

See Chap 5 of Da Prato's book "Intro to ∞-dim'l analysis".

⊛ $\Leftrightarrow\!\!\Rightarrow$ $\int \mathcal{L}_N(F(\vec{u}))\,\rho_N^{low}(d\vec{u}) = 0$, $\vec{u} = (u, v)$

In the following, we prove ⊛ (by a formal computation).

↑ dropping π_N, etc.

· Write $\widehat{u}(n) = a_n + i\, b_n$

$\widehat{v}(n) = p_n + i\, q_n$

$$\begin{cases} \partial_t u = v \\ \partial_t v = (\Delta - 1)\, u - \underbrace{N[u]}_{= u^k} - v + \sqrt{2}\,\xi \end{cases}$$

$$\partial_t a_n = p_n$$
$$\partial_t b_n = q_n$$
$$\partial_t p_n = -\langle n \rangle^2 a_n - \mathrm{Re}\,\widehat{N(\bar{a}, \bar{b})}(n) - p_n + \sqrt{2}\,\partial_t(\mathrm{Re}\, B_n)$$
$$\partial_t q_n = -\langle n \rangle^2 b_n - \mathrm{Im}\,\widehat{N(\bar{a}, \bar{b})}(n) - q_n + \sqrt{2}\,\partial_t(\mathrm{Im}\, B_n).$$

$\bar{a} = \{a_n\}_{n \in \mathbb{Z}^2}$, etc. (or $|n| \le N$)

Write $L_N = \underbrace{L_N^1}_{\text{NLW part}} + \underbrace{L_N^2}_{\text{OU part}}$.

· OU part: $L_N^2 F(\bar{a}, \bar{b}, \bar{p}, \bar{q}) = \sum_n \Big(-p_n \partial_{p_n} F + \frac{1}{2} \partial^2_{p_n} F - q_n \partial_{q_n} F + \frac{1}{2} \partial^2_{q_n} F \Big)$

$$\int \mathcal{L}_N^2 \, F(\bar{a}, \bar{b}, \bar{p}, \bar{q}) \, d\rho = 0 \; ?$$

really ρ_N^{low}

$$P(a,b) = \frac{1}{k+1} \int u^{k+1} dx$$

for fixed n //

$-p_n \partial_{p_n} F + \frac{1}{2} \partial_{p_n}^2 F$

$-q_n \partial_{q_n} F + \frac{1}{2} \partial_{q_n}^2 F$

$$e^{-P(a,b)} \; e^{-\frac{1}{2}\sum_n \langle n^2 \rangle a_n^2 + \langle n^2 \rangle b_n^2} \; d\bar{a}\, d\bar{b}$$

$$\times \; e^{-\frac{1}{2}\sum_n p_n^2 + q_n^2} \; d\bar{p}\, d\bar{q}$$

$$\int \frac{1}{2} \partial_{p_n}^2 F \; e^{-\frac{1}{2}\sum_n p_n^2} \, dp_n = -\frac{1}{2} \int \partial_{p_n} F \cdot (-2p_n) \, dp_n$$

$$= \int p_n \partial_{p_n} F \, dp_n$$

do the same for q_n

$$\Rightarrow \int \mathcal{L}_N^2 \, F(\bar{a}, \bar{b}, \bar{p}, \bar{q}) \, d\rho = 0$$

· NLW part:

$$\partial_t a_n = p_n$$
$$\partial_t b_n = q_n$$
$$\partial_t p_n = -\langle n\rangle^2 a_n - \mathrm{Re}\,\widehat{\mathcal{N}(\bar a, \bar b)}(n)$$
$$\partial_t q_n = -\langle n\rangle^2 b_n - \mathrm{Im}\,\widehat{\mathcal{N}(\bar a, \bar b)}(n)$$

chain rule

$$\mathcal{L}_N^1 F(\bar a, \bar b, \bar p, \bar q) = p_n \partial_{a_n} F + q_n \partial_{b_n} F$$
$$+ \left(-\langle n\rangle^2 a_n - \mathrm{Re}\,\widehat{\mathcal{N}(\bar a, \bar b)}(n)\right) \partial_{p_n} F$$
$$+ \left(-\langle n\rangle^2 b_n - \mathrm{Im}\,\widehat{\mathcal{N}(\bar a, \bar b)}(n)\right) \partial_{q_n} F.$$

Q: $\int \mathcal{L}_N^1 F(\bar a, \bar b, \bar p, \bar q)\, d\rho = 0$?

$$\int p_n \partial_{a_n} F \, d\rho \overset{\text{IBP}}{=} \int \langle n^2 \rangle a_n p_n F + \underbrace{\partial_{a_n} P(\bar{a},\bar{b}) p_n F}_{①} \, d\rho$$

$$\uparrow$$

$$e^{-P(\bar{a},\bar{b})} \; e^{-\frac{1}{2}\sum_n \langle n^2 \rangle a_n^2} \, d\bar{a}$$

$$\int q_n \partial_{b_n} F \, d\rho \overset{\text{IBP}}{=} \int \langle n \rangle b_n q_n F + \underbrace{\partial_{b_n} P(\bar{a},\bar{b}) q_n F}_{②} \, d\rho$$

$$\int \left(-\langle n^2 \rangle a_n - \widehat{\mathrm{Re}\, N(\bar{a},\bar{b})}(n)\right) \partial_{p_n} F \, d\rho \; e^{-\frac{1}{2}\sum p_n^2} \, d\bar{p}$$

$$\overset{\text{IBP}}{=} -\int \langle n^2 \rangle a_n p_n F \, d\rho - \underbrace{\int \widehat{\mathrm{Re}\, N(\bar{a},\bar{b})}(n) \, p_n F \, d\rho}_{③}$$

$$\int \left(-\langle n^2 \rangle b_n - \widehat{\mathrm{Im}\, N(\bar{a},\bar{b})}(n)\right) \partial_{q_n} F \, d\rho$$

$$\overset{\text{IBP}}{=} -\int \langle n \rangle b_n q_n F \, d\rho - \underbrace{\int \widehat{\mathrm{Im}\, N(\bar{a},\bar{b})}(n) \, q_n F \, d\rho}_{④}$$

$$P(\bar a,\bar b)=\frac{1}{k+1}\int u^{k+1}\,dx=\frac{1}{k+1}\sum_{n_1+\dots+n_{k+1}=0}\prod_{j=1}^{k+1}\hat u(n_j)$$

