Ring theoretic properties of quantum grassmannians

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Abstract

The $m \times n$ quantum grassmannian, $\mathcal{G}_q(m,n)$, with $m \leq n$, is the subalgebra of the algebra $\mathcal{O}_q(M_{mn})$ of quantum $m \times n$ matrices that is generated by the maximal $m \times m$ quantum minors. Several properties of $\mathcal{G}_q(m,n)$ are established. In particular, a k-basis of $\mathcal{G}_q(m,n)$ is obtained, and it is shown that $\mathcal{G}_q(m,n)$ is a noetherian domain of Gelfand-Kirillov dimension $m(n-m)+1$. The algebra $\mathcal{G}_q(m,n)$ is identified as the subalgebra of coinvariants of a natural left coaction of $\mathcal{O}_q(SL_m)$ on $\mathcal{O}_q(M_{mn})$ and it is shown that $\mathcal{G}_q(m,n)$ is a maximal order.

2000 Mathematics subject classification: 16W35, 16P40,16P90, 16S38, 17B37, 20G42

Introduction

Fix a base field k, a nonzero scalar $q \in k$ and positive integers m, n with $m \leq n$. The *coordinate ring of quantum $m \times n$ matrices*, $\mathcal{O}_q(M_{mn})$, is the k-algebra generated by mn indeterminates X_{ij}, $1 \leq i \leq m$ and $1 \leq j \leq n$, subject to the following relations:

$$\begin{array}{rcl} X_{ij}X_{il} & = & qX_{il}X_{ij}, \\ X_{ij}X_{kj} & = & qX_{kj}X_{ij}, \\ X_{il}X_{kj} & = & X_{kj}X_{il}, \\ X_{ij}X_{kl} - X_{kl}X_{ij} & = & (q-q^{-1})X_{il}X_{kj}, \end{array} \tag{1}$$

for $1 \leq i < k \leq m$ and $1 \leq j < l \leq n$. It is well-known that $\mathcal{O}_q(M_{mn})$ can be presented as an iterated skew polynomial algebra over k with the generators added in lexicographic order. As a consequence of this presentation, it is easy to establish that $\mathcal{O}_q(M_{mn})$ is a noetherian domain of Gelfand-Kirillov dimension mn.

We will usually write $\mathcal{O}_q(M_n)$ for the algebra $\mathcal{O}_q(M_{nn})$. In this algebra the *quantum determinant*, $D_q = \det_q$ is defined by

$$D_q := \sum_{\sigma \in S_n} (-q)^{l(\sigma)} X_{1,\sigma(1)} \dots X_{n,\sigma(n)};$$

*Some of the results in this paper appear in the first author's PhD thesis (Edinburgh, 2001). She thanks EPSRC for financial support.

†Part of this work was done while the third author was visiting the University of Edinburgh. He thanks the Edinburgh Mathematical Society for the financial support of this visit.

from [13, Theorem 4.6.1], we know that D_q is in the centre of $\mathcal{O}_q(M_n)$.

Following [6], we use the notation $[I \mid J]$ to denote the quantum determinant of the quantum matrix subalgebra $\mathcal{O}_q(M_{I,J})$ of $\mathcal{O}_q(M_{mn})$ generated by the elements X_{ij} with $i \in I$ and $j \in J$, where I and J are index sets with $|I| = |J|$. The element $[I|J]$ is the *quantum minor* determined by the index sets I and J. If $I = \{i_1, \ldots, i_s\}$ and $J = \{j_1, \ldots, j_s\}$ where the indices are written in ascending order, then we will often denote $[I \mid J]$ by $[i_1 \ldots i_s \mid j_1 \ldots j_s]$.

In this paper we are interested in studying the ring theoretic properties of a certain subalgebra of $\mathcal{O}_q(M_{mn})$, the *quantum deformation of the homogeneous coordinate ring of the* $m \times n$ *grassmannian*, $\mathcal{G}_q(m,n)$. This is a deformation of the classical homogeneous coordinate ring of the grassmannian of m-dimensional k-subspaces of n-dimensional k-space and is generated by the maximal quantum minors of $\mathcal{O}_q(M_{mn})$; to be more specific, $\mathcal{G}_q(m,n)$ is the subalgebra of $\mathcal{O}_q(M_{mn})$ generated by the $m \times m$ quantum minors of $\mathcal{O}_q(M_{mn})$. In the quantum grassmannian $\mathcal{G}_q(m,n)$, any $m \times m$ quantum minor will involve rows $1, \ldots, m$ of the quantum matrix (X_{ij}) associated to $\mathcal{O}_q(M_{mn})$. Thus, to simplify notation, we may denote a quantum minor by its columns only; that is, the quantum minor given by the row set $\{1, \ldots, m\}$ and column set J will be denoted by $[J]$.

Example $\mathcal{G}_q(2,4)$ is the k-algebra generated by the 2×2 minors of the 2×4 quantum matrix of $\mathcal{O}_q(M_{2,4})$: $[12], [13], [14], [23], [24]$ and $[34]$.
Using the relations for $\mathcal{O}_q(M_{mn})$ and [6, Lemma A.1] we can calculate the following commutation relations:

$$[12][13] = q[13][12], \quad [12][14] = q[14][12], \quad [12][23] = q[23][12],$$
$$[12][24] = q[24][12], \quad [12][34] = q^2[34][12], \quad [13][14] = q[14][13],$$
$$[13][23] = q[23][13], \quad [13][24] = [24][13] + \left(q - q^{-1}\right)[14][23],$$
$$[13][34] = q[34][13], \quad [14][23] = [23][14], \quad [14][24] = q[24][14],$$
$$[14][34] = q[34][14], \quad [23][24] = q[24][23], \quad [23][34] = q[34][23],$$
$$[24][34] = q[34][24],$$

and the Quantum Plücker relation

$$[12][34] - q[13][24] + q^2[14][23] = 0.$$

Remark Quantum matrices and quantum grassmannians can be defined in an exactly similar manner over any commutative ring R with an invertible element $q \in R$. In the next section, we shall need to consider quantum grassmannians defined over a Laurent polynomial extension either of a field or of the integers.

