Publ. Math. Debrecen
Manuscript (April 26, 2012)

The Sobolev inequality on the torus revisited

By Árpád Bényi and Tadahiro Oh

Abstract. We revisit the Sobolev inequality for periodic functions on the d-dimensional torus. We provide an elementary Fourier analytic proof of this inequality which highlights both the similarities and differences between the periodic setting and the classical d-dimensional Euclidean one.

1. Introduction: motivation and preliminaries

The Sobolev spaces are ubiquitous in harmonic analysis and PDEs, where they appear naturally in problems about regularity of solutions or well-posedness. Tightly connected to these problems are certain embedding theorems that relate the norms of Lebesgue and Sobolev spaces for appropriate indices. These theorems are known under the name of Sobolev inequalities; they are stated rigorously in Proposition 1.1 and Corollary 1.2; see also Subsection 2.2. In this note, we use tools from classical Fourier analysis and provide an elementary approach to such inequalities for periodic functions on the d-dimensional torus.

The appeal of Sobolev spaces is due to the simplicity of their definition which captures both the regularity and size of a distribution. If k is a positive integer and $1 \leq p \leq \infty$, let $L^p_k(\mathbb{R}^d)$ denote the space of all $u \in L^p(\mathbb{R}^d)$ such that the weak derivatives $D^\alpha u \in L^p(\mathbb{R}^d)$ for all $|\alpha| \leq k$. In the PDE literature, this space is often denoted by $W^{k,p}(\mathbb{R}^d)$. For non-integer values of $s > 0$, the complex interpolation of the integer order spaces $L^p_k(\mathbb{R}^d)$ yields the inhomogeneous (fractional) *Sobolev spaces*, or as they are also commonly referred to, inhomogeneous *Bessel potential*

Mathematics Subject Classification: Primary 42B35; Secondary 42B10, 42B25.
Key words and phrases: Sobolev spaces, Sobolev inequality, Fourier series, Bessel potential, Riesz potential, maximal function.

spaces. We denote them by $L^p_s(\mathbb{R}^d)$, $s \in \mathbb{R}^+$. In fact, on the Fourier side, these spaces can be defined for all $s \in \mathbb{R}$. As such, they are Banach spaces, endowed with the norm

$$\|u\|_{L^p_s(\mathbb{R}^d)} = \big\| \big(\langle\xi\rangle^s \hat{u}(\xi)\big)^\vee \big\|_{L^p(\mathbb{R}^d)}.$$

Here, $\langle\xi\rangle = (1 + 4\pi^2|\xi|^2)^{\frac{1}{2}}$, $\hat{u}$ denotes the Fourier transform defined by

$$\hat{u}(\xi) = \int_{\mathbb{R}^d} u(x) e^{-2\pi i x\cdot\xi} dx,$$

and $u^\vee$ denotes the inverse Fourier transform of u defined by $u^\vee(x) = \hat{u}(-x)$. We can also define the fractional inhomogeneous Sobolev spaces $W^{s,p}(\mathbb{R}^d)$ by applying the real interpolation method to the integer order spaces $W^{k,p}(\mathbb{R}^d)$. It is worth pointing out, however, that, due to the different methods of interpolation used (real and complex, respectively), we have $W^{s,p}(\mathbb{R}^d) \neq L^p_s(\mathbb{R}^d)$ unless s is an integer or $p = 2$. The spaces $W^{s,p}(\mathbb{R}^d)$ can also be characterized by the L^p-modulus of continuity, analogous to (1.13). See the books by Bergh and Löfström [3], Stein [12], and Tartar [15] for more detailed discussions on $L^p_s(\mathbb{R}^d)$ and $W^{s,p}(\mathbb{R}^d)$.

The homogeneous Sobolev spaces $\dot{L}^p_s(\mathbb{R}^d)$ are defined in a similar way, by replacing $\langle\cdot\rangle$ with $|\cdot|$ in the definition above[1]:

$$\|u\|_{\dot{L}^p_s(\mathbb{R}^d)} = \big\| \big(|\xi|^s \hat{u}(\xi)\big)^\vee \big\|_{L^p(\mathbb{R}^d)}.$$

When $p = 2$, we simply write $H^s(\mathbb{R}^d) = L^2_s(\mathbb{R}^d)$ or $\dot{H}^s(\mathbb{R}^d) = \dot{L}^2_s(\mathbb{R}^d)$.

Let now $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ denote the d-dimensional torus. In analogy with the definition of the Sobolev spaces on the d-dimensional Euclidean space $\mathbb{R}^d$, the inhomogeneous Sobolev (or Bessel potential) spaces $H^s(\mathbb{T}^d)$ and $L^p_s(\mathbb{T}^d)$ on the torus $\mathbb{T}^d$ are defined via the norms

$$\|u\|_{H^s(\mathbb{T}^d)} = \bigg(\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s} |\hat{u}(n)|^2\bigg)^{\frac{1}{2}}, \tag{1.1}$$

$$\|u\|_{L^p_s(\mathbb{T}^d)} = \big\| \big(\langle n\rangle^s \hat{u}(n)\big)^\vee \big\|_{L^p(\mathbb{T}^d)}. \tag{1.2}$$

Here, u denotes a periodic function on $\mathbb{T}^d$ and $\hat{u}(n)$, $n \in \mathbb{Z}^d$, are its Fourier coefficients defined by

$$\hat{u}(n) = \int_{\mathbb{T}^d} u(x) e^{-2\pi i n\cdot x} dx.$$

[1] Strictly speaking, the homogeneous spaces $\dot{L}^p_s(\mathbb{R}^d)$ are defined only for the equivalence classes modulo polynomials (corresponding to the distributions supported at the origin on the Fourier side).