$\hat u(n_j)$ = $a_{n_j}+\mathrm{i}\,b_{n_j}$

$$\frac{\partial}{\partial a_n}P(\bar a,\bar b)=\widehat{N(\bar a,\bar b)}(-n)$$

↑ prod. rule & chain rule

$$\frac{\partial}{\partial a_{-n}}P(\bar a,\bar b)=\widehat{N(\bar a,\bar b)}(n)$$

$N(u)=u^k$

① $$\sum_{n\in\mathbb{Z}^d}\frac{\partial}{\partial a_n}P(\bar a,\bar b)=\sum_{n\in I_*}\left(\frac{\partial}{\partial a_n}P(\bar a,\bar b)+\frac{\partial}{\partial a_{-n}}P(\bar a,\bar b)\right)+\frac{\partial}{\partial a_0}P(\bar a,\bar b)$$

$$=\sum_{n\in I_*}2\,\mathrm{Re}\,\widehat{N(\bar a,\bar b)}(n)+\widehat{u^k}(0)$$

$$\simeq-\sum_n ③$$

I_* = "$\mathbb{Z}^d/2\setminus\{\vec 0\}$"

$$=\bigcup_{k=0}^{d-1}\mathbb{Z}^k\times\mathbb{N}\times\{0\}^{d-k-1}$$

By a similar computation, we get

$$\sum_n (2) + \sum_n (4) = 0 \quad \left(\Leftarrow \operatorname{Im} \widehat{u^k}(0) = 0\right)$$

$\Rightarrow$ We showed $\int L_N^1 F(\bar{a}, \bar{b}, \bar{p}, \bar{q})\, d\rho = 0$

$$\Rightarrow \int L_N F(\bar{a}, \bar{b}, \bar{p}, \bar{q})\, d\rho = \sum_{j=1}^{2} \int L_N^j F(\bar{a}, \bar{b}, \bar{p}, \bar{q})\, d\rho$$

$$= 0.$$

see Subsec 5.2, Oh-Robert-Wang '21 (for SNLH)

$\Rightarrow$ We can make the computation above rigorous by inserting π_N and considering the phase space $\bar{a} = \{a_n\}_{|n| \le N}$, etc

$\Rightarrow$ ρ_N^{low} is invariant under (low) i.e. P_t = Markov semigroup for (low).

$$\int P_t F(\vec{\Phi})\, d\rho_N^{low} = \int F(\vec{\Phi})\, d\rho_N^{low}, \quad \forall F \in C_b.$$

Lec 7 : Dec 04, 2025 (THU)

- Φ^{k+1}_d - measure on $\mathbb{T}^d$: $d\rho = Z^{-1} e^{-\frac{1}{k+1}\int u^{k+1} dx} d\mu_1$, $k \in 2\mathbb{N}+1$.
  - $d\mu_1 = Z^{-1} e^{-\frac{1}{2}\|u\|^2_{H^1}} du$ = GFF on $\mathbb{T}^d$.

- Stochastic quantization: Parisi - Wu '81.
  - study stoch. dynamics which preserves a target measure

$\Rightarrow$ stochastic quantization eqn (SQE):

$$\partial_t u + (1-\Delta) u + u^k = \sqrt{2}\,\xi \quad \Leftarrow \text{SNLH.}$$

$d=2$: Da Prato - Debussche '03

$d=3$ & $k=3$: Hairer '14

also called

- dynamical Φ^{k+1}_d - model (parabolic Φ^{k+1}_d - model)

(Also, stoch Cahn-Hilliard eqn preserves the Φ^{k+1}_d - meas: Röckner - Yang - R. Zhu AAP '21.

· Canonical stoch. quantization: Ryang-Saito-Shigemoto '85.

Canonical SQE (= Hamiltonian SQE).

Langvin eqn

$$\begin{cases} \partial_t u = \dfrac{\partial E}{\partial v} \\ \partial_t v = -\dfrac{\partial E}{\partial u} - v + \sqrt{2}\xi \end{cases}$$

→ measure

Φ^{k+1}_d-meas on u ⊗ Uo on v → Gibbs meas.

ξ: white noise

ex: stoch damped NLW = hyperbolic Φ^{k+1}_d-model.

$$\partial_t^2 u + \partial_t u + (1-\Delta)u + u^k = \sqrt{2}\xi.$$

$d=2$: GKOT '22
$d=3$ & $k=3$: BDNY '24.

· In the complex-valued setting, NLS: $i\partial_t u + (1-\Delta)u + |u|^{k-1}u = 0$

preserves the Φ^{k+1}_d-measure (= Gibbs meas for NLS)

"dispersive Φ^{k+1}_d-model"

$d=2$: Bourgain '96

SPDE version? $i\partial_t u + (1-i)(1-\Delta)u + |u|^{k-1}u = \sqrt{2}\xi$ (1 − i: heat)

stoch complex Ginzburg-Landau eqn ⇐ analysis for SNLH applies

Chap 5 : 2-d stochastic NLW

Sec 5.1 : NLW with random initial data

$$\text{(NLW)} \quad \begin{cases} \partial_t^2 u + (1-\Delta) u + u^k = 0 \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases} \qquad \text{on } \mathbb{T}^2.$$

Main goal: Prove a.s. LWP w.r.t. the Gibbs measure ρ

↑ almost sure local existence of a uniq soln

- probabilistic conti dependence:
  Burq-Tzvetkov '14, Pocovnicu '17.

On $\mathbb{T}^2$, $d\rho = Z^{-1} e^{-\frac{1}{k+1}\int u^{k+1} dx} d\mu_1 \otimes d\mu_0 (u, \partial_t u)$.

↑ view it as a weight
(integrable w.r.t. $\mu_1 \otimes \mu_0$)

Need renormalization
(See Lec 18-19 from 2024)

- $\rho \ll \mu_1 \otimes \mu_0$ (In fact, $\rho \sim \mu_1 \otimes \mu_0$

$\rightsquigarrow$ LWP a.s. w.r.t. ρ $\Longleftrightarrow$ LWP a.s. w.r.t. $\mu_1 \otimes \mu_0$.

$$\begin{cases} u_0^\omega = \sum_{n \in \mathbb{Z}^2} \frac{g_n(\omega)}{\langle n \rangle} e^{in\cdot x} \\ u_1^\omega = \sum_{n \in \mathbb{Z}^2} h_n(\omega) e^{in\cdot x} \end{cases}$$

$\{g_n, h_n\}$, family of $\mathbb{C}$-valued std. Gaussian r.v.'s conditioned that

$g_{-n} = \overline{g_n}$

$h_{-n} = \overline{h_n}$

$\Rightarrow$ u_0^ω, u_1^ω are real-valued.

