Torus invariant primes in the quantum grassmannian

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• Joint work at various times with: Karel Casteels, Ann Kelly, Stéphane Launois, Laurent Rigal, Ewan Russell

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The quantum grassmannian $\mathcal{O}_q(G(k,n))$

- The quantum grassmannian $\mathcal{O}_q(G(k,n))$ is the subalgebra of $\mathcal{O}_q(\mathcal{M}(k,n))$ generated by the maximal $k \times k$ quantum minors.

- Denote by $[I]$ the quantum minor $[1 \ldots k|I]$.

- There is a torus action of $\mathcal{H} = \mathbb{C}^n$ given by column multiplication.

Example $\mathcal{O}_q(G(2,4))$ is generated by the six quantum minors

$$[12], [13], [14], [23], [24], [34]$$

Most quantum minors $q^{\bullet}$-commute, for example,

$$[12]\,[13] = q\,[13]\,[12]\,, \quad [12]\,[34] = q^2\,[34]\,[12]$$

However,

$$[13]\,[24] = [24]\,[13] + (q - q^{-1})\,[14]\,[23]$$

and there is a quantum Plücker relation

$$[12]\,[34] - q\,[13]\,[24] + q^2\,[14]\,[23] = 0.$$

Problem: Describe $\mathcal{H}-$Spec($\mathcal{O}_q(G(k,n))$)

Snag: Goodearl-Letzter theory can't be used directly since $\mathcal{O}_q(G(k,n))$ is not usually an iterated Öre extension (or a factor of one)

Nevertheless, one might hope to prove:

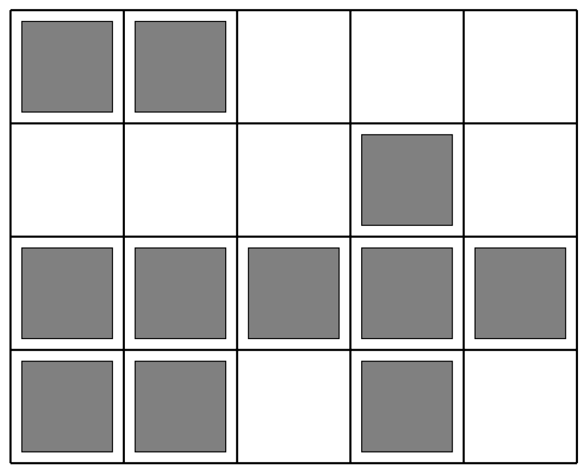
- There are only finitely many $\mathcal{H}$-primes
- All $\mathcal{H}$-primes are completely prime
- We can specify the quantum minors in a given $\mathcal{H}$-prime
- Each $\mathcal{H}$-prime is generated by the quantum minors that it contains
- Describe the containments between $\mathcal{H}$-primes

Launois, Lenagan and Rigal There is a bijection between $\mathcal{H}$−Spec($\mathcal{O}_q(G(k,n))$) (ignoring the irrelevant ideal) and Cauchon diagrams on Young diagrams that fit inside a $k \times (n-k)$ array

The theorem is proved by defining quantum algebras with a straightening law, quantum Schubert varieties, quantum Schubert cells, partition subalgebras of quantum matrices and using a non-commutative version of dehomogenisation.

Cauchon diagrams

A Young diagram with entries coloured black or white is said to be a **Cauchon diagram** if it satisfies the following rule: if there is a black in a given square then either each square to the left is also coloured black or each square above is also coloured black



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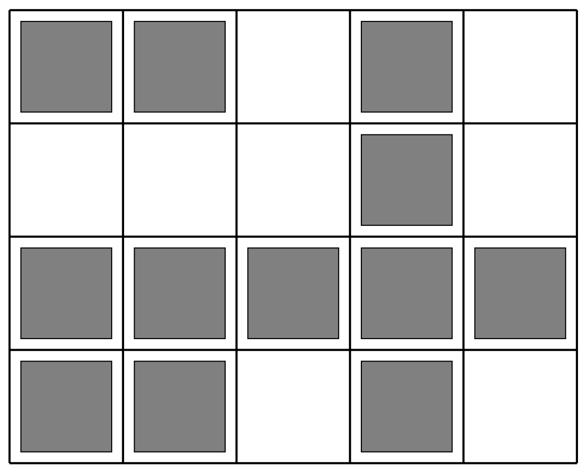