The fact that $H^s(\mathbb{T}^d) = L^2_s(\mathbb{T}^d)$ is a simple consequence of Plancherel's identity. Clearly, we can define the homogeneous Sobolev spaces on the torus in a similar way:

$$\|u\|_{\dot{H}^s(\mathbb{T}^d)} = \Bigg(\sum_{n\in\mathbb{Z}^d\setminus\{0\}} |n|^{2s} |\hat{u}(n)|^2 \Bigg)^{\frac{1}{2}}, \tag{1.3}$$

$$\|u\|_{\dot{L}^p_s(\mathbb{T}^d)} = \big\| \big(|n|^s \hat{u}(n)\big)^\vee \big\|_{L^p(\mathbb{T}^d)}. \tag{1.4}$$

The periodic Sobolev inequality inequality is part of the mathematical analysis folklore. It is essentially stated in Strichartz' paper [14], albeit with no proof. Due to the geometric and topological structure of the torus, the Sobolev inequality on $\mathbb{T}^d$ can be viewed as a particular case of a Sobolev inequality on a compact manifold; see, for example, [1] and [8]. However, our goal here is to provide what we believe is a very natural and elementary proof of this inequality via Fourier analysis which emphasizes the periodic nature of the Sobolev spaces involved; to the best of our knowledge, this argument is missing from the literature. It is plausible that one can infer other proofs of the Sobolev inequality on $\mathbb{T}^d$ from corresponding ones on $\mathbb{R}^d$ (such as the ones implied by the fundamental solution of the Laplacian or by isoperimetric inequalities). Our hope is that the expository and self-contained nature of this presentation makes it accessible to a large readership.

The Sobolev inequality on the torus can be understood as an embedding of a periodic Sobolev space into a periodic Lebesgue space. More precisely, we have the following.

Proposition 1.1. *Let u be a function on $\mathbb{T}^d$ with mean zero. Suppose that $s > 0$ and $1 < p < q < \infty$ satisfy*

$$\frac{s}{d} = \frac{1}{p} - \frac{1}{q}. \tag{1.5}$$

Then, we have

$$\|u\|_{L^q(\mathbb{T}^d)} \lesssim \|u\|_{\dot{L}^p_s(\mathbb{T}^d)}. \tag{1.6}$$

Here, and throughout this note, we use $A \lesssim B$ to denote an estimate of the form $A \leq cB$ for some $c > 0$ independent of A and B. Similarly, we use $A \sim B$ to denote $A \lesssim B$ and $B \lesssim A$.

An immediate consequence of Proposition 1.1 is the same inequality for the inhomogeneous Sobolev spaces $L^p_s(\mathbb{T}^d)$ with the natural condition on the indices.

Corollary 1.2. *Let u be a function on $\mathbb{T}^d$. Suppose that $s > 0$ and $1 < p < q < \infty$ satisfy*

$$\frac{s}{d} \geq \frac{1}{p} - \frac{1}{q}. \tag{1.7}$$

Then, we have

$$\|u\|_{L^q(\mathbb{T}^d)} \lesssim \|u\|_{L^p_s(\mathbb{T}^d)}. \tag{1.8}$$

Perhaps unsurprisingly, the appearance of Sobolev spaces on the torus is frequent in works that investigate, for example, nonlinear PDEs in periodic setting. Let us briefly discuss some applications of these spaces and of the periodic Sobolev inequality in the study of the Kortweg-de Vries (KdV) equation:

$$u_t + u_{xxx} + uu_x = 0, \quad (x, t) \in \mathbb{T} \times \mathbb{R}. \tag{1.9}$$

By the classical energy method, Kato [10, 11] proved local-in-time well-posedness of (1.9) in $H^s(\mathbb{T})$ for $s > 3/2$. This $3/2$ critical regularity arises from the Sobolev embedding theorem on $\mathbb{T}$ (see (2.1) for the continuous version) applied to the u_x term in the nonlinearity, since for each fixed t:

$$\|u_x(\cdot, t)\|_{L^\infty(\mathbb{T})} \lesssim \|u(\cdot, t)\|_{H^s(\mathbb{T})} \quad \text{for } s > 3/2.$$

In the seminal paper [2], Bourgain improved Kato's result and proved well-posedness of (1.9) in $L^2(\mathbb{T})$ by introducing a new weighted space-time Sobolev space $X^{s,b}(\mathbb{T} \times \mathbb{R})$ whose norm is given by

$$\|u\|_{X^{s,b}(\mathbb{T}\times\mathbb{R})} = \|\langle n\rangle^s \langle \tau - n^3\rangle^b \hat{u}(n, \tau)\|_{L^2_\tau \ell^2_n}, \quad s, b \in \mathbb{R}.$$

Ever since [2], this so-called Bourgain space $X^{s,b}$ and its variants have played a central role in the analysis of nonlinear (dispersive) PDEs and led to a significant development of the field. Let $S(t) = e^{-t\partial_x^3}$ denote the linear semigroup for (1.9). Then, the $X^{s,b}$-norm of a function u on $\mathbb{T} \times \mathbb{R}$ can be written as the usual space-time Sobolev $H^b_t H^s_x$-norm of its interaction representation $S(-t)u$:

$$\|u\|_{X^{s,b}} = \|S(-t)u\|_{H^b_t H^s_x}. \tag{1.10}$$

Now, in view of (1.10), the periodic Sobolev inequality (1.8) leads to the following estimate:

$$\|u\|_{L^2_t(\mathbb{R}; L^p_x(\mathbb{T}))} \lesssim \|u\|_{X^{s,0}(\mathbb{T}\times\mathbb{R})}$$

for $0 \leq s < 1/2$ and $2 \leq p \leq 2/(1 - 2s)$. Such estimates are widely used in multilinear estimates appearing in the I-method developed by Colliander, Keel, Staffilani, Takaoka, and Tao; see, for example, [4, Section 8].

Lastly, we present a heuristic argument indicating the connection between Bourgain's periodic L^4-Strichartz inequality:

$$\|u\|_{L^4_{x,t}(\mathbb{T}\times\mathbb{R})} \lesssim \|u\|_{X^{0,1/3}(\mathbb{T}\times\mathbb{R})} \tag{1.11}$$

and the Sobolev inequality. On the one hand, by the continuous Sobolev inequality (2.6) and (1.8), we have

$$\|u\|_{L^4_{x,t}(\mathbb{T}\times\mathbb{R})} \lesssim \|u\|_{X^{1/4,1/4}(\mathbb{T}\times\mathbb{R})}. \tag{1.12}$$

On the other hand, in view of the linear part of the equation (1.9), $u_t + u_{xxx} = 0$, we can formally view the three spatial derivatives as "equivalent" to one temporal derivative. Then, by formally moving the spatial derivative $s = 1/4$ in (1.12) to the temporal side, we obtain the temporal regularity $b = 1/3$ in (1.11), since $1/3 = 1/4 + (1/3)(1/4)$. Of course, this is merely a heuristic argument showing why $b = 1/3$ is the natural regularity in (1.11) and the actual proof is more complicated, see [2]. For various relations among the $L^p_t L^q_x$ spaces and $X^{s,b}$ spaces by the Sobolev inequality, the periodic L^4-and L^6-Strichartz inequalities and interpolation, the reader is referred to [5, Section 3].