$\mathrm{Law}(u_0^\omega, u_1^\omega) = \mu_1 \otimes \mu_0$

$$(u_0^\omega, u_1^\omega) \underset{\text{a.s.}}{\in} \mathcal{H}^{0-}(\mathbb{T}^2) \setminus \mathcal{H}^0(\mathbb{T}^2)$$

($\mathcal{H}^s = H^s \times H^{s-1}$

Consider the cubic NLW on $\mathbb{T}^2$:

$$\partial_t^2 u + (1-\Delta) u + u^3 = 0$$

- Critical regularity: $s_{scaling} = \frac{d}{2} - \frac{2}{k-1} = 0$ $\quad d=2,\ k=3$

$$s_{conf} = \frac{d+1}{4} - \frac{1}{k-1} = \frac{1}{4}$$

s_{conf}: critical regularity w.r.t. conformal invariance (Lorentzian invariance)

$\Rightarrow$ $s_{crit} = \max(s_{scaling}, s_{conf}, 0) = \frac{1}{4}$ when $d=2,\ k=3$

- cubic NLW on $\mathbb{T}^2$ is LWP in $\mathcal{H}^s(\mathbb{T}^2)$ for $s > \frac{1}{4}$
  - Strichartz estimate — Lindblad-Sogge '95, GKO '18
- focusing cubic NLW: $\partial_t^2 u + (1-\Delta) u - u^3 = 0$
  - ill-posed for $s < \frac{1}{4}$ — Tao's book

defocusing case? OPEN!!

· Probabilistic LWP for rougher data?

· $\text{Law}(u_0^\omega, u_1^\omega) = \mu_\alpha \otimes \mu_{\alpha-1}$.

$\alpha - \frac{d}{2} -$
$=$
$\alpha - 1 -$

$$\begin{cases} u_0^\omega = \sum_{n\in\mathbb{Z}^2} \frac{g_n}{\langle n\rangle^\alpha} e^{in\cdot x} \\ u_1^\omega = \sum_{n\in\mathbb{Z}^2} \frac{h_n}{\langle n\rangle^{\alpha-1}} e^{in\cdot x} \end{cases} \Rightarrow (u_0^\omega, u_1^\omega) \underset{\text{a.s.}}{\in} \mathcal{H}^{\alpha-1-}(\mathbb{T}^2)$$

$\Rightarrow$ Take $\alpha > 1$ s.t. $(u_0^\omega, u_1^\omega) \underset{\text{a.s.}}{\in} \mathcal{H}^0(\mathbb{T}^2)$

· Can view u_0^ω & u_1^ω as randomization of deterministic functions:

$$\begin{cases} u_0 = \sum_{n\in\mathbb{Z}^2} \frac{\perp}{\langle n\rangle^\alpha} e^{in\cdot x} \\ u_1 = \sum_{n\in\mathbb{Z}^2} \frac{\perp}{\langle n\rangle^{\alpha-1}} e^{in\cdot x} \end{cases} \Rightarrow (u_0, u_1) \in \mathcal{H}^{\alpha-1-}(\mathbb{T}^2).$$

· Start with a pair of (determ) functions:

$$u_0 = \sum_{n\in\mathbb{Z}^2} a_n e^{in\cdot x}, \qquad u_1 = \sum_{n\in\mathbb{Z}^2} b_n e^{in\cdot x}$$

and consider their randomization: Burq-Tzvetkov '08.

$$u_0^\omega = \sum_{n\in\mathbb{Z}^2} g_n a_n e^{in\cdot x}, \qquad u_1^\omega = \sum_{n\in\mathbb{Z}^2} h_n b_n e^{in\cdot x}$$

u_0^ω: u_0 * white noise

· If $(u_0, u_1) \in \mathcal{H}^s \setminus \mathcal{H}^{s+\varepsilon}$ (for some ε),

then $(u_0^\omega, u_1^\omega) \underset{a.s.}{\in} \mathcal{H}^s \setminus \mathcal{H}^{s+\varepsilon}$ No smoothing under randomization.

but $(u_0^\omega, u_1^\omega) \underset{a.s.}{\in} W^{s,r} \times W^{s-1,r}$ $\forall\, r < \infty$ gain of integrability.

· Linear soln : $W(t)(u_0, u_1) = \partial_t S(t) u_0 + S(t) u_1$

$$S(t) = \frac{\sin(t\langle\nabla\rangle)}{\langle\nabla\rangle}$$

· **Lem 5.1.1.:** $(u_0, u_1) \in \mathcal{H}^s$. Then,

$T < \infty$, $q, r < \infty$

$$P\left(\| W(t)(u_0^\omega, u_1^\omega) \|_{L_T^q W_x^{s,r}} > \lambda \right) \le C \exp\left(- c \frac{\lambda^2}{T^{\frac{2}{q}} \|(u_0,u_1)\|_{\mathcal{H}^s}^2} \right) \quad \forall\, 0 < \lambda < \infty$$

$$\Rightarrow \quad W(t)(u_0^\omega, u_1^\omega) \underset{a.s.}{\in} L_T^q W_x^{s,r}$$

Pf:

$p \ge q \vee r$

$$\| \| W(t)(u_0^\omega, u_1^\omega) \|_{L_T^q W_x^{s,r}} \|_{L^p(\Omega)} \overset{\text{Mink}}{\le} \| \| \langle\nabla\rangle^s W(t)(u_0^\omega, u_1^\omega) \|_{L^p(\Omega)} \|_{L_T^q L_x^r}$$

$$\le p^{1/2} \| \| \langle\nabla\rangle^s W(t)(u_0^\omega, u_1^\omega) \|_{L^2(\Omega)} \|_{L_T^q L_x^r}$$

$$\sim p^{1/2} T^{\frac{1}{q}} \| (u_0, u_1) \|_{\mathcal{H}^s}$$

□

Rmk: By Sobolev, $W(t)(u_0^\omega, u_1^\omega) \in L_T^q W_x^{s-\varepsilon,\infty}$, $\forall \varepsilon > 0$.

Thm 5.1.2: $(u_0, u_1) \in \mathcal{H}^0(\mathbb{T}^2)$.