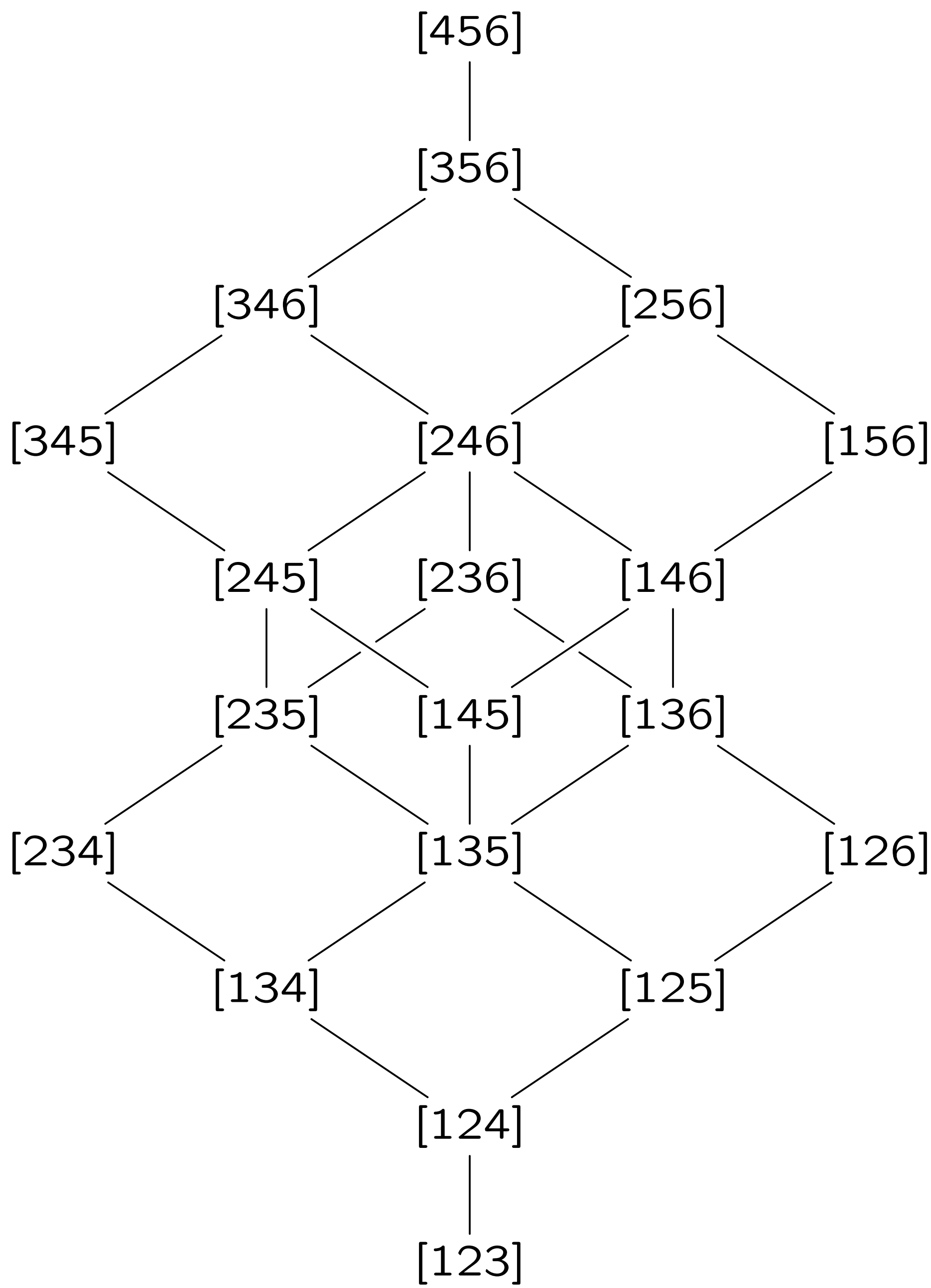
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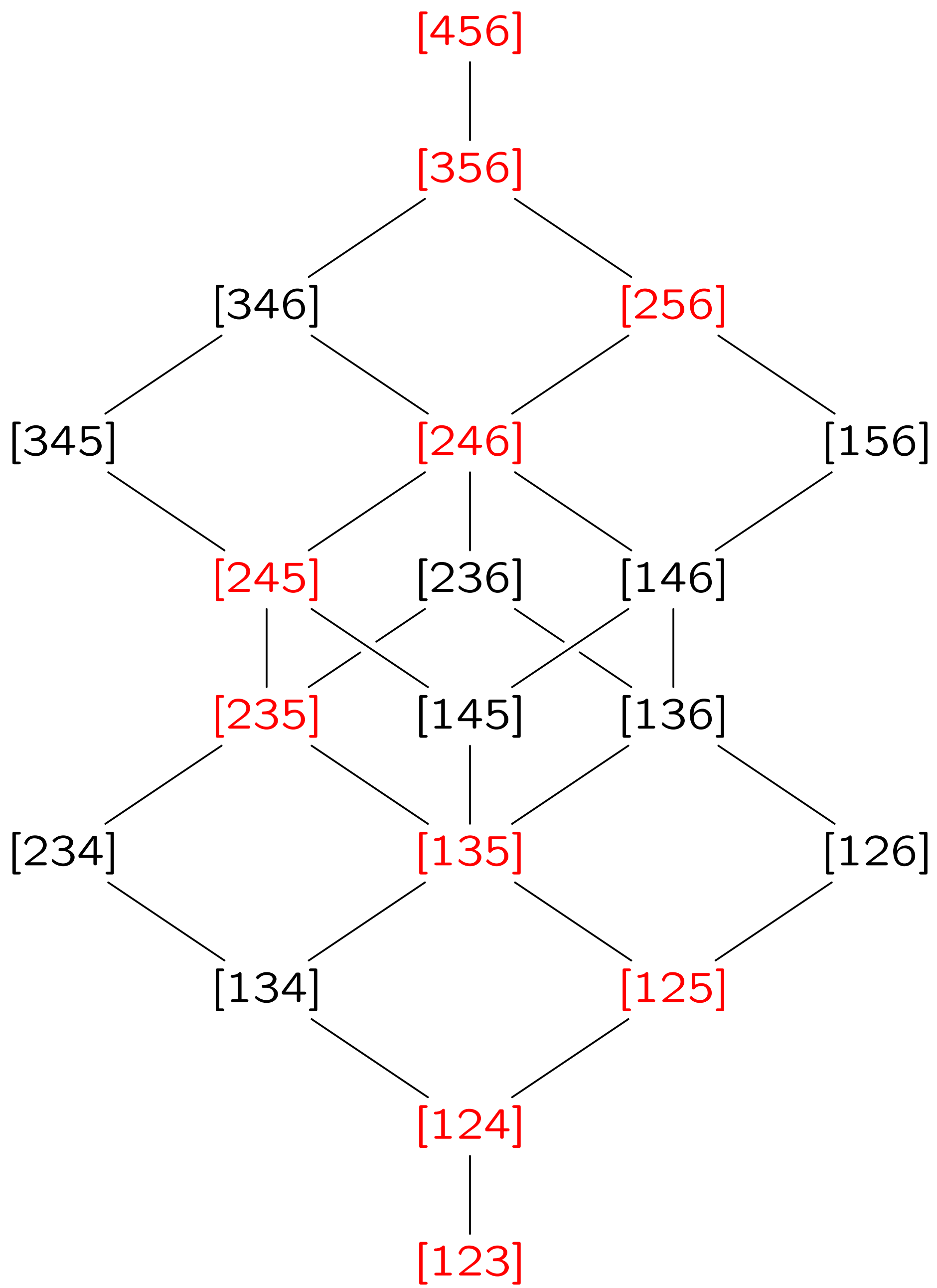
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