Having discussed the usefulness of the Sobolev inequality in periodic setting, the next natural question that arises is how it differs from its Euclidean counterpart. We postpone the answer to this question to the following section. However, in anticipation of this answer, we provide the reader with the following insight: the periodic Sobolev spaces are intrinsically more delicate in nature than the non-periodic ones, and thus the proofs in the periodic case require a more careful analysis. In order to justify this claim, let us take a closer look at the difference (and analogy) between the homogeneous Bessel potential spaces $\dot{H}^s(\mathbb{R}^d)$ and $\dot{H}^s(\mathbb{T}^d)$.

We begin by recalling the following characterization of the $\dot{H}^s(\mathbb{R}^d)$ norm by the L^2-modulus of continuity; see Hörmander's monograph [9]:

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi = c \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} dxdy. \tag{1.13}$$

The proof of (1.13) goes as follows. By the change of variables $x \mapsto x + y$, the double integral in (1.13) is

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x+y) - u(y)|^2}{|x|^{d+2s}} dxdy = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|e^{2\pi i x\cdot\xi} - 1|^2}{|x|^{d+2s}} dx\, |\hat{u}(\xi)|^2 d\xi,$$

where we used the fact that, for fixed x, the Fourier transform of $u(x+y)-u(y)$ as a function of y is given by $(e^{2\pi i x\cdot\xi}-1)\hat{u}(\xi)$. Now, define $A(\xi)$ by

$$A(\xi) = |\xi|^{-2s}\int_{\mathbb{R}^d}\frac{|e^{2\pi i x\cdot\xi}-1|^2}{|x|^{d+2s}}dx = |\xi|^{-2s}\int_{\mathbb{R}^d}\frac{\sin^2(\pi x\cdot\xi)}{4|x|^{d+2s}}dx. \tag{1.14}$$

Then, by the change of variables $x\mapsto tx$, we have $A(t\xi)=A(\xi)$. Hence, $A(\xi)=A$ is independent of ξ. Moreover, with $\xi=(1,0,\dots,0)$, we have

$$A = \int_{\mathbb{R}^d}\frac{\sin^2(\pi x_1)}{4|x|^{d+2s}}dx.$$

Noting that $\frac{\sin^2(\pi x_1)}{|x|^{d+2s}} \lesssim |x|^{-d+2(1-s)}$ near the origin and $\frac{\sin^2(\pi x_1)}{|x|^{d+2s}} \leq |x|^{-d-2s}$ near infinity, we have $A<\infty$. Hence, (1.13) follows from (1.14) by choosing $c=A$.

Remark 1.1. Given $u\in L^p(\mathbb{R}^d)$, $\omega_p(t)=\|u(x+t)-u(x)\|_{L^p_x}$ is called the L^p-modulus of continuity. Hence, we can view (1.13) as the characterization of the $\dot{H}^s(\mathbb{R}^d)$-norm by the L^2-modulus of continuity. There is an analogous result for the characterization of the $\dot{W}^{s,p}(\mathbb{R}^d)$-norm by the L^p-modulus of continuity; see Stein's book [12, p.141].

We note immediately that the double integral expression in (1.13) is not quite meaningful for periodic functions on $\mathbb{T}^d$ even if we only integrate over $\mathbb{T}^d$. Nonetheless, we have an analogue of (1.13) for $\dot{H}^s(\mathbb{T}^d)$, but the details of the proof are already a little more delicate.

Proposition 1.3. *Let $0<s<1$. Then, for $u\in\dot{H}^s(\mathbb{T}^d)$, we have*

$$\|u\|^2_{\dot{H}^s(\mathbb{T}^d)} \sim \int_{\mathbb{T}^d}\int_{[-\frac12,\frac12)^d}\frac{|u(x+y)-u(y)|^2}{|x|^{d+2s}}dxdy. \tag{1.15}$$

Proof. As before, we have

$$\int_{\mathbb{T}^d}\int_{[-\frac12,\frac12)^d}\frac{|u(x+y)-u(y)|^2}{|x|^{d+2s}}dxdy = \sum_{n\in\mathbb{Z}^d\setminus\{0\}}\int_{[-\frac12,\frac12)^d}\frac{|e^{2\pi i x\cdot n}-1|^2}{|x|^{d+2s}}dx\,|\hat{u}(n)|^2.$$

It remains to show that $B(n)$ given by

$$B(n) = |n|^{-2s}\int_{[-\frac12,\frac12)^d}\frac{|e^{2\pi i x\cdot n}-1|^2}{|x|^{d+2s}}dx = |n|^{-2s}\int_{[-\frac12,\frac12)^d}\frac{\sin^2(\pi x\cdot n)}{4|x|^{d+2s}}dx \tag{1.16}$$

is bounded both from above and below uniformly in $n \in \mathbb{Z}^d \setminus \{0\}$. Note that, in this case, we can not use a change of variables to show that $B(n)$ is independent of n. Of course, by extending the integration to $\mathbb{R}^d$, we have $B(n) \leq A(n) = A < \infty$, where $A(n)$ is, as in (1.14), independent of n.