Then, the cubic NLW is a.s. LWP w.r.t. the randomized initial data

$$(u_0^\omega, u_1^\omega) = \Big(\sum_{n\in\mathbb{Z}^2} g_n a_n e^{in\cdot x}, \sum_{n\in\mathbb{Z}^2} h_n b_n e^{in\cdot x} \Big).$$

Rmk: · a.s. LWP for Law $(u_0^\omega, u_1^\omega) = \mu_\alpha \otimes \mu_{\alpha-1}$, $\alpha > 1$

· Gibbs meas : $\alpha = 1$.

Pf: First order expansion : $u = \underbrace{W(t)(u_0^\omega, u_1^\omega)}_{= z} + v$.

Then, v satisfies

$$\text{(NLW')} \quad \begin{cases} \partial_t^2 v + (1-\Delta)v + (v+z)^3 = 0. \\ (v, \partial_t v)|_{t=0} = (0,0). \end{cases}$$

- $\Gamma(v) = \int_0^t \frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle} (v+z)^3(t')\, dt'$.

Set $\vec{\Gamma}(v, \partial_t v) = (\vec{\Gamma}(v), \partial_t \vec{\Gamma}(v))$

$$\Rightarrow \|\vec{\Gamma}(v, \partial_t v)\|_{C_T \mathcal{H}^1_x} \leq T^{1/2} \|(v+z)^3\|_{L^2_{T,x}}$$

$$(v+z)^3 = v^3 + 3v^2 z + 3v z^2 + z^3.$$

$0 < T \leq 1$

- $\|v\|_{C_T L^6_x} \overset{\text{Sob}}{\lesssim} \|v\|_{C_T H^1_x} \leq \|\vec{v}\|_{C_T \mathcal{H}^1_x}$, $\vec{v} = (v, \partial_t v)$
- $\|z\|_{L^6_{T,x}} \leq \|z\|_{L^6([0,1]; L^6_x)} \leq C_\omega < \infty$ a.s.

$$\overset{\text{Young}}{\leq} C \|\vec{v}\|^3_{C_T \mathcal{H}^1_x} + C'_\omega$$

- A diff. estimate follows from a similar computation.

$\Rightarrow \exists!$ soln $(v, \partial_t v)$ to (NLW′) in $C_T \mathcal{H}^1_x$. $T_\omega = T(\|z^\omega\|_{L^6([0,1]; L^6_x)}) > 0$

□

Thm 5.1.3: Let $s > 0$. $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^2)$.

Then, the cubic NLW is a.s. GWP with the same randomized initial data as in Thm 5.1.2.

Pf: The same proof as in the 1-d case (Thm 4.2.7. from Lec 5) holds. Namely, compute $\partial_t E(\vec{v})$ and apply Gronwall.

Need $\|z\|_{L^\infty_T L^b_x} \underset{a.s.}{<} \infty$ ← See also Prop 1.8.5 from Lec 8, 2024

$\partial_t^2 z = -\langle \nabla \rangle^2 z.$

↑ Sobolev in time. Then, convert a loss of time derivative to that of spatial derivative.

$\Rightarrow$ Need $s > 0$.

□

Rmk: Thm 5.1.3 holds true for $s = 0$.

Need a Yudovich-type argument (i.e. double Gronwall)

Burq-Tzvetkov '14

Q: a.s. LWP in negative Sobolev spaces?

· Need renormalization. (Wick renormalization

Consider

$$\begin{cases} \partial_t^2 u_N + (1-\Delta) u_N + \pi_N \left((\pi_N u_N)^k \right) = 0 \\ (u_N, \partial_t u_N)\big|_{t=0} = (\pi_N u_0^\omega, \pi_N u_1^\omega) \\ \text{Law}\,(u_0^\omega, u_1^\omega) = \mu_1 \otimes \mu_0 \end{cases}$$

or u_N^k

$z(t) \in H^{0-}$

· 1st order expansion : $u_N = z_N + v_N$, $z_N = \pi_N z$

$$\Rightarrow \partial_t^2 v_N + (1-\Delta) v_N = - \sum_{\ell=0}^{k} \binom{k}{\ell} v_N^{k-\ell} \underline{\underline{z_N^\ell}}$$

Σ issue in taking a limit as $N \to \infty$.

Wick power : $:Z_N^\ell(t,x):\; = H_\ell(Z_N(t,x)\,;\,\sigma_N)$ ← Hermite polynomial see p.15

$$\sigma_N = \mathbb{E}[Z_N^2(t,x)] \qquad Z_N = \sum_{|n|\le N} \frac{\cos(t\langle n\rangle)\,g_n + \sin(t\langle n\rangle)\,h_n}{\langle n\rangle}\, e^{in\cdot x}$$

$$= \sum_{|n|\le N} \frac{\cos^2(t\langle n\rangle) + \sin^2(t\langle n\rangle)}{\langle n\rangle^2}$$

$$= \sum_{|n|\le N} \frac{1}{\langle n\rangle^2} \sim \log N \to \infty.$$

Claim: $:Z_N^\ell:$ converges to some limit, denoted by $:Z^\ell:$

- a.s. in $C_T W_x^{-\varepsilon,\infty}$
- in $L^p(\Omega\,;\,C_T W_x^{-\varepsilon,\infty})$ $\quad p<\infty$.

· So, we instead consider

$$\partial_t^2 v_N + (1-\Delta) v_N = -\sum_{\ell=0}^{k} \binom{k}{\ell} v_N^{k-\ell} :z_N^\ell:$$

Wick power

$$\downarrow N \to \infty$$

$$\partial_t^2 v + (1-\Delta) v = -\sum_{\ell=0}^{k} \binom{k}{\ell} v^{k-\ell} :z^\ell:$$

$:z^\ell: \in C_T W_x^{-\varepsilon,\infty}$ a.s.

We can study LWP of this eqn. for $\vec{v} \in C_T \mathcal{H}_x^{1-\varepsilon}$

(and $\vec{v}_N \to \vec{v}$.