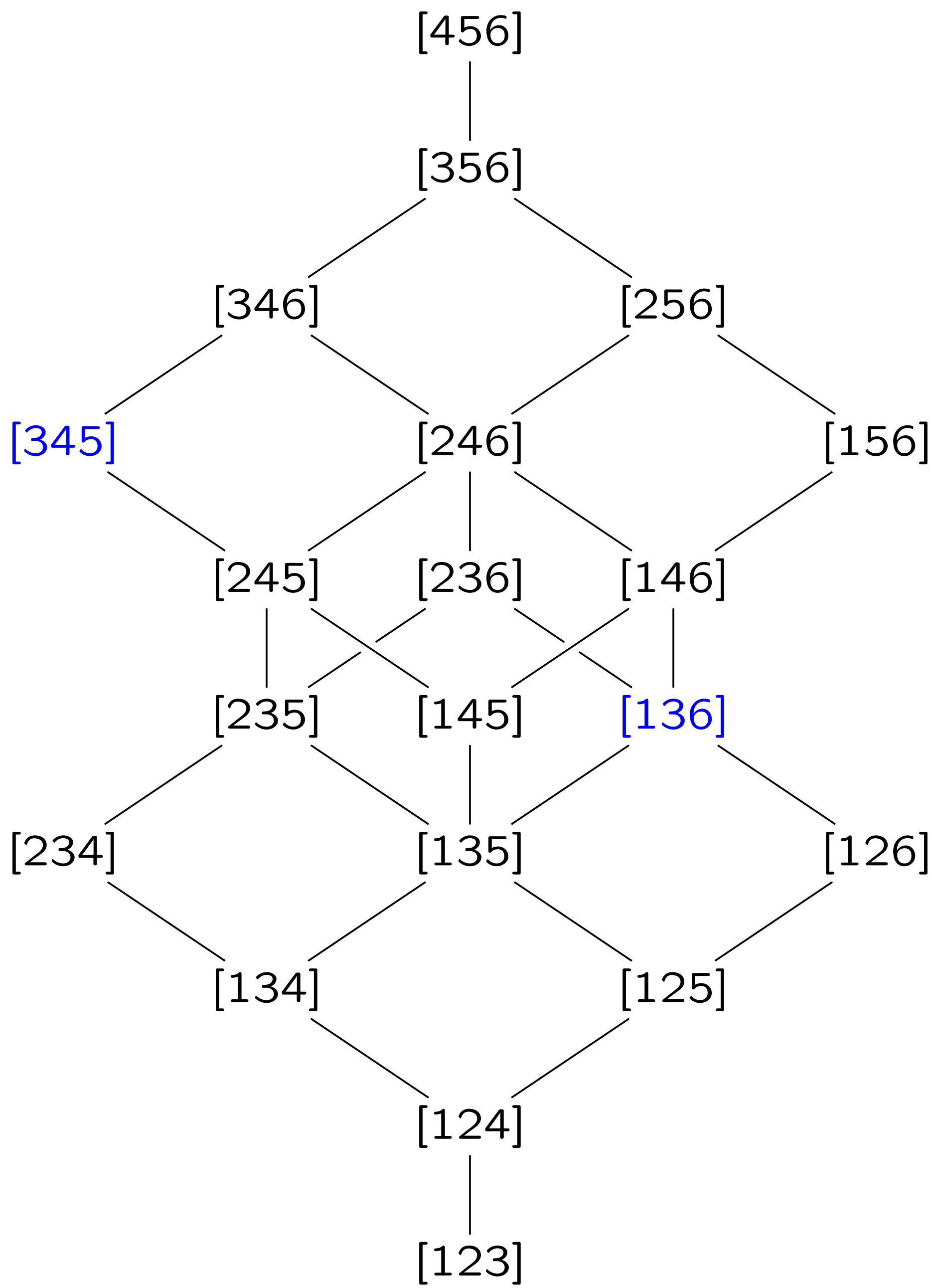
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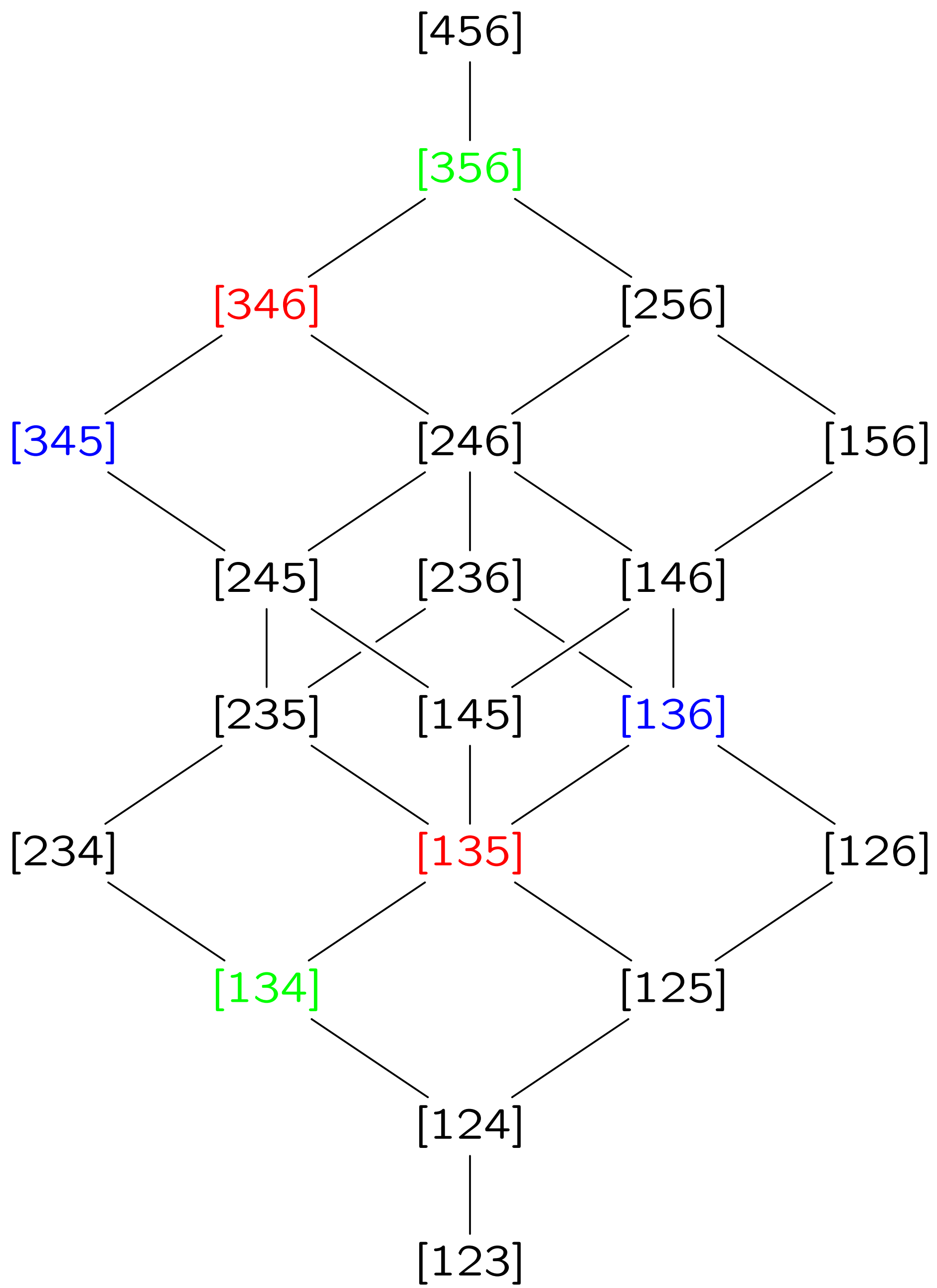