1 Fioresi's commutation relations

In [3], Fioresi has developed useful commutation relations for the $m \times m$ quantum minors which generate $\mathcal{G}_q(m,n)$. However, Fioresi works in the following setting. The field k that she considers is required to be algebraically closed of characteristic zero, and the quantum matrix algebra that she considers is generated as an algebra over the ring $k[q,q^{-1}]$, where q is transcendental over k. The first thing that we need to do is to observe that these commutation relations hold over any field k and for any $0 \neq q \in k$. A couple of warnings about notation for readers comparing [3] with this paper. First, because of the choice of relations for $\mathcal{O}_q(M_{mn})$, it is necessary to replace q by q^{-1} in any relation taken from [3]. Secondly, Fioresi works with the quantum grassmannian defined by the maximal $m \times m$ minors of $\mathcal{O}_q(M_{nm})$; thus, in any maximal minor, she uses all of the m columns, and a generating quantum minor of the grassmannian is specified by choosing m rows. To deal with this second difference, we can think of both versions of the quantum grassmannian as being subalgebras in the quantum matrix algebra $\mathcal{O}_q(M_n)$ and observe that the transpose automorphism, τ, see [13, 3.7.1], transforms Fioresi's quantum grassmannian to our quantum grassmannian.

Recall the following total *lexicographic ordering* on quantum minors: $[j_1 j_2 \dots j_m] <_{\mathrm{lex}} [i_1 i_2 \dots i_m]$ if and only if there exists an index α such that $j_l = i_l$ for $l < \alpha$, but $j_\alpha < i_\alpha$.

Let $[I] = [i_1 \dots i_m]$ denote an $m \times m$ quantum minor. If $[I] \neq [1 \dots m]$, consider the least integer s such that $i_s > s$. Let $\sigma([I])$ be the quantum minor obtained from $[I]$ by replacing i_s by $i_s - 1$ and leaving the other indices unchanged. Obviously, $\sigma([I]) <_{\mathrm{lex}} [I]$. The *standard tower* of $[I]$ is the sequence of quantum minors $[I_N] >_{\mathrm{lex}} [I_{N-1}] >_{\mathrm{lex}} \dots >_{\mathrm{lex}} [I_1] >_{\mathrm{lex}} [I_0]$ where $[I_N] = [I]$, $[I_{l-1}] = \sigma([I_l])$, and $[I_0] = [1, \dots, r]$. If $[I] = [1 \dots r]$ then the standard tower is defined to be the single quantum minor $[I]$.

We will denote the version of the $m \times n$ quantum grassmannian constructed by Fioresi by $\mathcal{G}_h(m,n)$. Note also that the relations in [3] use h where we would use h^{-1}; thus we should interchange h and h^{-1}.

Proposition 1.1 *Let K be an algebraically closed field of characteristic zero, and let h be an indeterminate over K. Set $\mathcal{G}_h(m,n)$ to be the quantum grassmannian subalgebra of $\mathcal{O}_h(M(K[h,h^{-1}])_{mn})$. Let $I, J \subseteq \{1, \dots, n\}$ with $|I| = |J| = m$, and $[I] <_{\mathrm{lex}} [J]$. Set $s = m - |I \cap J|$. Then in $\mathcal{G}_h(m,n)$,*

$$[I]\,[J] = h^s\,[J]\,[I] + \sum_{[L] <_{\mathrm{lex}} [I]} \lambda_{[L]} \left(h - h^{-1}\right)^{i_{[L]}} (-h)^{j_{[L]}}\,[L]\,[L'] \,,$$

where $i_{[L]}, j_{[L]} \in \mathbb{N}$ and $\lambda_{[L]}$ is either 0 or 1, while L' is the set $(I \cap J) \cup ((I \cup J) \setminus L)$.

Proof In [3, Proposition 2.21 and Theorem 3.6], Fioresi obtains commutation relations of the above form, but with the products $[L]\,[L']$ on the right hand side of the equation above more carefully stated. In Proposition 2.21 she first obtains the result for the case that $I \cap J = \emptyset$. In this case, the quantum minors $[L]$ involved are members of the standard tower of $[I]$, and so $[L] <_{\mathrm{lex}} [I]$, as we require. The general case where $I \cap J \neq \emptyset$ is dealt with in Theorem 3.6. Set $[\widetilde{I}]$ to be the quantum minor obtained from columns $I\backslash(I\cap J)$, and similarly, define $[\widetilde{J}]$. Proposition 2.21 provides a commutation rule for $[\widetilde{I}][\widetilde{J}]$ with terms on the right hand side $[\widetilde{L}][\widetilde{L'}]$ where $[\widetilde{L}] <_{\mathrm{lex}} [\widetilde{I}]$. In Theorem 3.6, a commutation rule with the same coefficients is then obtained for $[I]\,[J]$ by replacing each $[\widetilde{L}][\widetilde{L'}]$ by $[\widetilde{L}\cup(I\cap J)][\widetilde{L'}\cup(I\cap J)]$. Thus, all that needs to be done is to make the easy observation that if $[\widetilde{L}] <_{\mathrm{lex}} [\widetilde{I}]$ then $[\widetilde{L}\cup(I\cap J)] <_{\mathrm{lex}} [\widetilde{I}\cup(I\cap J)] = [I]$. ∎

Corollary 1.2 *Let k be any field and q any nonzero element of k. Set $\mathcal{G}_q(m,n)$ to be the quantum grassmannian subalgebra of $\mathcal{O}_q(M_{mn})$. Let $I, J \subseteq \{1,\ldots,n\}$ with $|I| = |J| = m$, and $[I] <_{\mathrm{lex}} [J]$. Set $s = m - |I\cap J|$. Then in $\mathcal{G}_q(m,n)$,*

$$[I]\,[J] = q^s\,[J]\,[I] + \sum_{[L]<_{\mathrm{lex}}[I]} \lambda_{[L]}\left(q-q^{-1}\right)^{i_{[L]}}(-q)^{j_{[L]}}\,[L]\,[L'] \,,$$

where $\lambda_{[L]} \in k$, $i_{[L]}, j_{[L]} \in \mathbb{N}$ and $\lambda_{[L]}$ is either 0 or 1, while L' is the set $(I\cap J)\cup((I\cup J)\setminus L)$.

Proof Proposition 1.1 applies in the case that $K = \mathbb{C}$. In this case, observe that the coefficients of the monomials in the maximal minors are all in $\mathbb{Z}[h,h^{-1}]$; so that these relations hold in the quantum grassmannian over $\mathbb{Z}[h,h^{-1}]$. There is then a natural homomorphism from this quantum grassmannian to $\mathcal{G}_q(m,n)$, such that $z \mapsto z1_k$ for $z \in \mathbb{Z}$ and $h \mapsto q$, which produces the required relations. ∎

Recall that an element a of an algebra A is a *normal* element if $aA = Aa$. The next result follows immediately from the previous Corollary.