Renormalized NLW: $\boxed{u = z + v}$

$$:u^k: = \sum_{\ell=0}^{k} \binom{k}{\ell} v^{k-\ell} :z^\ell:$$

$$\begin{cases} \partial_t^2 u + (1-\Delta) u + :u^k: = 0 \\ \text{Law}(u(0), \partial_t u(0)) = \mu_1 \otimes \mu_0 \end{cases}$$

- Renormalized NLW is a "formal" equation

$k=2$: $:u^2: \;=\; u^2 - \sigma_\infty \;=\; u^2 - \infty$

$k=3$: $:u^3: \;=\; u^3 - 3\sigma_\infty u \;=\; u^3 - 3\cdot\infty\cdot u$

- Hermite polynomial $H_k(x;\sigma)$ with variance parameter $\sigma>0$.
  - defined via the following generating func:

$$e^{tx-\frac{1}{2}\sigma^2 t} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x;\sigma)$$

- $H_0(x;\sigma)=1$, $H_1(x,\sigma)=x$, $H_2(x;\sigma)=x^2-\sigma$, $H_3(x;\sigma)=x^3-3\sigma x$.
- $\left\{ H_k(x;\sigma) \Big/ \sqrt{k!\,\sigma^k} \right\}_{k=0}^{\infty}$ forms an O.N. basis of $L^2\left(\mathbb{R}; \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-x^2}{2\sigma}} dx\right)$.

↑ Can obtain these by apply Gram-Schmidt to $1, x, x^2, \cdots$ in this space

Next lecture : 10 AM UK time on

Dec 8, 2025 (MON)

(last lecture of the year 2025.)

Lec 8 : Dec 08, 2025 (MON)

· NLW on $\mathbb{T}^2$

$k \geq 2$, integer

$$\begin{cases} \partial_t^2 u + (1-\Delta) u + u^k = 0 \\ (u, \partial_t u)|_{t=0} = (u_0^\omega, u_1^\omega) \quad \text{with Law}\,(u_0^\omega, u_1^\omega) = \mu_1 \otimes \mu_0 \end{cases}$$

① freq-truncate the random initial data

$$\begin{cases} \partial_t^2 u_N + (1-\Delta) u_N + u_N^k = 0 \\ (u_N, \partial_t u_N)|_{t=0} = (\pi_N u_0^\omega, \pi_N u_1^\omega) \end{cases}$$

② first order expansion: $u_N = z_N + v_N$.

$z_N = \pi_N z$
$= \pi_N W(t)(u_0^\omega, u_1^\omega)$.

$$\begin{cases} \partial_t^2 v_N + (1-\Delta) v_N + \sum_{\ell=0}^{k} \binom{k}{\ell} v_N^{k-\ell} z_N^\ell = 0 \\ (v_N, \partial_t v_N)|_{t=0} = (0,0). \end{cases}$$

↑ The limit does not exist as $N \to \infty$.

③ Wick renormalization

$$:Z_N^\ell: \;=\; H_\ell(Z_N;\sigma_N) \longrightarrow :Z^\ell: \quad \text{a.s. in } C_T W_x^{-\varepsilon,\infty} \text{ (later)}$$

$$\sigma_N = \mathbb{E}\big[Z_N^2(t,x)\big] = \sum_{|n|\in N} \frac{1}{\langle n\rangle^2} \sim \log N \to \infty$$

↑ indep of t and x

for the current problem due to the invariance of $\mu_1 \otimes \mu_0$ under the linear dynamics

$$(NLW_N') \begin{cases} \partial_t^2 v_N + (1-\Delta) v_N + \sum_{\ell=0}^{k} \binom{k}{\ell} v_N^{k-\ell} \underline{\underline{:Z_N^\ell:}} = 0 \\ (v_N, \partial_t v_N)|_{t=0} = (0,0) \end{cases}$$

④ Take $N \to \infty$

$$\begin{cases} \partial_t^2 v + (1-\Delta) v + \boxed{\sum_{\ell=0}^{k} \binom{k}{\ell} v^{k-\ell} \underline{\underline{:Z^\ell:}}} = 0 \\ (v, \partial_t v)|_{t=0} = (0,0) \end{cases} \qquad = :u^k:$$

(5) Undo the first order expansion: $u = z + v$

$$\begin{cases} \partial_t^2 u + (1-\Delta) u + :u^k: = 0 \\ (u, \partial_t u)|_{t=0} = (u_0^\omega, u_1^\omega) \end{cases}$$

renormalized NLW
Wick ordered NLW

- This is just a formal equation

$$k=2: \quad :u^2: = u^2 - \sigma_\infty = u^2 - \infty$$
$$k=3: \quad :u^3: = u^3 - 3 \cdot \infty \cdot u$$

$(NLW_N') \Rightarrow u_N = z_N + v_N$

$$H_k(x+y;\sigma) = \sum_{\ell=0}^{k} \binom{k}{\ell} x^{k-\ell} H_\ell(y;\sigma)$$

$$\begin{cases} \partial_t^2 u_N + (1-\Delta) u_N + H_k(u_N; \sigma_N) = 0 \\ (u_N, \partial_t u_N)|_{t=0} = (\pi_N u_0^\omega, \pi_N u_1^\omega) \end{cases}$$

↑ ptwise renormalization

$$\sigma_N = \mathbb{E}[z_N^2(t,x)]$$

$:u^k:$ in the limit.

· What if we do not renormalize?

renormalized truncated NLW ($k=3$):

$$\begin{cases} \partial_t^2 u_N + (1-\Delta) u_N + u_N^3 - 3\sigma_N u_N = 0 \\ (u_N, \partial_t u_N)|_{t=0} = (\pi_N u_0^\omega, \pi_N u_1^\omega) \end{cases} \qquad \sigma_N \to \infty$$

· If we instead consider

$$\begin{cases} \partial_t^2 u_N + (1-\Delta) u_N + u_N^3 = 0 \\ (u_N, \partial_t u_N)|_{t=0} = (\pi_N u_0^\omega, \pi_N u_1^\omega) \end{cases}$$

No renormalization

← LWP for each $N \in \mathbb{N}$

BUT $u_N \to 0$.

triviality

(oh-Okamoto-Robert, SPA '20 for SdNLW

⇒ Renormalization is needed to see any non-trivial limiting behavior.

· see also weak universality in GKO '18

Prop 5.1.4: $l \in \mathbb{N}$, $\varepsilon > 0$, $T > 0$.

$:z_N^l:$ converges to a limit, denoted by $:z^l:$

- a.s. in $C_T W_x^{-\varepsilon,\infty}(\mathbb{T}^2)$
- in $L^p(\Omega; C_T W_x^{-\varepsilon,\infty})$, $p < \infty$.