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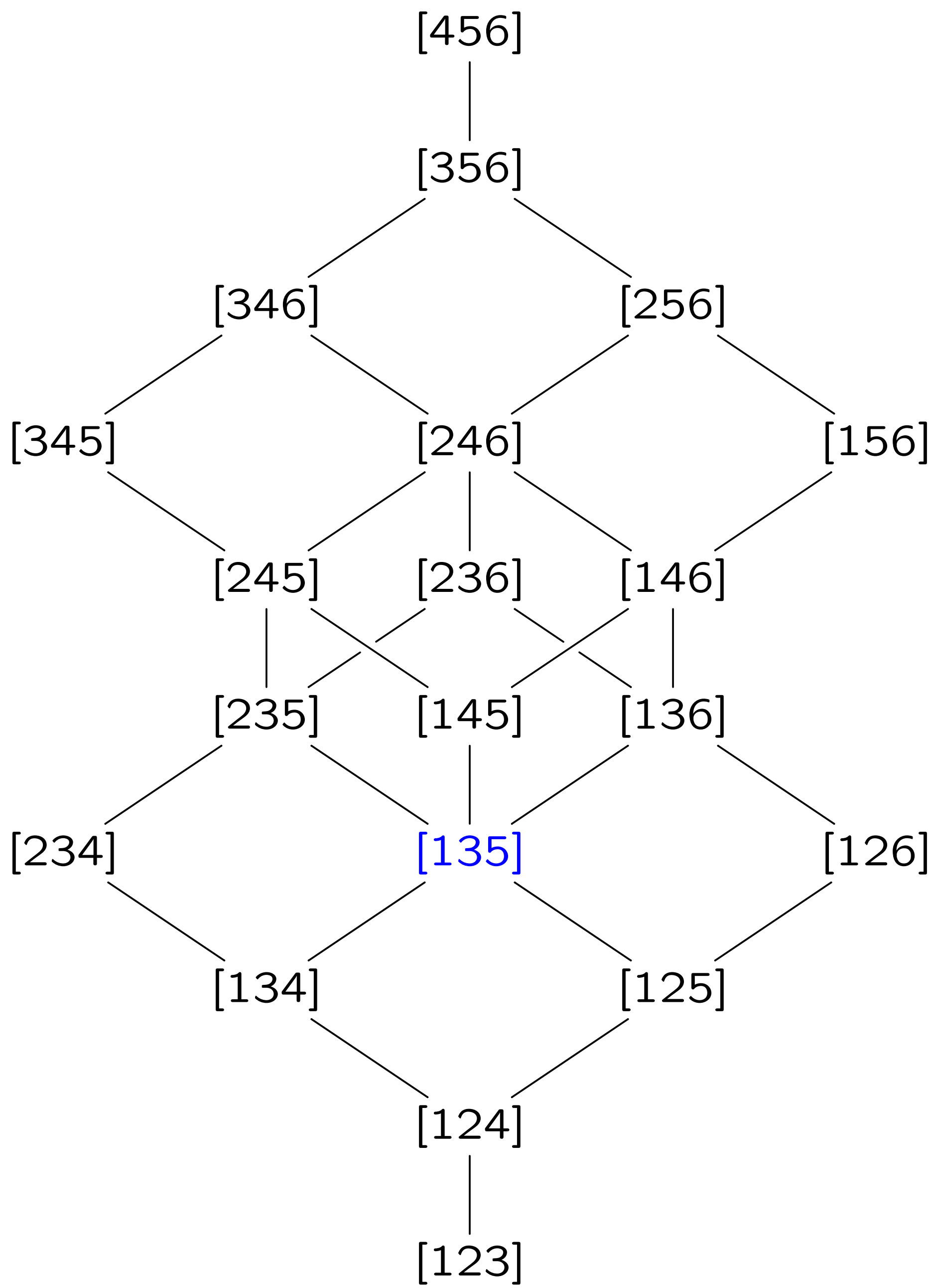
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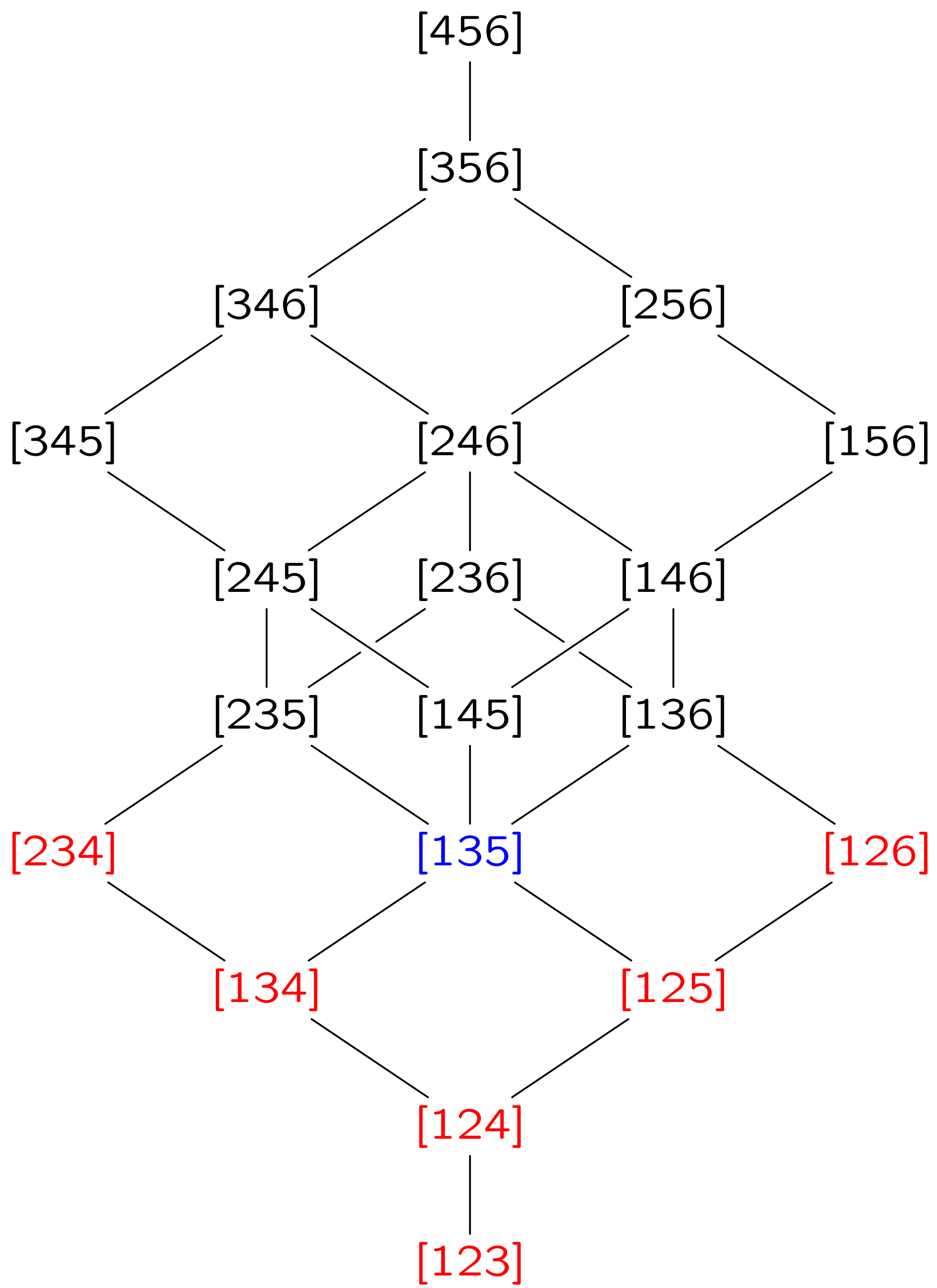