Corollary 1.3 *An $m\times m$ quantum minor $[I] \in \mathcal{G}_q(m,n)$ is normal modulo the ideal generated by the set $\{[J] \mid [J] <_{\mathrm{lex}} [I]\}$.*

The algebra $\mathcal{O}_q(M_{mn})$ is a connected $\mathbb{N}$-graded algebra, graded by the total degree in the canonical generators. Since $\mathcal{G}_q(m,n)$ is a subalgebra generated by homogeneous elements of degree m with respect to this grading, $\mathcal{G}_q(m,n)$ inherits a connected $\mathbb{N}$-graded structure in which its canonical generators have degree one.

Theorem 1.4 *The quantum grassmannian $\mathcal{G}_q(m,n)$ is a noetherian domain.*

Proof The quantum grassmannian $\mathcal{G}_q(m,n)$ is generated by the $\binom{n}{m}$ quantum minors of size m in $\mathcal{O}_q(M_{mn})$. Denote these quantum minors by $u_1 <_{\mathrm{lex}} u_2 <_{\mathrm{lex}} \cdots <_{\mathrm{lex}} u_{\binom{n}{m}}$. Then by Corollary 1.3, $\{u_1,\ldots,u_{\binom{n}{m}}\}$ is a normalising sequence of $\mathcal{G}_q(m,n)$; that is, u_1 is normal and u_l is normal modulo the ideal generated by $\{u_1,\ldots,u_{l-1}\}$, for $l > 1$. The factor by the ideal generated by this normalising sequence is the base field; so the fact that $\mathcal{G}_q(m,n)$ is noetherian follows by repeated use of [1, Lemma 8.2].

Finally, $\mathcal{G}_q(m,n)$ is a domain since it is a subalgebra of $\mathcal{O}_q(M_{mn})$ which is a domain. ∎

Remark If A is a noetherian, connected $\mathbb{N}$-graded k-algebra such that every non-simple graded prime factor ring A/P contains a nonzero homogeneous normal element in $\oplus_{i\geq 1}(A/P)_i$ then we say that A has *enough normal elements* ([14]). Thus, the two previous results show that the quantum grassmannian has enough normal elements.

There is a useful isomorphism between $\mathcal{G}_q(m,n)$ and $\mathcal{G}_{q^{-1}}(m,n)$ which we now describe. Notice that, if $1 \leq i_1 < \cdots < i_m \leq n$, $\mathcal{G}_q(m,n)$ is isomorphic to the subalgebra of $\mathcal{O}_q(M_n)$ generated by the $m \times m$ minors that use rows $i_1, \ldots, i_m$, that is, the minors $[I|J]$ with $I = \{i_1, \ldots, i_m\}$ and $J \subseteq \{1, \ldots, n\}$, $|J| = m$. Let $A := \mathcal{O}_q(M_n)$ with generators X_{ij} and $A' := \mathcal{O}_{q^{-1}}(M_n)$ with generators X'_{ij}. Take a copy R of $\mathcal{G}_q(m,n)$ inside A generated by the $m \times m$ quantum minors that use the first m rows of A, and take a copy R' of $\mathcal{G}_{q^{-1}}(m,n)$ that uses the last m rows of A'. Following the proof of [7, Corollary 5.9], we see that there is an isomorphism $\delta : A \longrightarrow A'$ which takes $[I|J]$ to $[\omega_0 I|\omega_0 J]'$, where $[-|-]'$ denotes a quantum minor in $A' := \mathcal{O}_{q^{-1}}(M_n)$ and ω_0 is the longest element of the symmetric group S_n; that is, $\omega_0(i) = n - i + 1$. Note that the isomorphism δ restricted to R produces an isomorphism from R to R' that takes a generating minor $[I]$ to the minor $[\omega_0 I]'$. In particular, note that under this isomorphism, $[12 \ldots m]$, the leftmost minor of $R = \mathcal{G}_q(m,n)$, is translated into the rightmost minor $[n-m+1 \ldots n]'$ of the quantum grassmannian $R' = \mathcal{G}_{q^{-1}}(m,n)$. We denote this induced isomorphism from $\mathcal{G}_q(m,n)$ to $\mathcal{G}_{q^{-1}}(m,n)$ by δ also.

As an example of the use of the isomorphism δ, we record the following lemma which we need later.

Lemma 1.5 *Let $I \subseteq \{1, \ldots, n\}$ with $|I| = m$. Then*

$$[I]\,[n-m+1 \ldots n] = q^s [n-m+1 \ldots n]\,[I]$$

where $s = m - |I \cap \{n-m+1, \ldots, n\}|$, and thus $[n-m+1 \ldots n]$ is normal in $\mathcal{G}_q(m,n)$.

Proof Note that $\omega_0\{n-m+1, \ldots, n\} = \{1, \ldots, m\}$. Note also that $|I \cap \{n-m+1, \ldots, n\}| = |\omega_0 I \cap \omega_0\{n-m+1, \ldots, n\}| = |\omega_0 I \cap \{1, \ldots, m\}|$.

By Corollary 1.2, $[1 \ldots m][\omega_0 I] = q^s[\omega_0 I][1 \ldots m]$. Applying δ to this equation gives $[n-m+1 \ldots n]'[I]' = q^s[I]'[n-m+1 \ldots n]'$ in $\mathcal{G}_{q^{-1}}(m,n)$. This can be rewritten as $[I]'[n-m+1 \ldots n]' = q^{-s}[n-m+1 \ldots n]'[I]'$ in $\mathcal{G}_{q^{-1}}(m,n)$. Finally, replacing q^{-1} by q, we obtain

$$[I][n-m+1 \ldots n] = q^s[n-m+1 \ldots n][I]$$

in $\mathcal{G}_q(m,n)$. ∎

2 A basis for $\mathcal{G}_q(m,n)$

In this section, we obtain a basis for $\mathcal{G}_q(m,n)$. This basis is a subset of the basis of preferred products of $\mathcal{O}_q(M_{mn})$ obtained in [6, Section 1]. First, we adapt the language used in that paper to the grassmannian subalgebra $\mathcal{G}_q(m,n)$. Recall from Section 1 that if J is an m-element subset of $\{1, \ldots, n\}$ then $[J]$ denotes the quantum minor $[1, \ldots, m \mid J]$

of $\mathcal{O}_q(M_{mn})$. Thus, let $m, n \in \mathbb{N}^*$ with $n \geq m$. We define a partial ordering on m-element subsets of $\{1, \ldots, n\}$.