Next, we show that $B(n)$ is bounded below by a positive constant, independent of $n = (n_1, n_2, ..., n_d)$. Rearrange n_j such that $n_1, \dots, n_m$ are non-zero and $n_{m+1} = \cdots = n_d = 0$. By symmetry, assume that n_1 is positive and that $n_1 = \max(n_1, |n_2|, \dots, |n_d|)$. Now, we restrict the integral in (1.16) to

$$D = \big\{x \in [-\tfrac{1}{2}, \tfrac{1}{2}]^d : |x| < \tfrac{1}{2|n|},\ n_j x_j > 0, j = 1, \dots, m\big\} \\ \cap \big\{|x_1| = \max|x_j|\big\} \subset [-\tfrac{1}{2}, \tfrac{1}{2})^d.$$

We have $n_1 x_1 \leq n \cdot x \leq \frac{1}{2}$ on D. Since $2y \leq \sin \pi y$ for $y \in [0, \frac{1}{2}]$, we have

$$\sin^2(\pi x \cdot n) \gtrsim (n_1 x_1)^2 \gtrsim |n|^2 |x|^2,$$

where the last inequality follows from $|n_1| \gtrsim |n|$ and $|x_1| \gtrsim |x|$. Then, by integration in the polar coordinates, we obtain

$$B(n) \gtrsim |n|^{2-2s} \int_D |x|^{-d+2-2s} dx \gtrsim |n|^{2-2s} \int_{|x|<\frac{1}{2|n|}} |x|^{-d+2-2s} dx \\ \sim |n|^{2-2s} \int_0^{\frac{1}{2|n|}} r^{1-2s} dr \gtrsim 1.$$

This completes the proof of (1.15). $\square$

2. The Sobolev inequality

This section is devoted to a discussion of the Sobolev inequality on the d-dimensional torus. As already pointed out in the previous section, the inequality is widely used for periodic PDEs.

2.1. Sobolev's embedding theorem. Sobolev's embedding theorem states that, for $sp > d$,

$$\|u\|_{L^\infty(\mathbb{R}^d)} \lesssim \|u\|_{L^p_s(\mathbb{R}^d)}. \tag{2.1}$$

Notice that the condition $sp > d$ is equivalent to $\frac{s}{d} > \frac{1}{p} = \frac{1}{p} - \frac{1}{\infty}$; compare this also with (1.5). When $p \leq 2$, (2.1) follows from Hölder's inequality and

Hausdorff-Young's inequality. Indeed,

$$
\begin{aligned}
|u(x)| &\leq \int_{\mathbb{R}^d} |\hat{u}(\xi)| d\xi \leq \bigg(\int_{\mathbb{R}^d} \langle \xi \rangle^{-ps} d\xi \bigg)^{\frac{1}{p}} \| \langle \xi \rangle^s \hat{u}(\xi) \|_{L^{p'}(\mathbb{R}^d)} \\
&\lesssim \big\| \big(\langle \xi \rangle^s \hat{u}(\xi) \big)^\vee \big\|_{L^p(\mathbb{R}^d)} = \|u\|_{L^p_s(\mathbb{R}^d)}.
\end{aligned}
$$

This argument, in particular, shows that $\hat{u} \in L^1(\mathbb{R}^d)$. Hence, it follows from Riemann-Lebesgue lemma that u is uniformly continuous on $\mathbb{R}^d$, vanishing at infinity. The same argument yields the corresponding result on $\mathbb{T}^d$.

When $p > 2$, we need to proceed differently. We borrow some ideas from the nice exposition in Grafakos' books [6, 7]. Define G_s by

$$
G_s(x) = \big(\langle \xi \rangle^{-s} \big)^\vee (x). \tag{2.2}
$$

Note that G_s is the convolution kernel of the Bessel potential $J_s = (I - \Delta)^{-\frac{s}{2}}$ of order s, i.e. $J_s(f) = f * G_s$. Then, the following estimates hold for G_s (see [7, Proposition 6.1.5]):

$$
G_s(x) \leq C(s,d) e^{-\frac{|x|}{2}} \quad \text{for } |x| \geq 2, \tag{2.3}
$$

while for $|x| \leq 2$, we have

$$
G_s(x) \leq c(s,d) \begin{cases} |x|^{s-d} + 1 + O(|x|^{s-d+2}), & \text{for } 0 < s < d, \\ \log \frac{2}{|x|} + 1 + O(|x|^2), & \text{for } s = d, \\ 1 + O(|x|^{s-d}), & \text{for } s > d. \end{cases} \tag{2.4}
$$

When $s \geq d$, $G_s \in L^{p'}(\mathbb{R}^d)$, while when $s < d$, we have $G_s \in L^{p'}(\mathbb{R}^d)$ (near the origin) if and only if $sp > d$. Thus, by Young's inequality, we obtain

$$
\|f * G_s\|_{L^\infty(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}. \tag{2.5}
$$

This proves (2.1) since (2.5) is equivalent to it. Note also that Young's inequality implies that $u = f * G_s$ is uniformly continuous on $\mathbb{R}^d$.

We will briefly describe an argument for $p > 2$ on $\mathbb{T}^d$ at the end of the next subsection.

2.2. The Sobolev inequality. Let $s > 0$ and $1 < p < q < \infty$ satisfy (1.5). Sobolev's inequality on $\mathbb{R}^d$ states that for all such s, p, q we have

$$
\|u\|_{L^q(\mathbb{R}^d)} \lesssim \|u\|_{\dot{L}^p_s(\mathbb{R}^d)}. \tag{2.6}
$$

This is equivalent to the following Hardy-Littlewood-Sobolev inequality:

$$\|I_s(f)\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}, \tag{2.7}$$

where $s > 0$ and $1 < p < q < \infty$ satisfy (1.5), and $I^s = (-\Delta)^{-\frac{s}{2}}$ denotes the Riesz potential of order s.

Using [6, Theorem 2.4.6], we have

$$(|\xi|^z)^\vee = \pi^{-\frac{2z+d}{2}} \frac{\Gamma(\frac{d+z}{2})}{\Gamma(\frac{-z}{2})} |x|^{-z-d}, \tag{2.8}$$

where the equality holds in the sense of distributions (indeed, when $\operatorname{Re} z < 0$, the expression in (2.8) is in $L^1_{\text{loc}}(\mathbb{R}^d)$ and can be made sense as a function). Now, (2.8) allows us to write

$$I_s(f)(x) = 2^{-s}\pi^{-\frac{d}{2}} \frac{\Gamma(\frac{d-s}{2})}{\Gamma(\frac{s}{2})} \int_{\mathbb{R}^d} f(x-y)|y|^{-d+s} dy. \tag{2.9}$$

See Subsection 6.1.1 in [7]. Then, one can prove (2.7) by an argument on the physical side, using (2.9); see [7, Theorem 6.1.3], and also the proof of Proposition 1.1 below. We note in passing that, having established (2.9), one may view (2.7) as the "endpoint" case of Young's inequality. Indeed, by a simple application of Young's inequality, one would obtain

$$\|I_s(f)\|_{L^q(\mathbb{R}^d)} \lesssim \big\||x|^{-d+s}\big\|_{L^{\frac{d}{d-s}}(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)},$$

where the first factor on the right-hand side is infinite since $|x|^{-d+s}$ barely misses to be in $L^{\frac{d}{d-s}}(\mathbb{R}^d)$.