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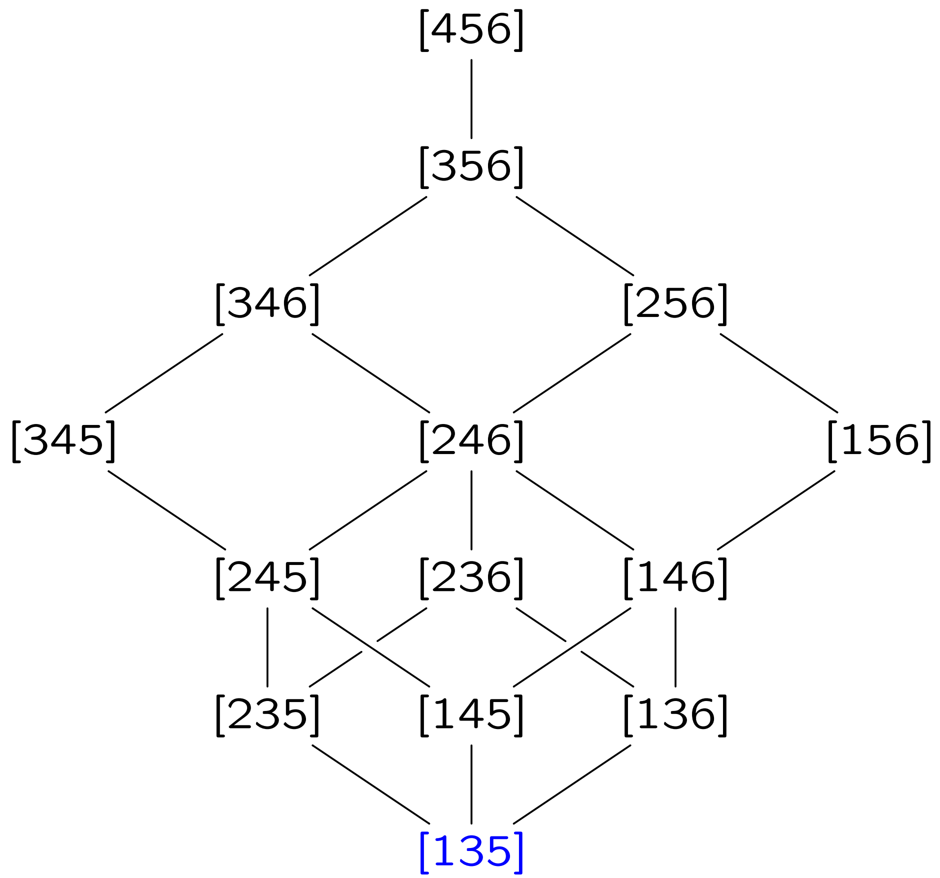
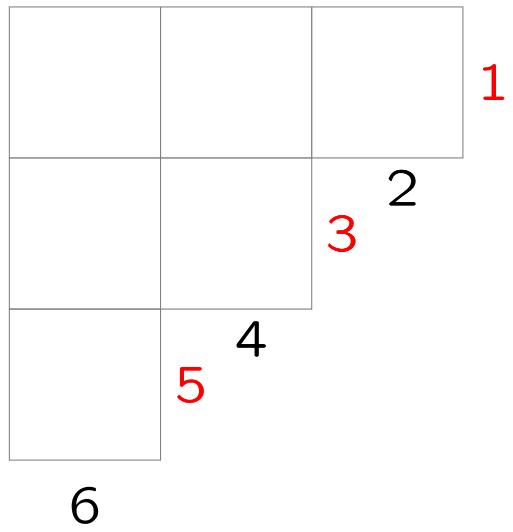
Quantum Schubert variety corresp to [135]