For now, we assume Prop 5.1.4 and prove a.s LWP of

(NLW_ν)
$$\begin{cases} \partial_t^2 v + (1-\Delta) v = -\sum_{l=0}^{k} \binom{k}{l} v^{k-l} \underbrace{:z^l:}_{\in C_T W_x^{-\varepsilon,\infty} \text{ a.s.}} \\ (v, \partial_t v)\big|_{t=0} = \underbrace{(v_0, v_1)}_{\text{deterministic initial data}} \end{cases}$$

↑ pretend they are deterministic.

Consider

$$\circledast \quad \begin{cases} \partial_t^2 v + (1-\Delta) v = -\sum_{\ell=0}^{k} \binom{k}{\ell} v^{k-\ell} \mathcal{Z}_\ell & \mathcal{Z}_0 \equiv 1 \\ (v, \partial_t v)\big|_{t=0} = (v_0, v_1) \in \mathcal{H}^s(\mathbb{T}^2) & \text{for some } s > 0. \end{cases}$$

($\mathcal{Z}_\ell$: deterministic)

Enhanced data set : $\Xi = (v_0, v_1, \mathcal{Z}_1, \cdots, \mathcal{Z}_k)$.

$$\mathcal{X}^s = \mathcal{H}^s(\mathbb{T}^2) \times \left(C([0,1]; \mathcal{W}^{-\varepsilon,\infty}(\mathbb{T}^2)) \right)^{\otimes k}$$

Prop 5.1.5: Given $\varepsilon > 0$, let $s = 1 - \varepsilon$.

Then, $\circledast$ is LWP. More precisely, $\exists\, T = T(\|\Xi\|_{\mathcal{X}^s}) > 0$

and a unique soln $(v, \partial_t v) \in C_T \mathcal{H}^s_x$ to $\circledast$.

Moreover, the map : $\Xi \in \mathcal{X}^s \longmapsto (v, \partial_t v) \in C_T \mathcal{H}^s_x$ is continuous.

As a corollary, we obtain

Thm 5.1.6: NLW on $\mathbb{T}^2$ with Law $((u(0), \partial_t u(0))) = \mu_1 \otimes \mu_0$

is a.s. LWP in the sense that the v-eqn (NLW_v) is a.s. LWP.
(with $(v_0, v_1) = (0,0)$)

Moreover, if v_N satisfies

$$\begin{cases} \partial_t^2 v_N + (1-\Delta) v_N = -\sum_{\ell=0}^{k} \binom{k}{\ell} v_N^{k-\ell} :z_N^\ell: \\ (v_N, \partial_t v_N)|_{t=0} = (0,0), \end{cases}$$

$:z_N^\ell:$ — Prop 5.1.4 → $:z^\ell:$

Prop 5.1.4 & Prop 5.1.5 →

then $(v_N, \partial_t v_N) \to (v, \partial_t v)$, a.s. in $C_T \mathcal{H}^{1-\varepsilon}_x$

$T = T_\omega$ ↑ random local existence time, a.s. positive

$\Rightarrow (u_N, \partial_t u_N) \mapsto (u, \partial_t u)$, a.s. in $C_T \mathcal{H}^{-\varepsilon}_x$.

· **Proof of Prop 5.1.5:**

$$\Gamma(v)(t) = W(t)(v_0, v_1) - \sum_{l=0}^{k} \binom{k}{l} \int_0^t \underbrace{\frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle}}_{S(t-t')} \left(v^{k-l}(t') Z_l(t')\right) dt'.$$

$0 < T \leq 1$

$Z_0 = 1.$

· $l = 0$: $\left\| \int_0^t \underbrace{S(t-t')}_{S} v^k(t') dt' \right\|_{C_T H_x^{1-\varepsilon}}$

$$\lesssim T \| v^k \|_{C_T H_x^{-\varepsilon}} \qquad \frac{\varepsilon}{2} = \frac{1+\varepsilon}{2} - \frac{1}{2} \qquad \frac{s}{d} = \frac{1}{p} - \frac{1}{q}$$

$$\overset{\text{Sob}}{\underset{\text{Hölder}}{\lesssim}} T \| v \|_{C_T L_x^{\frac{2k}{1+\varepsilon}}} \qquad \frac{1}{2} - \frac{1+\varepsilon}{2k} = \frac{k-1-\varepsilon}{2k} \leq \frac{1-\varepsilon}{2}$$

$k \geq 2.$

$$\lesssim T \| v \|^k_{C_T H_x^{1-\varepsilon}}$$

· $l = k$:

$$\left\| \int_0^t S(t-t') Z_k(t') dt' \right\|_{C_T H_x^{1-\varepsilon}} \lesssim T \| Z_k \|_{C_T H_x^{-\varepsilon}} \leq T \| \Xi \|_{X^s}.$$

Lem 5.1.7: $s \geq 0$, $1 < p \leq \infty$, $1 < q, r < \infty$

$q, r' \geq p'$ (for our purpose, $p = \infty \Rightarrow p' = 1$ — trivially satisfied)

Benyi-Oh-Zhao PAMS '24 (GKO '18) Also on $\mathbb{R}^d$

s.t. $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{r} + \frac{s}{d}$.

Then,
$$\| \langle \nabla \rangle^{-s} (fg) \|_{L^r(\mathbb{T}^d)} \lesssim \| \langle \nabla \rangle^{-s} f \|_{L^p(\mathbb{T}^d)} \| \langle \nabla \rangle^{s} g \|_{L^q(\mathbb{T}^d)}$$

Moral: The product fg "can be defined" if the sum of regularities is non-negative.

- $\ell = 1, \dots, k-1$:
$$\| \int_0^t S(t-t') (v^{k-\ell}(t') Z_\ell(t')) dt' \|_{C_T H^{1-\varepsilon}_x} \lesssim T \| v^{k-\ell} Z_\ell \|_{C_T H^{-\varepsilon}_x}$$

Lem 5.1.7 ($q = r = 2$, $p = \infty$)
$$\lesssim T \| Z_\ell \|_{C_T W^{-\varepsilon,\infty}_x} \| \langle \nabla \rangle^{\varepsilon} (v^{k-\ell}) \|_{C_T L^2_x}$$

frac. Leibniz
$$\lesssim \| v \|^{k-\ell}_{C_T W^{\varepsilon, 2(k-\ell)}_x} \overset{\text{Sob}}{\lesssim} \| v \|^{k-\ell}_{C_T H^{1-\varepsilon}_x}$$

$\frac{1}{2} - \frac{1}{2(k-1)} \leq \frac{1-\varepsilon}{2}$

$$\Rightarrow \|\vec{\Gamma}(\vec{v})\|_{C_T\mathcal{H}_x^{1-\varepsilon}} \lesssim \|(v_0,v_1)\|_{\mathcal{H}^{1-\varepsilon}} \leq \|\Xi\|_{X^{1-\varepsilon}}. \qquad s = 1-\varepsilon$$

$$+ T\left(\|\vec{v}\|^k_{C_T\mathcal{H}_x^{1-\varepsilon}} + \sum_{\ell=1}^{k} \|\vec{v}\|^{k-\ell}_{C_T\mathcal{H}_x^{1-\varepsilon}} \|\Xi\|_{X^s} \right)$$

- A similar computation yields the difference estimate.