Definition 2.1 *Let $A, B \subseteq \{1, \ldots, n\}$ with $|A| = m = |B|$. We define a partial ordering, denoted by $\leq_*$. Write A and B in ascending order:*

$$A = \{a_1 < a_2 < \cdots < a_m\} \qquad \text{and} \qquad B = \{b_1 < b_2 < \cdots < b_m\}.$$

Define $A \leq_* B$ to mean that $a_i \leq b_i$ for $i = 1, \ldots, m$.

This naturally defines a partial ordering on the generators of $\mathcal{G}_q(m, n)$.

Definition 2.2 *Let $[I]$ and $[J]$ belong to the generating set of $\mathcal{G}_q(m, n)$. Then we write that $[I] \leq_c [J]$ if and only if $I \leq_* J$.*

For example, Figure 1 shows the ordering on generators of $\mathcal{G}_q(3, 6)$.

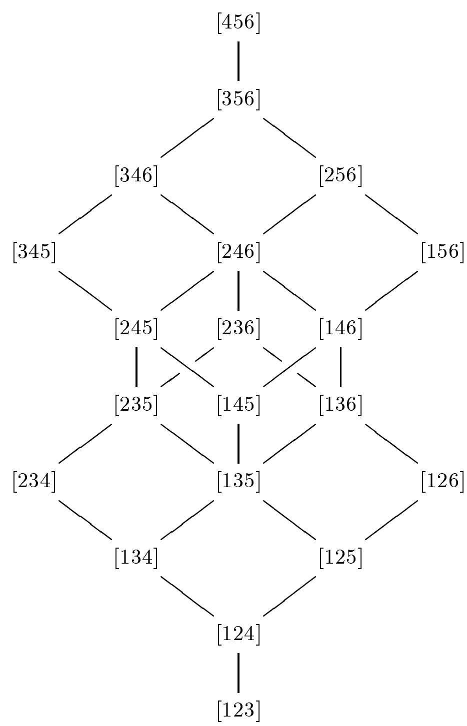

Figure 1: The partial ordering $\leq_c$ on $\mathcal{G}_q(3, 6)$

Recall that a *tableau* is a Young diagram with entries in each box. If each row of a tableau T has length m then we will say that T is an *m-tableau*. Here, we consider tableaux with entries from $\{1, \ldots, n\}$ and no repetitions in each row. An *allowable m-tableau* T is an

m-tableau with strictly increasing rows. If an allowable m-tableau T has rows $J_1, \ldots, J_s$, then T is *preferred* if and only if $J_1 \leq_* J_2 \leq_* \ldots \leq_* J_s$.

Let $I = \{m, m-1, \ldots, 1\}$ and let S be an m-tableau which has the same number of rows as T and such that each row of S is I. Then T is an allowable (preferred) m-tableau if and only if the bitableau $(S \mid T)$ is allowable (preferred) in the sense of [6]. With this in mind, we define the following ordering on allowable m-tableau. Let

$$T = \begin{pmatrix} J_1 \\ J_2 \\ \vdots \\ J_t \end{pmatrix}, \qquad S = \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_s \end{pmatrix}.$$

Then $T \prec S$ if $t > s$, or if $s = t$ and

$$\{J_1, \ldots, J_t\} <_{\text{lex}} \{L_1, \ldots, L_s\};$$

that is, there exists an index i such that $J_\alpha = L_\alpha$ for $\alpha < i$, but $J_i <_* L_i$.

Any allowable m-tableau determines a product of quantum minors in the quantum grassmannian as follows.

Definition 2.3 *For any (allowable) m-tableau*

$$T = \begin{pmatrix} J_1 \\ J_2 \\ \vdots \\ J_s \end{pmatrix},$$

define $[T] = [J_1][J_2] \ldots [J_s]$.

Definition 2.4 *The* content *of an m-tableau T is the multiset* $\{1^{t_1}, 2^{t_2}, \cdots, n^{t_n}\}$, *where* t_i *is the number of times i appears in T.*

We will use the content of a tableau to define a natural $\mathbb{Z}^n$-grading on the $m \times n$ quantum grassmannian. There is a $\mathbb{Z}^n$-grading on $\mathcal{O}_q(M_{mn})$ defined by assigning degree ε_j to X_{ij}, where ε_j for $j = 1, \ldots, n$ form the natural basis of $\mathbb{Z}^n$. Since the maximal minors of $\mathcal{O}_q(M_{mn})$ are homogeneous with respect to this basis, there is an induced $\mathbb{Z}^n$-grading on $\mathcal{G}_q(m,n)$: consider a product of minors $[T]$ in $\mathcal{G}_q(m,n)$, if the tableau T has content $\{1^{t_1}, 2^{t_2}, \ldots, n^{t_n}\}$, then $[T]$ is homogeneous of degree $(t_1, t_2, \ldots, t_n)$. Thus, the degree of a product is dependent on the number of times each column of the $m \times n$ quantum matrix appears in it.

Theorem 2.5 (Generalised Quantum Plücker Relations for Quantum grassmannians) *Let* $J_1, J_2, K \subseteq \{1, 2, \ldots, n\}$ *be such that* $|J_1|, |J_2| \leq m$ *and* $|K| = 2m - |J_1| - |J_2| > m$. *Then*

$$\sum_{K' \sqcup K'' = K} (-q)^{\ell(J_1;K') + \ell(K';K'') + \ell(K'';J_2)} [J_1 \sqcup K'][K'' \sqcup J_2] = 0,$$

where $\ell(I; J) = |\{(i,j) \in I \times J : i > j\}|$.