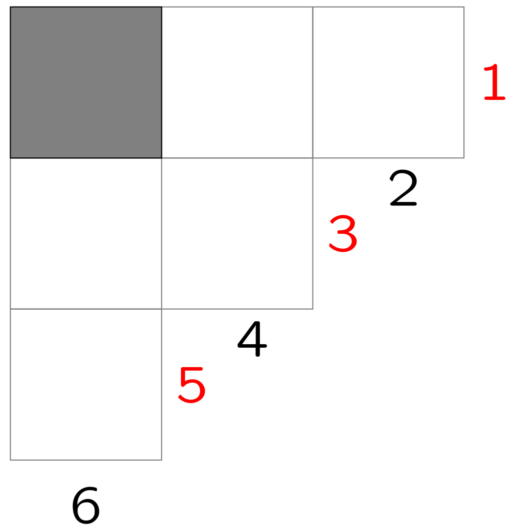


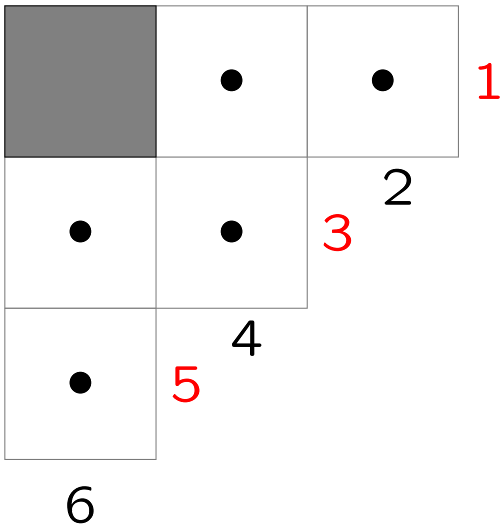
Schubert cell: use noncommutative dehomogenisation at [135]

Schubert cell for [135]



$\mathcal{H}$**-prime in Schubert cell** [135]