We arrive at last to the Sobolev inequality for periodic functions on $\mathbb{T}^d$ stated in Proposition 1.1, which we prove next. As before, (1.6) is equivalent to the following Hardy-Littlewood-Sobolev inequality on $\mathbb{T}^d$:

$$\|I_s(f)\|_{L^q(\mathbb{T}^d)} \lesssim \|f\|_{L^p(\mathbb{T}^d)}, \tag{2.10}$$

where f has mean zero.

The proof of (2.10) follows along the same lines as the proof of (2.7) on $\mathbb{R}^d$ (c.f. [7, Theorem 6.1.3]) once we obtain a formula analogous to (2.8) relating $|n|^{-s}$ and $|x|^{-d+s}$ for $n \in \mathbb{Z}^d$ and $x \in \mathbb{T}^d$. Indeed, our argument has a simple structure. First, we see that the operator I_s is realizable as a convolution with the distribution on $\mathbb{T}^d$ whose Fourier series is given by $\sum_{n\in\mathbb{Z}^d\setminus\{0\}}|n|^{-s}e^{2\pi in\cdot x}$. Then, a refinement of the Poisson summation formula, based on some smooth cut-off techniques, gives that, modulo an additive smooth function on $\mathbb{T}^d$, this distribution behaves like a constant multiple of $|x|^{-d+s}$. While this is a known fact, see Stein and Weiss [13, Theorem 2.17, p. 256], for the convenience of the reader we err on the side of a longer and different argument than in [13] by providing full details of the careful analysis required here. Finally, we notice that the (L^p, L^q) mapping property of the convolution operator I_s can be reduced to that of the convolution operator with $|x|^{-d+s}$ which roughly follows as in the d-dimensional Euclidean case once we do a correct splitting of the d-dimensional torus.

2.3. Proof of Proposition 1.1. We start by recalling the Poisson summation formula.

Lemma 2.1 (Theorem 3.1.17 in [6]). *Suppose that $f, \hat{f} \in L^1(\mathbb{R}^d)$ satisfy*

$$|f(x)| + |\hat{f}(x)| \le C(1+|x|)^{-d-\delta}$$

for some $C, \delta > 0$. Then, f and $\hat{f}$ are continuous and, for all $x \in \mathbb{R}^d$, we have

$$\sum_{n\in\mathbb{Z}^d}\hat{f}(n)e^{2\pi in\cdot x} = \sum_{n\in\mathbb{Z}^d} f(x+n). \tag{2.11}$$

Let now η be a smooth function on $\mathbb{R}^d$ such that $\eta(\xi) = 1$ for $|\xi| \ge \frac{1}{2}$ and $\eta(\xi) = 0$ for $|\xi| \le \frac{1}{4}$. For $0 < \operatorname{Re} s < d$, define $g(x) = \big(\eta(\xi)|\xi|^{-s}\big)^{\wedge}(x)$. Then, it is known (see [6, Example 2.4.9]) that g decays faster than the reciprocal of any polynomial at infinity. Let

$$h(x) = g(x) - G(x), \quad \text{where} \quad G(x) = \pi^{s-\frac{d}{2}}\frac{\Gamma(\frac{d-s}{2})}{\Gamma(\frac{s}{2})}|x|^{s-d}. \tag{2.12}$$

Then, $h \in C^\infty(\mathbb{R}^d)$. We would like to apply now Lemma 2.1 to g and $\hat{g} = \eta(\xi)|\xi|^{-s}$. However, the decay of $\hat{g}$ at infinity is not fast enough (since (1.5) implies $s < d$) and we have $\hat{g} \notin L^1(\mathbb{R}^d)$. Hence, Lemma 2.1 is not applicable.

Fix $\phi \in \mathcal{S}(\mathbb{R}^d)$ supported on $[-\frac{1}{2}, \frac{1}{2})^d$ such that $\int_{\mathbb{R}^d}\phi(x)dx = 1$, and let $\phi_\varepsilon(x) = \varepsilon^{-d}\phi(\varepsilon^{-1}x)$, $\varepsilon > 0$. The family $\{\phi_\varepsilon\}$ is an approximation of the identity. If we now let $g_\varepsilon = g * \phi_\varepsilon$, then $\hat{g}_\varepsilon(\xi) = \hat{g}(\xi)\hat{\phi}_\varepsilon(\xi) = \hat{\phi}_\varepsilon(\xi)\eta(\xi)|\xi|^{-s}$ satisfies the desired decay $|\hat{g}_\varepsilon(\xi)| \le C(1+|\xi|)^{-d-\delta}$ for some $\delta > 0$. Clearly, $|g_\varepsilon(x)| \le C(1+|x|)^{-d-\delta}$ near infinity thanks to the rapid decay of g at infinity. Also, g_ε is bounded near the origin since $|x|^{s-d}$ is integrable near the origin (and thus, g_ε is a C^∞ function.)