Proof We work in the algebra $\mathcal{O}_q(M_n)$ and apply [6, Proposition B2(a)] with $I_1 = I_2 = \{1, \ldots, m\} =: I$. Thus,

$$\sum_{K' \sqcup K'' = K} (-q)^{\ell(J_1;K') + \ell(K';K'') + \ell(K'';J_2)} [I | J_1 \sqcup K'][I | K'' \sqcup J_2] = 0,$$

since $|K| > m = |I_1 \cup I_2|$, see [6, B3]. This is the desired relation. ▪

Lemma 2.6 *Let T be an m-tableau with content γ and suppose that T is not preferred. Then*

(a) *T is not minimal with respect to $\prec$ among m-tableaux with content γ;*

(b) *$[T]$ can be expressed as a linear combination of products $[S]$, where each S is an m-tableau with content γ such that $S \prec T$.*

Proof Follow the proof of [6, Lemma 1.7]. Note that in the proof the only place where the shape of a bitableau might change is near the end of the proof where the right-hand side of the Exchange Formula is considered. In our situation, the right-hand side is zero, as noted in Theorem 2.5. ▪

Note that fixing the content of an m-tableau fixes its shape and thus fixes the number of rows in the m-tableau.

Let $\partial = (c_1, \ldots, c_n) \in \mathbb{N}^n$. Let V be the homogeneous component of degree ∂ in $\mathcal{G}_q(m, n)$. Note that V might be zero, and that this is the case if and only if there is no product $[T]$ where T is an m-tableau of content $(1^{c_1} \ldots n^{c_n})$. Also, an element of $\mathcal{G}_q(m, n)$ belongs to V if and only if it is a linear combination of products $[T]$, where T runs over all m-tableau with content $(1^{c_1} \ldots n^{c_n})$; that is, the products $[T]$, where T runs over all m-tableau with content $(1^{c_1} \ldots n^{c_n})$ span V.

Theorem 2.7 *Let $\partial = (c_1, \ldots, c_n) \in \mathbb{N}^n$, let V be the homogeneous component of $\mathcal{G}_q(m, n)$ with degree ∂, and set $\gamma = (1^{c_1} 2^{c_2} \cdots n^{c_n})$. The products $[T]$, as T runs over all preferred m-tableau with content γ, form a basis for V.*

Proof It is enough to prove that for any m-tableau T with content γ the product $[T]$ is a linear combination of products $[S]$ where S is a preferred m-tableau with content γ. Let $\mathcal{E}$ be the set of m-tableau with content γ; clearly, $\mathcal{E}$ is a finite set and we order it by $\prec$. We use induction on $\prec$ to show the result. Let $U \in \mathcal{E}$. If U is minimal, then it is preferred, by part (a) of the previous result. Otherwise, by part (b) of the previous result, $[U]$ is a linear combination of products $[S]$, where $S \in \mathcal{E}$ and $S \prec U$. Thus, by an induction argument applied to S, we may conclude that $[U]$ is a linear combination of products $[S]$ where S is a preferred m-tableau with content γ.

Recall that $\mathcal{G}_q(m, n)$ is a subalgebra of $\mathcal{O}_q(M_{mn})$ and notice that the products $[T]$, as T runs over all preferred m-tableaux of content γ, form a subset of the basis of $\mathcal{O}_q(M_{mn})$ constructed in [6]. Therefore, they are linearly independent and we have the result. ▪

Corollary 2.8 *The products $[T]$, as T runs over all preferred m-tableaux, form a basis for $\mathcal{G}_q(m,n)$.*

This basis can be used to calculate the Gelfand-Kirillov dimension of the $m \times n$ quantum grassmannian.

Consider the partial ordering $\leq_c$ on the generating minors of $\mathcal{G}_q(m,n)$. A *saturated path* between two minors $a <_c b$ will be an 'upwards path' $a = a_1 <_c a_2 <_c \ldots <_c a_l = b$ of minors such that no additional terms can be added; that is, for any index i there is no minor d such that $a_i <_c d <_c a_{i+1}$. The *length* of such a saturated path is defined to be l. For example, a saturated path between the minors [134] and [256] in $\mathcal{G}_q(3,6)$ is

$$[134] <_c [234] <_c [235] <_c [236] <_c [246] <_c [256] .$$

The length of this saturated path is 6.

A *maximal path* is a saturated path between the two minors $[1 \ldots m]$ and $[n-m+1 \ldots n]$. It is easy to check that any maximal path has length $m(n-m)+1$.

Proposition 2.9 *Let $G = \mathcal{G}_q(m,n)$ and let α be the length of a maximal path in G. Then*

$$\mathrm{GKdim}(\mathcal{G}_q(m,n)) = \alpha = m(n-m)+1.$$

Proof Let V be the k-subspace of G spanned by the $m \times m$ minors which generate G. Then $\mathrm{GKdim}(G) = \overline{\lim} \log_n d_V(n)$ where $d_V(n) = \dim_k (\sum_{i=0}^n V^i)$. Let $a_1, a_2, \ldots, a_\alpha$ be a maximal path in G. Then $a_1^{s_1} a_2^{s_2} \ldots a_\alpha^{s_\alpha} \in V^{n+1}$ whenever $\sum_{i=1}^\alpha s_i = n+1$. The set $\{a_1^{s_1} a_2^{s_2} \ldots a_\alpha^{s_\alpha} \mid \Sigma s_i = n+1\}$ is linearly independent. Therefore

$$\dim_k \left(V^{n+1}\right) \geq |\{a_1^{s_1} a_2^{s_2} \ldots a_\alpha^{s_\alpha} \mid \Sigma s_i = n+1\}| = \binom{n+\alpha}{\alpha-1}$$

which is a polynomial in n of degree $\alpha - 1$. It follows that $\mathrm{GKdim}(G) \geq \alpha$.

Let $a_{i_1} \ldots a_{i_n} \in V^n$. By Theorem 2.7, $a_{i_1} \ldots a_{i_n}$ may be rewritten as a linear combination of preferred products from V^n.