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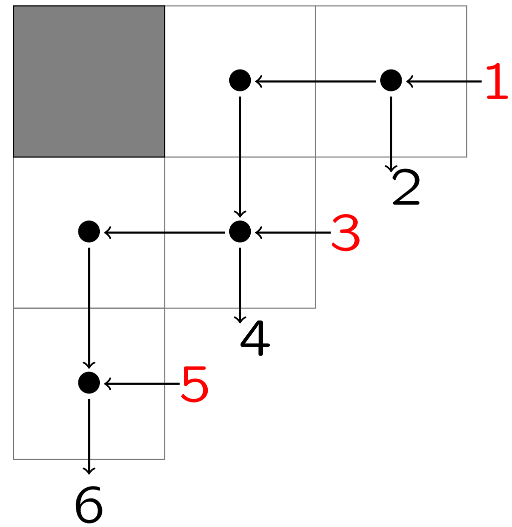
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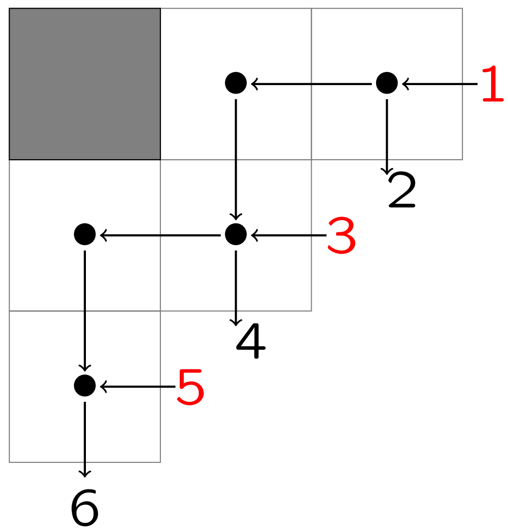
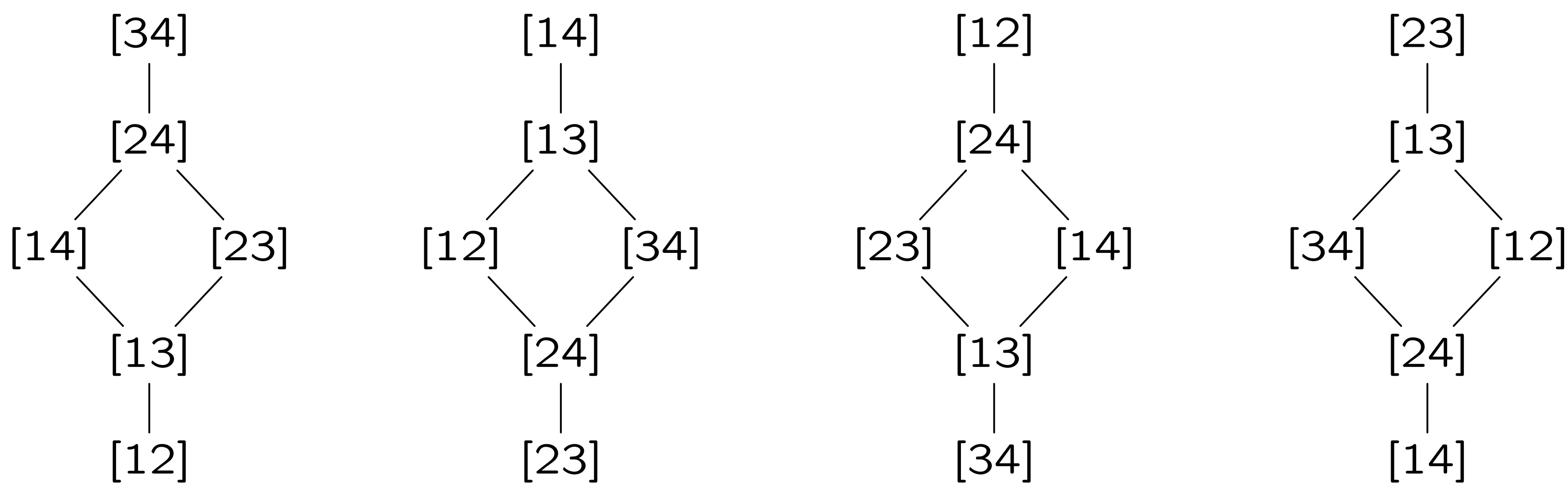


There is a vertex disjoint set of paths from $\{1, 3\}$ to $\{2, 4\}$ so [245] is not in the prime.

There is no vertex disjoint set of paths from $\{1, 3\}$ to $\{4, 6\}$ so [456] is in the prime.

The i-**order**: $i \leq_i i+1 \leq_i \ldots \leq_i n \leq_i 1 \leq_i \ldots \leq_i i-1$



The four orderings on $\mathcal{O}_q(G(2,4))$

• The quantum grassmannian is a quantum algebra with a straightening law with respect to each of the n orderings

Fix an invariant prime P in $\mathcal{O}_q(G(k,n))$

• For each i-order there is a unique quantum minor $[I_i]$ such that $[I_i] \notin P$ but $[J] \in P$ for each $J \not\geq_i I_i$

Let Π_i denote $\{[J] \mid J \not\geq_i I_i\}$. Then

$$\Pi(P) := \cup_{i=1}^{n} \Pi_i \subseteq P$$

Conjecture $\Pi(P)$ is the set of quantum minors belonging to P, and P is generated as an ideal by $\Pi(P)$

• We hope to prove this conjecture by using the path methods that Karel Casteels will describe in his talk

Continuing with the notation on the previous slide:

- The quantum minors $I_1, I_2, \ldots, I_n$ form a **Grassmann necklace,** Neck(P)