Let $x \in [-\frac{1}{2}, \frac{1}{2})^d$. By Lemma 2.1 we have

$$\sum_{n \in \mathbb{Z}^d \setminus \{0\}} \frac{\hat{\phi}_\varepsilon(n) e^{2\pi i n \cdot x}}{|n|^s} = \sum_{n \in \mathbb{Z}^d} \frac{\hat{\phi}_\varepsilon(n) \eta(n) e^{2\pi i n \cdot x}}{|n|^s}$$
$$= \sum_{\max |n_j| \leq 1} g_\varepsilon(x+n) + \sum_{\substack{n \in \mathbb{Z}^d \\ \max |n_j| \geq 2}} g_\varepsilon(x+n). \tag{2.13}$$

Note that, for $x, y \in [-\frac{1}{2}, \frac{1}{2})^d$ and $n \in \mathbb{Z}^d$, we have

$$\begin{aligned} &|x - y + n| \geq 1, && \text{if} \quad \max |n_j| \geq 2, \\ &|x - y + n| \leq 3\sqrt{d}, && \text{if} \quad \max |n_j| \leq 1. \end{aligned}$$

Let $r \geq 1$. Since $g(x)$ is a smooth rapidly decreasing function on $|x| \geq 1$, we have

$$\begin{aligned} &\Big\| \sum_{\substack{n \in \mathbb{Z}^d \\ \max |n_j| \geq 2}} g_\varepsilon(x+n) \Big\|_{L^r([-\frac{1}{2}, \frac{1}{2})^d)} \\ &\qquad = \Big\| \sum_{\substack{n \in \mathbb{Z}^d \\ \max |n_j| \geq 2}} \int_{[-\frac{1}{2}, \frac{1}{2})^d} g(x - y + n) \phi_\varepsilon(y) dy \Big\|_{L^r([-\frac{1}{2}, \frac{1}{2})^d)} \\ &\qquad \leq \|\phi_\varepsilon\|_{L^1([-\frac{1}{2}, \frac{1}{2})^d)} \|g(x)\|_{L^r(|x| \geq 1)} = \|g(x)\|_{L^r(|x| \geq 1)} < \infty. \end{aligned}$$

Also, since h in (2.12) is a smooth function, we have

$$\begin{aligned} &\Big\| \sum_{\max |n_j| \leq 1} h * \phi_\varepsilon(x+n) \Big\|_{L^r([-\frac{1}{2}, \frac{1}{2})^d)} \\ &\qquad = \Big\| \sum_{\max |n_j| \leq 1} \int_{[-\frac{1}{2}, \frac{1}{2})^d} h(x - y + n) \phi_\varepsilon(y) dy \Big\|_{L^r([-\frac{1}{2}, \frac{1}{2})^d)} \\ &\qquad \leq \|\phi_\varepsilon\|_{L^1([-\frac{1}{2}, \frac{1}{2})^d)} \|h(x)\|_{L^r(|x| \leq 3\sqrt{d})} = \|h(x)\|_{L^r(|x| \leq 3\sqrt{d})} < \infty. \end{aligned}$$

Motivated by these two estimates, we let

$$H_\varepsilon(x) = \sum_{\substack{n \in \mathbb{Z}^d \\ \max |n_j| \geq 2}} g_\varepsilon(x+n) + \sum_{\max |n_j| \leq 1} h * \phi_\varepsilon(x+n).$$

Then, H_ε is smooth on $[-\frac{1}{2}, \frac{1}{2})^d$ and

$$\|H_\varepsilon\|_{L^r([-\frac{1}{2},\frac{1}{2})^d)} \leq C < \infty, \tag{2.14}$$

where the constant C is independent of $\varepsilon > 0$.

Moreover, from (2.13), we have

$$\sum_{n\in\mathbb{Z}^d\setminus\{0\}} \frac{\hat{\phi}_\varepsilon(n)e^{2\pi i n\cdot x}}{|n|^s} = \sum_{\max|n_j|\leq 1} G*\phi_\varepsilon(x+n) + H_\varepsilon(x), \tag{2.15}$$

for $x \in [-\frac{1}{2}, \frac{1}{2})^d$, where H_ε is smooth, satisfying (2.14).

We are now ready to prove (2.10). Let $\varepsilon > 0$. We will first prove

$$\|\phi_\varepsilon * I_s(f)\|_{L^q(\mathbb{T}^d)} \lesssim \|f\|_{L^p(\mathbb{T}^d)} \tag{2.16}$$

for smooth f with mean zero on $\mathbb{T}^d$, where the implicit constant is independent of $\varepsilon > 0$. In the following, we often view f as a periodic function defined on $\mathbb{R}^d$. By (2.15), we have

$$\begin{aligned}
\phi_\varepsilon * I_s(f)(x) &= (2\pi)^{-s} \sum_{n\in\mathbb{Z}^d\setminus\{0\}} \hat{f}(n)\hat{\phi}_\varepsilon(n)|n|^{-s}e^{2\pi i n\cdot x}\\
&= (2\pi)^{-s}\int_{\mathbb{T}^d} f(y) \sum_{n\in\mathbb{Z}^d\setminus\{0\}} \hat{\phi}_\varepsilon(n)|n|^{-s}e^{2\pi i n\cdot(x-y)}dy\\
&\sim \sum_{\max|n_j|\leq 1}\int_{\mathbb{T}^d} f(y)\big(G*\phi_\varepsilon\big)(x-y+n)dy + \int_{\mathbb{T}^d} f(y)H_\varepsilon(x-y)dy\\
&=: \mathrm{I}(x) + \mathrm{II}(x)
\end{aligned} \tag{2.17}$$

for $x \in [-\frac{1}{2}, \frac{1}{2})^d$. Here, for fixed $x \in [-\frac{1}{2}, \frac{1}{2})^d$, y ranges over $x + (-\frac{1}{2}, \frac{1}{2}]^d$ such that $x - y \in [-\frac{1}{2}, \frac{1}{2})^d$. By Young's inequality with $\frac{1}{r} = 1 + \frac{1}{q} - \frac{1}{p}$, we have

$$\|\mathrm{II}\|_{L^q(\mathbb{T}^d)} \leq \|H_\varepsilon\|_{L^r(\mathbb{T}^d)}\|f\|_{L^p(\mathbb{T}^d)} \lesssim \|f\|_{L^p(\mathbb{T}^d)}, \tag{2.18}$$

where the implicit constant is independent of $\varepsilon > 0$ thanks to (2.14).