There are finitely many maximal paths in $\mathcal{G}_q(m,n)$. Suppose there are c such paths and index them $1, \ldots, c$. Let $a_1 <_c a_2 <_c \ldots <_c a_\alpha$ be the ith maximal path and let $W_i^{(n)}$ denote the subspace generated by monomials $a_1^{s_1} \ldots a_\alpha^{s_\alpha}$ such that $\Sigma s_j = n$. The above observation shows that $V^n \subseteq \sum_{i=1}^c W_i^{(n)}$. Consider $\dim(W_i^{(n)})$. The products $a_1^{s_1} \ldots a_\alpha^{s_\alpha}$ such that $\Sigma s_j = n$ are linearly independent. Therefore

$$\dim(W_i^{(n)}) \quad = \quad |\{a_1^{s_1} \ldots a_\alpha^{s_\alpha} \mid \sum s_i = n\}| = |\{(s_1, \ldots, s_\alpha) \in \mathbb{N}^\alpha \mid \sum s_i = n\}|.$$

Therefore $\dim(W_i^{(n)}) = \dim(W_j^{(n)})$ for all $i, j \in \{1, \ldots, c\}$. Thus

$$\dim(V^n) \leq \dim\left(\sum_{i=1}^c W_i^{(n)}\right) \leq c \dim\left(W_1^{(n)}\right) = c \binom{n+\alpha-1}{\alpha-1}$$

and $d_V(n) \leq c \sum_{i=0}^n \binom{i+\alpha-1}{\alpha-1}$, a polynomial of degree α. It follows that $\mathrm{GKdim}(G) \leq \alpha$. Hence, $\mathrm{GKdim}(G) = \alpha = m(n-m)+1$. ∎

For example, $\mathrm{GKdim}(\mathcal{G}_q(2,4)) = 2(4-2)+1 = 5$.

3 Noncommutative Dehomogenisation

If R is a commutative $\mathbb{N}$-graded algebra, and x is a homogeneous nonzerodivisor in degree one, then the *dehomogenisation* of R at x is usually defined to be the factor algebra $R/(x-1)R$, [2, Appendix 16.D]. This definition is unsuitable in a noncommutative algebra if the element x is merely normal rather than central: in this case, the factor algebra is often too small to be useful. For example, let R be the quantum plane $k_q[x,y]$ with $xy = qyx$ and $q \neq 1$. Setting $x = 1$ forces $y = 0$; so that the factor algebra $R/\langle x-1\rangle$ is isomorphic to k rather than being an algebra of Gelfand-Kirillov dimension 1, as one might hope. However, in the commutative case, an alternative approach is to observe that the localised algebra $S := R[x^{-1}]$ is $\mathbb{Z}$-graded, $S = \oplus_{i\in\mathbb{Z}} S_i$, and that $S_0 \cong R/(x-1)R$. If x is a normal nonzerodivisor of degree one in a noncommutative $\mathbb{N}$-graded algebra $R = \oplus_{i\in\mathbb{N}} R_i$, then one can form the Ore localisation $R[x^{-1}] =: S$, and then this second approach does yield a useful algebra in the noncommutative case. Indeed, for $i, j \in \mathbb{N}$ denote by $R_i x^{-j}$ the k-subspace of elements of S that can be written as rx^{-j} with $r \in R_i$; clearly, $R_i x^{-j} \subseteq R_{i+1}x^{-(j+1)}$. For $l \in \mathbb{Z}$, set $S_l = \sum_{t\geq 0} R_{l+t}x^{-t} = \cup_{t\geq 0} R_{l+t}x^{-t}$. Then S is a $\mathbb{Z}$-graded algebra with $S = \oplus_{l\in\mathbb{Z}} S_l$.

Definition 3.1 *Let $R = \oplus R_i$ be an $\mathbb{N}$-graded k-algebra and let x be a regular homogeneous normal element of R of degree one. Then the* dehomogenisation *of R at x, written* $\mathrm{Dhom}(R,x)$*, is defined to be the zero degree subalgebra S_0 of the $\mathbb{Z}$-graded algebra $S := R[x^{-1}]$.*

It is easy to check that $\mathrm{Dhom}(R,x) = \sum_{i=0}^{\infty} R_i x^{-i} = \cup_{i=0}^{\infty} R_i x^{-i}$. In particular, if $R = k[R_1]$ then $\mathrm{Dhom}(R,x) = \sum_{i=0}^{\infty} (R_1x^{-1})^i = \cup_{i=0}^{\infty} (R_1x^{-1})^i$, and further, if $R_1 = ka_1 + \cdots + ka_s$ then $\mathrm{Dhom}(R,x) = k[a_1x^{-1}, \ldots, a_sx^{-1}]$.

Denote by σ the automorphism of S given by $\sigma(s) = xsx^{-1}$ for $s \in S$. Note that σ induces an automorphism of S_0, also denoted by σ.

Lemma 3.2 *Let R be an $\mathbb{N}$-graded algebra and let x be a regular normal homogeneous element of degree 1. Then there is an isomorphism*

$$\theta : \mathrm{Dhom}(R,x)[y, y^{-1}; \sigma] \longrightarrow R[x^{-1}]$$

which is the identity on $\mathrm{Dhom}(R,x)$ *and sends y to x.*

Proof The existence of θ is clear from the universal property of skew-Laurent extensions. It is easy to check that θ is an isomorphism. ▪

Some properties of dehomogenisation follow in an elementary way from this result.

Corollary 3.3 *Let $R = \oplus_{i\geq 0} R_i$ be an $\mathbb{N}$-graded algebra and let x be a regular homogeneous normal element of degree one.*
(i) R is a domain if and only if $\mathrm{Dhom}(R,x)$ *is a domain.*
(ii) If R is noetherian then $\mathrm{Dhom}(R,x)$ *is noetherian.*
(iii) If R is locally finite (that is, $\dim(R_i) < \infty$ *for all $i \in \mathbb{N}$) then* $\mathrm{GKdim}(R) = \mathrm{GKdim}(\mathrm{Dhom}(R,x)) + 1$.

Proof Point (i) follows at once from the isomorphism in Lemma 3.2.
(ii) If R is noetherian then so is $R[x^{-1}]$ and thus $\mathrm{Dhom}(R,x)[y,y^{-1};\sigma]$ is noetherian, by Lemma 3.2. As is well-known, since σ is an automorphism of $\mathrm{Dhom}(R,x)$, this implies that $\mathrm{Dhom}(R,x)$ is noetherian.
(iii) Let σ be the automorphism of R induced by conjugation by x. It is clear that σ is a graded automorphism; and so from the local finiteness of R, we see that the elements x^i, for $i \geq 1$, are local normal elements in the sense of [9, p168]. By using [9, 12.4.4], it follows that $\mathrm{GKdim}(R[x^{-1}]) = \mathrm{GKdim}(R)$. On the other hand, the automorphism σ induced on S_0 by conjugation by x in S is locally algebraic in the sense of [9, p164]. Indeed, $S_0 = \cup_{t\geq 0} R_t x^{-t}$ and for all $t \in \mathbb{N}$ the k-subspace $R_t x^{-t}$ is a finite dimensional σ-stable subspace of S_0. It follows from [9, p164] that $\mathrm{GKdim}(S_0[y,y^{-1};\sigma]) = \mathrm{GKdim}(S_0)+1$. The conclusion follows from Lemma 3.2. ▪