□

Rmk: By using the Strichartz estimate, we can lower the regularity of initial data.

ex: $k=3$. $(v_0,v_1) \in \mathcal{H}^{\frac{1}{4}+\varepsilon}$ as in the deterministic case

see GKO '18.

Thm 5.1.8: The defocusing renormalized NLW on $\mathbb{T}^2$ ($k \in 2\mathbb{N}+1$) is a.s. GWP w.r.t. the Gibbsian initial data, and the Gibbs meas is invariant under the resulting NLW dynamics.

Truncated NLW:
$$\begin{cases} \partial_t^2 u_N + (1-\Delta) u_N + \pi_N \left(:(\pi_N u_N)^k: \right) = 0, \quad k \in 2\mathbb{N}+1 \\ \text{Law}\left((u_N(0), \partial_t u_N(0))\right) = \rho_N \end{cases}$$

$$= \left(\rho_N^{low} \otimes (\pi_N^{-1})_* \mu_1 \right) \otimes \mu_0$$

- nonlinear dynamics for $|n| \le N$ $\Leftarrow$ Thm 5.1.6 applies (after small modification)
- lin. dynamics for $|n| \ge N$

$\Rightarrow$ $(u_N, \partial_t u_N) \longrightarrow (u, \partial_t u)$ locally in time.

Then, apply Bourgain's inv. meas. argument. See the notes from 2024

One key diff: we study v_N, v eqns.

$N \in \mathbb{N}$. · key prop: $\forall\, T, \varepsilon > 0$, $\exists\, \Omega_N = \Omega_N(T, \varepsilon)$ $\subset \mathcal{H}^{-\varepsilon}(\mathbb{T}^2)$

① $\rho_N(\Omega_N^c) < \varepsilon$

② $\| (v_N, \partial_t v_N) \|_{C_T \mathcal{H}^{1-\varepsilon}_x} \lesssim \left(\log \frac{1}{\varepsilon}\right)^{k/2}$ ← unif in N

See Oh-Robert-Tzvetkov, AHL '23 (Sec 5)

Rmk: $k = 3$: We can prove pathwise GWP (GKOT, '22).

SNLH: pathwise GWP for any $k \in 2\mathbb{N}+1$ (Mourrat-Weber, AOP '17)
Also, on $\mathbb{R}^2$

NLW (or Sd NLW) on $\mathbb{R}^2$:
Oh-Tolomeo-Wang-Zheng, CMP '25+

· Back to the construction of Wick powers $:z^\ell:$ (Prop 5.1.4 on page 5.

· hypercontractivity & Wiener chaos estimate

Fix $d \in \mathbb{N}$. $H = L^2(\mathbb{R}^d; \mu_d)$ $d\mu_d = \frac{1}{(2\pi)^{d/2}} e^{-|x|^2/2} dx$

$x = (x_1, \cdots, x_d)$

homog Wiener chaos of order k.

$$\mathcal{H}_k = \overline{\left\{ \prod_{j=1}^{d} H_{k_j}(x_j) \;:\; k = k_1 + \cdots + k_d \right\}}^{\|\cdot\|_H}$$

or $L^p(\mathbb{R}^d, \mu_d)$ $p < \infty$

↑ When $d = \infty$, $k_j = 0$ except for finitely many j's.

Ito - Wiener decomposition

$$H = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$$

· Ornstein-Uhlenbeck operator $L = \Delta - x \cdot \nabla$

See the note "OU process, ..." on my website.

$F \in \mathcal{H}_k$ is an eigenfunc of L with eigenvalue $-k$.

· Lem 5.1.9: hypercontractivity of the OU semigroup e^{tL}

$\infty >$ $p \geq q > 1$. Then, for $t \geq \frac{1}{2} \log\left(\frac{p-1}{q-1}\right)$, we have

$$\| e^{tL} f \|_{L^p(\mathbb{R}^d, \mu_d)} \leq \| f \|_{L^q(\mathbb{R}^d, \mu_d)}$$

↑ indep of $d \in \mathbb{N} \Rightarrow$ holds true for $d = \infty$.

($\Leftrightarrow$ log Sobolev ineq. (Gross '75

· $q = 2$, $f \in \mathcal{H}_k \Rightarrow e^{tL} f = e^{-tk} f$. Take $t = \frac{1}{2} \log(p-1)$

$$\| f \|_{L^p(\mathbb{R}^d, \mu_d)} \leq e^{tk} \| f \|_{L^2} = (p-1)^{k/2} \| f \|_{L^2}.$$

Let $\mathcal{H}_{\le k} = \bigoplus_{j=0}^{k} \mathcal{H}_j$.

Simon's book. Thm I.22 '74.

Lem 5.1.10: Wiener chaos estimate $\quad 2 \le p < \infty$

$$\|F\|_{L^p(\Omega)} \le (p-1)^{k/2} \|F\|_{L^2(\Omega)}$$

for any $F \in \mathcal{H}_{\le k}$.

Lem 5.1.11: f, g, jointly Gaussian, mean 0, variance σ_f, σ_g.

$$\mathbb{E}\left[H_k(f;\sigma_f)\, H_\ell(g;\sigma_g) \right] = \delta_{k\ell} \cdot k! \left(\mathbb{E}[fg] \right)^k.$$

· Proof of Prop 5.1.4

$$Z_N(t,x) = \sum_{|n|\le N} \frac{\cos(t\langle n\rangle)\, g_n + \sin(t\langle n\rangle)\, h_n}{\langle n\rangle} e^{in\cdot x}.$$

· $\mathbb{E}\big[|\langle\nabla\rangle^{-\varepsilon} Z_N(t,x)|^2\big] = \sum_{|n|\le N} \frac{1}{\langle n\rangle^{2+2\varepsilon}} \lesssim 1$, unif in $N\in\mathbb{N}$.