Next, we estimate I. First, note that for $x - y \in [-\frac{1}{2}, \frac{1}{2})^d$, $\max|n_j| \leq 1$, and $|z| > 2\sqrt{d}$, we have $x - y + n - z \notin [-\frac{1}{2}, \frac{1}{2})^d$. Then, recalling (2.12) and changing the order of integration, we have

$$\begin{aligned}
|\mathrm{I}(x)| &\lesssim \bigg|\sum_{\max|n_j|\leq 1}\int_{x+(-\frac{1}{2},\frac{1}{2}]^d} f(y)\int_{|z|\leq 2\sqrt{d}} |z|^{s-d}\phi_\varepsilon(x-y+n-z)dzdy\bigg|\\
&\lesssim \int_{|z|\leq 2\sqrt{d}} |z|^{s-d}\sum_{\max|n_j|\leq 1}\int_{x+(-\frac{1}{2},\frac{1}{2}]^d} |f(y)|\phi_\varepsilon(x-y-z+n)dydz\\
&\lesssim \int_{|z|\leq 2\sqrt{d}} |z|^{s-d}F_\varepsilon(x-z)dz,
\end{aligned}$$

where F_ε is defined by

$$F_\varepsilon(z) = \sum_{\max|n_j|\leq 1} \int_{\mathbb{R}^d} |f(y)|\phi_\varepsilon(z-y+n)dy. \tag{2.19}$$

Here, we are viewing f as a periodic function defined on $\mathbb{R}^d$. Although the domain of integration in (2.19) is $\mathbb{R}^d$, the actual integration is over a bounded domain since ϕ_ε is supported on $[-\frac{1}{2}, \frac{1}{2})^d$. Making a change of variables in (2.19) and using the periodicity of f, we have

$$\begin{aligned} F_\varepsilon(z) = \sum_{\max|n_j|\leq 1} \int_{\mathbb{R}^d} |f(y+n)|\phi_\varepsilon(z-y)dy = c\int_{z-y\in[-\frac{1}{2},\frac{1}{2})^d} |f(y)|\phi_\varepsilon(z-y)dy \\ = c\int_{[-\frac{1}{2},\frac{1}{2})^d} |f(z-y)|\phi_\varepsilon(y)dy. \end{aligned}$$

As such, the expression defining $F_\varepsilon(z)$ is now periodic in z. Going now back to the estimate on $|\mathrm{I}(x)|$, we divide it into two parts:

$$\begin{aligned} |\mathrm{I}(x)| &\lesssim \int_{|z|\leq R} |z|^{s-d} F_\varepsilon(x-z)dz + \int_{R<|z|\leq 2\sqrt{d}} |z|^{s-d} F_\varepsilon(x-z)dz \\ &=: J_1(F_\varepsilon)(x) + J_2(F_\varepsilon)(x), \end{aligned}$$

where $R = R(x) > 0$ is to be chosen later.

Note that J_1 is given by a convolution with $|z|^{s-d}\chi_{|z|\leq R}(z)$, which is radial, integrable, and symmetrically decreasing about the origin. Hence, it can be bounded from above by the uncentered Hardy-Littlewood maximal function $M(F_\varepsilon)$ (see [6, Theorem 2.1.10]):

$$J_1(F_\varepsilon)(x) \leq M(F_\varepsilon)(x)\bigg(\int_{|z|\leq R} |z|^{s-d}dz\bigg) = cR^s M(F_\varepsilon)(x). \tag{2.20}$$

We recall that $M(f)(x)$ is defined as the supremum of the averages of $|f|$ over all balls $B(y,\delta) := \{z \in \mathbb{R}^d : |z-y| < \delta\}$ that contain the point x, that is

$$M(f)(x) = c_d \sup_{\substack{\delta>0 \\ |x-y|<\delta}} \frac{1}{\delta^d} \int_{|z-y|<\delta} |f(z)|\,dz.$$

By Hölder's inequality followed by Minkowski's integral inequality, we have

$$\begin{aligned} J_2(F_\varepsilon)(x) &\leq \bigg(\int_{|z|>R} |z|^{(s-d)p'}dz\bigg)^{\frac{1}{p'}} \bigg(\int_{|z|\leq 2\sqrt{d}} \big(F_\varepsilon(x-z)\big)^p dz\bigg)^{\frac{1}{p}} \\ &\lesssim R^{-\frac{d}{q}} \|f\|_{L^p(\mathbb{T}^d)} \end{aligned} \tag{2.21}$$

for $x \in [-\frac{1}{2}, \frac{1}{2})^d$. Note also that here we use the condition (1.5) on the indices s, p, q, in particular the fact that $(s-d)p' = -d - \frac{dp'}{q}$. From (2.20) and (2.21), we obtain

$$|\,\mathrm{I}\,(x)| \lesssim R^s M(F_\varepsilon)(x) + R^{-\frac{d}{q}} \|f\|_{L^p(\mathbb{T}^d)}. \tag{2.22}$$

Choose now $R = R(x) > 0$ that minimizes (2.22). With

$$R = c\|f\|_{L^p(\mathbb{T}^d)}^{\frac{p}{d}} \big(M(F_\varepsilon)(x)\big)^{-\frac{p}{d}}, \tag{2.23}$$

we have[2]

$$|\,\mathrm{I}\,(x)| \lesssim \big(M(F_\varepsilon)(x)\big)^{\frac{p}{q}} \|f\|_{L^p(\mathbb{T}^d)}^{1-\frac{p}{q}}. \tag{2.24}$$

By taking the L^q-norm of both sides and then using the boundedness of the Hardy-Littlewood maximal operator $M(F_\epsilon)$ on $L^p(\mathbb{T}^d)$ (see [6, Theorem 2.1.6]), we obtain $\|I\|_{L^q(\mathbb{T}^d)} \lesssim \|f\|_{L^p(\mathbb{T}^d)}$. Combining this with (2.18), we get (2.16). Finally, (2.10) follows from (2.16) by taking $\varepsilon \to 0$. This completes the proof of Proposition 1.1.

2.4. Proof of (2.1) **on** $\mathbb{T}^d$**.** In the remainder of this section, we briefly describe the proof of (2.1) for $p > 2$ in the periodic setting. As pointed out at the beginning of Subsection 2.1, the proof of (2.1) for $p \le 2$ is identical to the one on $\mathbb{R}^d$.