4 Dehomogenisation of $\mathcal{G}_q(m,n)$

In the classical commutative theory it is a well-known and basic result that the dehomogenisation of the homogeneous coordinate ring of the $m \times n$ grassmannian at the minor $[n-m+1,\dots,n]$ is isomorphic to the coordinate ring of $m \times (n-m)$ matrices; that is,

$$\frac{\mathcal{O}(\mathcal{G}(m,n))}{\langle [n-m+1,\dots,n]-1\rangle} \cong \mathcal{O}(M_{m,n-m}(k)).$$

In this section, we show that the corresponding result holds for $\mathcal{G}_q(m,n)$ when we use the noncommutative dehomogenisation defined in the previous section. Recall from Lemma 1.5 that $[n-m+1,\dots,n]$ is a normal element of $\mathcal{G}_q(m,n)$: in fact, it q-commutes with the other maximal minors, and this will be important in calculations.

Recall that we may consider $\mathcal{G}_q(m,n)$ to be a $\mathbb{N}$-graded algebra with each $m \times m$ quantum minor given degree 1. Set $x = [n-m+1,\dots,n]$ and $S := \mathcal{G}_q(m,n)[x^{-1}]$, and note that $\mathrm{Dhom}(\mathcal{G}_q(m,n),[n-m+1,\dots,n]) = S_0$ is generated by elements of the form $\{I\} := [I]\,[n-m+1,\dots,n]^{-1}$ with $I \subseteq \{1,\dots,n\}$ and $|I| = m$, see Section 3.

Now let u be a positive integer and consider $\mathcal{O}_q(M_u)$. If $I \subseteq \{1,\dots,u\}$ then $\widetilde{I} := \{1,\dots,u\}\backslash I$. In an exponent I denotes the sum of the indices occuring in the index set I.

Let D_q be the quantum determinant of $\mathcal{O}_q(M_u)$. Since D_q is a central element, we can invert it to form the $u \times u$ *quantum general linear group* $\mathcal{O}_q(GL_u) := \mathcal{O}_q(M_u)[D_q^{-1}]$. The algebra $\mathcal{O}_q(GL_u)$ is a Hopf algebra, with antipode S, and counit ε.

There is a useful antiendomorphism $\Gamma : \mathcal{O}_q(M_u) \longrightarrow \mathcal{O}_q(M_u)$ defined on generators by $\Gamma(X_{ij}) = (-q)^{i-j}[\widetilde{\{j\}}|\widetilde{\{i\}}]$, see [13, Corollary 5.2.2]. We need to know the effect of Γ on quantum minors. This is given in the following lemma, which is presumably well-known but we give a proof since we have been unable to find a clear exposition. Recall that $\Delta([I|J]) = \sum_{|K|=|I|}[I|K]\otimes[K|J]$, where Δ is the comultiplication map on $\mathcal{O}_q(M_u)$, by [12, (1.9)]. Recall also that $\varepsilon([I|J])$ equals 1 if $I = J$ and 0 otherwise.

Lemma 4.1 *Let $[I|J]$ be an $r\times r$ quantum minor in $\mathcal{O}_q(M_u)$. Then,*
(i) $S([I|J])=(-q)^{I-J}[\widetilde{J}|\widetilde{I}]D_q^{-1}$
(ii) $\Gamma([I|J])=(-q)^{I-J}[\widetilde{J}|\widetilde{I}]D_q^{r-1}$

Proof We establish the first claim by calculating the expression

$$\sum_{K,L}(-q)^{L-J}S([I|K])[K|L][\widetilde{J}|\widetilde{L}]D_q^{-1}$$

in two different ways.

First,

$$\begin{aligned}\sum_{K,L}(-q)^{L-J}S([I|K])[K|L][\widetilde{J}|\widetilde{L}]D_q^{-1} &= \sum_{K}S([I|K])\left\{\sum_{L}(-q)^{L-J}[K|L][\widetilde{J}|\widetilde{L}]D_q^{-1}\right\}\\ &= \sum_{K}S([I|K])\varepsilon([K|J])1=S([I|J]),\end{aligned}$$

by using the first equality of [13, 4.4.3].

Secondly,

$$\begin{aligned}\sum_{K,L}(-q)^{L-J}S([I|K])[K|L][\widetilde{J}|\widetilde{L}]D_q^{-1} &= \sum_{L}\left\{\sum_{K}S([I|K])[K|L]\right\}(-q)^{L-J}[\widetilde{J}|\widetilde{L}]D_q^{-1}\\ &= \sum_{L}\varepsilon([I|L])(-q)^{L-J}[\widetilde{J}|\widetilde{L}]D_q^{-1}\\ &= (-q)^{I-J}[\widetilde{J}|\widetilde{I}]D_q^{-1},\end{aligned}$$

by using the defining property of the antipode.

The second claim follows easily from the first, since $S([I|J])=\Gamma([I|J])D_q^{-r}$ for $r\times r$ quantum minors $[I|J]$. This is easily established from the fact that it holds on the generators X_{ij} and that S and Γ are anti-endomorphisms. ▪

We will need the anti-endomorphism $\Gamma\circ\tau:\mathcal{O}_q(M_u)\longrightarrow\mathcal{O}_q(M_u)$ defined by $\Gamma\circ\tau(X_{ij})=(-q)^{j-i}[\widetilde{\{i\}}|\widetilde{\{j\}}]$ for $1\le i,j\le u$. Here, τ is the transposition automorphism given in [13, Proposition 3.7.1(1)]. Note that, by Lemma 4.1, the effect of $\Gamma\circ\tau$ on the $r\times r$ quantum minor $[I\mid J]$ is given by $\Gamma\circ\tau([I\mid J])=(-q)^{J-I}[\widetilde{I}\mid\widetilde{J}]D_q^{r-1}$.

Given $I=\{i_1,\dots,i_m\}\subseteq\{1,\dots,n\}$ the set $I\backslash\{i_k\}$ is denoted by $\{i_1,\dots,\widehat{i_k},\dots,i_m\}$. Given two sets $I,J\subseteq\{1,\dots,n\}$ recall that

$$\ell\,(I;J):=|\{(i,j)\in I\times J:i>j\}|.$$

In the next proof, and throughout the paper, $(-q)^\bullet$ denotes a power of $-q$ that is not necessary to keep track of explicitly.