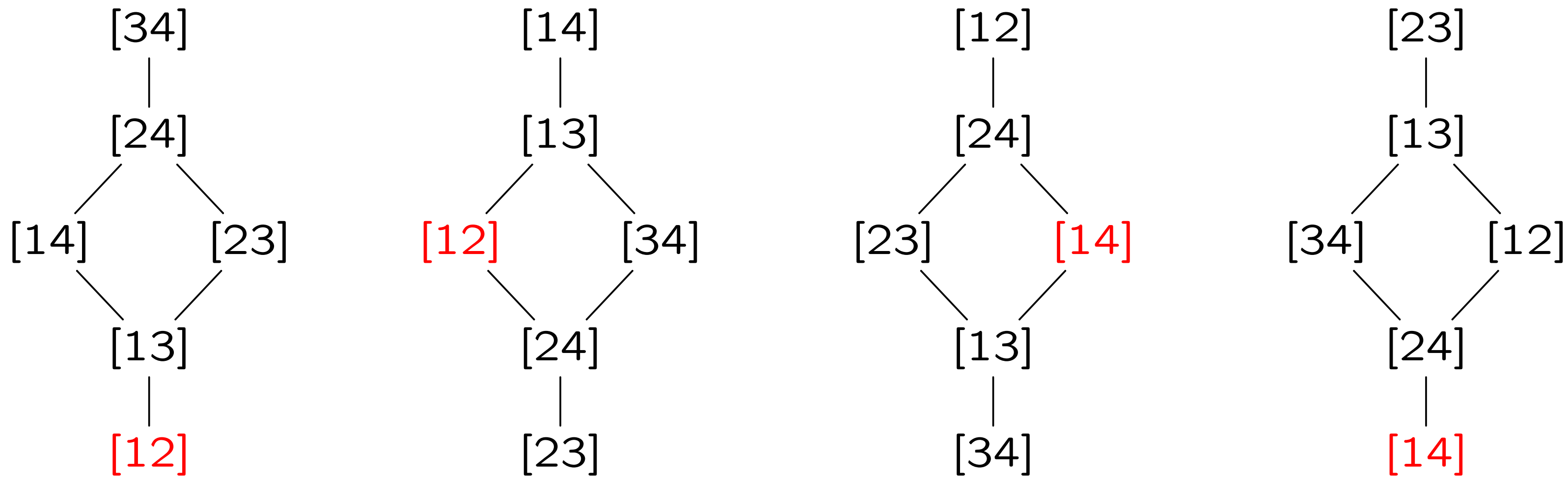
- Given a Cauchon diagram for an invariant prime P, we can construct Neck(P)

- If $P' \subseteq P$ then Neck(P') $\leq$ Neck(P)

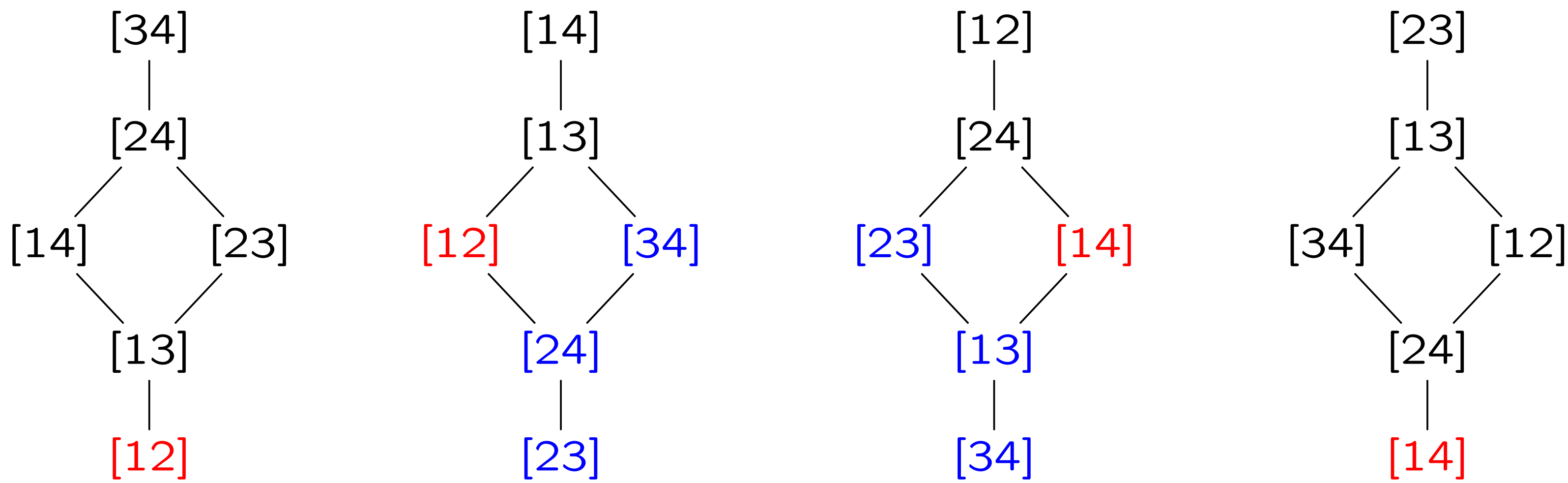
Conjecture The converse is true

- In $\mathcal{O}_q(G(2,4))$ consider the Grassmann necklace

$$(I_1, I_2, I_3, I_4) = (12, 12, 14, 14)$$



• Grassmann necklace: $(I_1, I_2, I_3, I_4) = (12, 12, 14, 14)$



• The $\mathcal{H}$-prime P with this necklace is $P = \langle [13], [23], [24], [34] \rangle$

• Note that $\mathcal{O}_q(G(2,4))/P \cong \mathbb{C}\Big[[12], [14]\Big]$ is a quantum plane, so P is prime.