If $s > \frac{d}{2}$ and $p \ge 2$, then from (2.1) (for $p = 2$) and $L^p(\mathbb{T}^d) \subset L^2(\mathbb{T}^d)$, we have

$$\|u\|_{L^\infty(\mathbb{T}^d)} \lesssim \|u\|_{H^s(\mathbb{T}^d)} \le \|u\|_{L^p_s(\mathbb{T}^d)}.$$

Now, consider the case $s \le \frac{d}{2}$ and $p > 2$. With G_s as in (2.2), it follows from (2.3) and (2.4) that

$$G_s(x) \lesssim |x|^{s-d}, \quad \text{for } |x| \le 2\sqrt{d}. \tag{2.25}$$

As in the proof of Proposition 1.1, the main point is to transfer the relation $\hat{G}_s(\xi) = \langle \xi \rangle^{-s}$ to the periodic domain $\mathbb{T}^d$. This is done by Poisson's summation formula, Lemma 2.1. However, the decay of $\langle \xi \rangle^{-s}$ at infinity is not fast enough, i.e. $\langle \xi \rangle^{-s} \notin L^1(\mathbb{R}^d)$, and thus, Lemma 2.1 is not directly applicable, Hence, we need to go through a similar modification as before. We omit this part of the argument. Once we do that, the main objective is to estimate the expression I in (2.17) with G_s in place of G:

$$|\,\mathrm{I}\,(x)| \le \int_{|z| \le 2\sqrt{d}} G_s(z) F_\varepsilon(x-z)dz.$$

[2]If $R \ge 2\sqrt{d}$ in (2.23), then there is no contribution from $J_2(F_\varepsilon)(x)$. In this case, (2.22) becomes $|\,\mathrm{I}\,(x)| \lesssim R^s M(F_\varepsilon)(x)$. Since $s > 0$, setting $R = 2\sqrt{d}$ yields (2.24).

Since $sp > d$, we have $(s-d)p' + d = p'(s - \frac{d}{p}) > 0$. Thus, using (2.25), Hölder's inequality and Minkowski's integral inequality, we get

$$\begin{aligned} |\,\mathrm{I}\,(x)| &\lesssim \bigg(\int_{|z|\le 2\sqrt{d}} |z|^{(s-d)p'} dz \bigg)^{\frac{1}{p'}} \bigg(\int_{|z|\le 2\sqrt{d}} \big(F_\varepsilon(x-z)\big)^p dz \bigg)^{\frac{1}{p}} \\ &\lesssim \|f\|_{L^p(\mathbb{T}^d)} \quad \text{for} \quad x \in [-\tfrac{1}{2}, \tfrac{1}{2})^d. \end{aligned}$$

Hence, we obtain $\|\,\mathrm{I}\,\|_{L^\infty(\mathbb{T}^d)} \lesssim \|f\|_{L^p(\mathbb{T}^d)}$. As pointed out in the proof of Proposition 1.1, the estimate $\|\mathrm{II}\|_{L^\infty(\mathbb{T}^d)} \lesssim \|f\|_{L^p(\mathbb{T}^d)}$ is rather straightforward. Combining these two estimates, we obtain (2.1).

Acknowledgments. The authors would like to thank the anonymous referee for thoughtful comments. The first named author is also grateful to Diego Maldonado, Virginia Naibo, and Pietro Poggi-Corradini for their hospitality during his Fall 2011 visit at Kansas State University and for useful discussions and suggestions that improved the exposition of this note. Á. Bényi is partially supported by Simons Foundation Grant No. 246024. T. Oh acknowledges support from an AMS-Simons Travel Grant.

References

[1] T. Aubin, Nonlinear analysis on manifolds. Monge-Ampère equations, Vol. 252, Grundleren der Mathematischen Wisserchaften, *Springer-Verlag*, *Berlin-New York*, 1982.

[2] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II. The KdV-equation, *Geom. Funct. Anal.* **3**, no. 3 (1993), 209–262.

[3] J. Bergh, J. Löfström, Interpolation spaces: an introduction, *Springer-Verlag*, *Berlin-New York*, 1976.

[4] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, Sharp global well-posedness for KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$, *J. Amer. Math. Soc.* **16**, no. 3 (2003), 705–749.

[5] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, Multilinear estimates for periodic KdV equations, and applications, *J. Funct. Anal.* **211**, no. 1 (2004), 173–218.

[6] L. Grafakos, Classical Fourier analysis, Vol. 249, Second edition. Graduate Texts in Mathematics, *Springer*, *New York*, 2008.

[7] L. Grafakos, Modern Fourier analysis, Vol. 250, Second edition. Graduate Texts in Mathematics, *Springer*, *New York*, 2009.

[8] E. Hebey, Nonlinear analysis on manifolds: Sobolev spaces and inequalities, *American Mathematical Society*, 1999.

[9] L. Hörmander, The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis, *Springer-Verlag*, *Berlin*, 2003.

[10] T. Kato, On the Korteweg-de Vries equation, *Manuscripta Math.* **28** (1979), 88–99.

[11] T. Kato, Quasi-linear equations of evolution, with application to partial differential equations, Vol. 448, Lecture notes in Math., *Springer-Verlag*, 1975, 25–70.

[12] E. Stein, Singular integrals and differentiability properties of functions, Vol. 30, Princeton Mathematical Series, *Princeton University Press*, *Princeton, N.J.*, 1970.

[13] E. Stein, G. Weiss, Introduction to Fourier analysis on Euclidean spaces, Vol. 32, Princeton Mathematical Series, *Princeton University Press*, *Princeton, N.J.*, 1972.

[14] R. Strichartz, Improved Sobolev inequalities, *Trans. Amer. Math. Soc.* **279**, no. 1 (1983), 397–409.

[15] L. Tartar, An introduction to Sobolev spaces and interpolation spaces, Vol. 3, Lecture Notes of the Unione Matematica Italiana, *Springer Berlin; UMI, Bologna*, 2007.

DEPARTMENT OF MATHEMATICS
WESTERN WASHINGTON UNIVERSITY
516 HIGH STREET
BELLINGHAM, WA 98226, USA

E-mail: arpad.benyi@wwu.edu

SCHOOL OF MATHEMATICS
THE UNIVERSITY OF EDINBURGH
JAMES CLERK MAXWELL BUILDING
THE KINGS BUILDINGS
MAYFIELD ROAD
EDINBURGH, EH9 3JZ, SCOTLAND, AND
DEPARTMENT OF MATHEMATICS
PRINCETON UNIVERSITY
FINE HALL, WASHINGTON RD
PRINCETON, NJ 08544-1000, USA

E-mail: hirooh@math.princeton.edu