· $\mathbb{E}\big[:Z_N^\ell(t,x):\ :Z_N^\ell(t,y):\big]$

$$\overset{\text{Lem 5.1.11}}{=} \ell!\Big(\mathbb{E}\big[Z_N(t,x)\,Z_N(t,y)\big]\Big)^\ell$$

$e_{n_1}(x-y)\cdots e_{n_\ell}(x-y)$

$$= \ell! \sum_{|n_1|,\cdots,|n_\ell|\le N} \prod_{j=1}^{\ell} \frac{1}{\langle n_j\rangle^2}\, e_{n_1+\cdots+n_\ell}(x-y)$$

· Apply $\langle\nabla_x\rangle^{-\varepsilon}$, $\langle\nabla_y\rangle^{-\varepsilon}$, and set $x=y$.

$$\Rightarrow \mathbb{E}\big[|\langle\nabla\rangle^{-\varepsilon}:Z_N^\ell(t,x):|^2\big] \lesssim \sum_{|n_1|,\cdots,|n_\ell|\le N} \frac{1}{\langle n_1+\cdots+n_\ell\rangle^{2\varepsilon}} \prod_{j=1}^{\ell}\frac{1}{\langle n_j\rangle} \lesssim 1, \text{ unif in } N$$

$\Rightarrow \quad \| \; \| : z_N^\ell(t) : \|_{W_x^{-\varepsilon, r}} \|_{L^p(\Omega)}$ $\quad r < \infty$ $\quad$ Minkowski's integral ineq

$p < \infty$

$\lesssim p^{\ell/2}$, unif in N. (Also for $r = \infty$, by Sobolev)

$\Rightarrow \quad : z_N^\ell(t) :$ is bdd in $L^p(\Omega; W_x^{-\varepsilon, \infty})$ $\quad p < \infty$, $\varepsilon > 0$.

· time regularity: $\delta_h f(t) = f(t+h) - f(t)$.

$$\frac{1}{\ell!} \mathbb{E}\left[\delta_h : z_N^\ell(t,x) : \; \delta_h : z_N^\ell(t,y) : \right]$$

$$\underset{\text{Lem 5.1.11}}{=} \left(\mathbb{E}\left[z_N(t+h,x)\, z_N(t+h,y) \right] \right)^\ell - \left(\mathbb{E}\left[z_N(t,x)\, z_N(t+h,y) \right] \right)^\ell$$

$$- \left(\mathbb{E}\left[z_N(t+h,x)\, z_N(t,y) \right] \right)^\ell + \left(\mathbb{E}\left[z_N(t,x)\, z_N(t,y) \right] \right)^\ell$$

$$\mathbb{E}\left[\left|\langle\nabla\rangle^{\xi}\,\delta_h : z_N^{\ell}(t,x) :\right|^2\right]$$

$$\sim \sum_{|n_1,\cdots,n_\ell|\le N} \frac{1}{\langle n_1+\cdots+n_\ell\rangle^{2\xi}} \prod_{j=1}^{\ell} G_j(t,n_j,h)$$

$$G_1(t,h,n) = \mathbb{E}\left[\delta_h \widehat{z}_N(t_1,n)\cdot\overline{\widehat{z}_N(t_1,n)}\right]$$

$j\neq 1$ $\quad G_j(t,h,n) = \mathbb{E}\left[\widehat{z}_N(t_1,n)\,\overline{\widehat{z}_N(t_2,n)}\right]$

$t_1, t_2 = t$ or $t+h$.

$$|G_j(t,h,n)| \lesssim \frac{1}{\langle n\rangle^2}, \quad j=1,\cdots,\ell.$$

$j=1$: By MVT,

$$|G_1(t,h,n)| \lesssim \frac{|h|}{\langle n\rangle}$$

$$|\cos((t+h)\langle n\rangle) - \cos(t\langle n\rangle)| \overset{MVT}{\lesssim} |h|\langle n\rangle$$

$$\Rightarrow |G_1(t,h,n)| \lesssim \frac{|h|^{\alpha}}{\langle n\rangle^{2-\alpha}}, \quad 0\le\alpha\le 1.$$

Given $\varepsilon > 0$, choose $2\varepsilon - \alpha > 0$ s.t.

we can sum over $n_1, \cdots, n_\ell$.

$$(**) \Rightarrow \Big\| \, \| \delta_h : z_N^\ell(t) : \|_{W^{-\varepsilon, r}} \Big\|^p_{L^p(\Omega)} \lesssim p^{p \cdot \frac{\ell}{2}} \, |h|^{\frac{\alpha p}{2}}$$

> 1 by choosing $p \gg 1$.

$$\frac{\alpha p}{2} = 1 + \left(\frac{\alpha p}{2} - 1\right)$$

for $r < \infty$ (and for $r = \infty$, by Sobolev.)

$\Rightarrow$ By Kolmogorov's continuity criterion,

$$: z_N^\ell : \; \in C_T W_x^{-\varepsilon, \infty}, \text{ a.s.}$$

in fact $C_T^\beta W_x^{-\varepsilon, \infty}$ $\beta < \varepsilon$.

A similar computation on $: z_M^\ell : - : z_N^\ell :$ for $M \geq N \geq 1$

shows this is Cauchy a.s. in $C_T W_x^{-\varepsilon, \infty}$ (also in $L^p(\Omega; C_T W_x^{-\varepsilon, \infty})$)

$\square$

Kolmogorov's continuity criterion (Exercise 8.2 in "Stoch. Processes".)

$$(**) \Rightarrow P\left(\sup_{t_1 \neq t_2} \frac{\| :Z_N^l(t_1): - :Z_N^l(t_2): \|_{W_x^{-\varepsilon,\infty}}}{|t_1 - t_2|^{\frac{\alpha}{2} - \frac{1}{p}}} > \lambda \right) \leq \frac{C}{\lambda^p}$$

for any finite $p \geq 1$.

$\Rightarrow$ $:Z_N^l: \in L^p(\Omega; C_T W_x^{-\varepsilon,\infty})$ with a unif bound in $N \in \mathbb{N}$

- A similar computation yields convergence of $:Z_N^l:$ in $L^p(\Omega; C_T W_x^{-\varepsilon,\infty})$.

Lec 9