Lemma 4.2 *The k-algebra* $\mathrm{Dhom}(\mathcal{G}_q(m,n),[n-m+1,\dots,n]) = S_0$ *is generated as an algebra by the elements* $\{j \quad n-m+1\dots\widehat{i}\dots n\}$ *for* $1 \le j \le n-m < i \le n$.

Proof We know that S_0 is generated by the elements $\{I\} := [I]\,[n-m+1,\dots,n]^{-1}$, where $I \subseteq \{1,\dots,n\}$ and $|I| = m$. We show that each such element can be expressed as a k-linear combination of products of elements of the form $\{j \quad n-m+1\dots\widehat{i}\dots n\}$, where $1 \le j \le n-m < i \le n$. Denote by A the subalgebra of S_0 generated by the elements $\{j \quad n-m+1\dots\widehat{i}\dots n\}$.

Let $I = \{i_1 \le \dots \le i_m\} \ne \{n-m+1,\dots,n\}$ be an ordered subset of $\{1,\dots,n\}$ and let $2 \le t \le m+1$ be such that $i_t \ge n-m+1$ but $i_{t-1} < n-m+1$; that is, $I \cap \{1,\dots,n-m\} = \{i_1,\dots i_{t-1}\}$. We will use induction on t to show that $\{I\} \in A$.

If $t = 2$, then I is of the form $\{j \quad n-m+1\dots\widehat{i}\dots n\}$ and so $\{I\} \in A$. Consider a fixed $t \in \{3,\dots,m+1\}$ and suppose that the result is true for $t-1$. Now consider $[I] = [i_1\dots i_m]$ with $I \cap \{1,\dots,n-m\} = \{i_1,\dots i_{t-1}\}$. We use the generalised Quantum Plücker relations (Theorem 2.5) to rewrite the product $[n-m+1,\dots,n]\,[i_1\dots i_m]$.

Let $K = \{i_1, n-m+1,\dots,n\}$, $J_1 = \emptyset$ and $J_2 = \{i_2,\dots,i_m\}$. Then

$$\sum_{K'\sqcup K''=K} (-q)^{\ell(K';K'')+\ell(K'';J_2)}\,[K']\,[K''\sqcup J_2] = 0$$

where either

$$K' = \{n-m+1,\dots,n\} \text{ and } K'' = \{i_1\},$$

or

$$K' = \{i_1\}\cup\{n-m+1,\dots,\widehat{l},\dots,n\} \text{ and } K'' = \{l\}$$

where $n-m+1 \le l \le n$ and $l \notin \{i_2,\dots,i_m\}$. Let $S = \{n-m+1,\dots,n\}\setminus\{i_2,\dots,i_m\}$. By re-arranging the above equation, we obtain

$$[n-m+1,\dots,n]\,[i_1\dots i_m] = -\sum_{l\in S}(-q)^{\bullet}\left[i_1\; n-m+1\dots\widehat{l}\dots n\right][l\; i_2\dots i_m]\,.$$

Multiplying through by $[n-m+1,\dots,n]^{-2}$ from the right, and using Lemma 1.5 gives

$$\{i_1\dots i_m\} = \sum_{l\in S}\pm(-q)^{\bullet}\{i_1\; n-m+1\dots\widehat{l}\dots n\}\{l\; i_2\dots i_m\}.$$

Now $\{l, i_2,\dots,i_m\}\cap\{1,\dots,n-m\} = \{i_2,\dots,i_{t-1}\}$ and so, by the inductive hypothesis, $\{l\; i_2\dots i_m\} \in A$. Clearly $\{i_1\; n-m+1\dots\widehat{l}\dots n\} \in A$, therefore $\{i_1\dots i_m\} \in A$. This completes the inductive step and the result follows. ■

Theorem 4.3 *There is an isomorphism*

$$\rho : \mathcal{O}_q(M_{m,n-m}) \longrightarrow \mathrm{Dhom}(\mathcal{G}_q(m,n),[n-m+1,\dots,n])$$

which is defined on generators by $\rho(X_{ij}) = \{j\; n-m+1\dots\widehat{n-i+1}\dots n\}$, *for* $1 \le i \le m$ *and* $1 \le j \le n-m$.

Proof In order to show that ρ is a homomorphism we have to show that the images of the X_{ij} under ρ still obey the relevant commutation relations. We will make repeated use of the anti-endomorphism $\Gamma \circ \tau$ defined before Lemma 4.2. There are four types of products to consider.

(1) Let $1 \leq i < l \leq m$ and $1 \leq j \leq n-m$. Then $X_{ij}X_{lj} = qX_{lj}X_{ij}$, and so we must show that $\rho(X_{ij})\,\rho(X_{lj}) = q\rho(X_{lj})\,\rho(X_{ij})$. Let $t = n+1-i$ and $s = n+1-l$. Note that $s < t$, and consider the product

$$\left[j \quad n-m+1 \ldots \widehat{t} \ldots n\right]\left[j \quad n-m+1 \ldots \widehat{s} \ldots n\right]$$

in $\mathcal{G}_q(m,n)$. We can think of this as a product in $\mathcal{O}_q(M_{m+1})$ where the rows are indexed by $1,\ldots,m+1$ and the columns by $j, n-m+1,\ldots,n$. Apply the anti-endomorphism $\Gamma\circ\tau$ to the commutation relation $X_{m+1,s}X_{m+1,t} = qX_{m+1,t}X_{m+1,s}$ we obtain:

$$\left[j\ n-m+1 \ldots \widehat{t} \ldots n\right]\left[j\ n-m+1 \ldots \widehat{s} \ldots n\right]$$
$$= q\left[j\ n-m+1 \ldots \widehat{s} \ldots n\right]\left[j\ n-m+1 \ldots \widehat{t} \ldots n\right].$$

Multiplying through this equation on the right by $[n-m+1,\ldots,n]^{-2}$ on each side and using Lemma 1.5 gives

$$\{j\ n-m+1 \ldots \widehat{t} \ldots n\}\{j\ n-m+1 \ldots \widehat{s} \ldots n\}$$
$$= q\{j\ n-m+1 \ldots \widehat{s} \ldots n\}\{j\ n-m+1 \ldots \widehat{t} \ldots n\};$$

that is, $\rho(X_{ij})\,\rho(X_{lj}) = q\rho(X_{lj})\,\rho(X_{ij})$.

(2) Let $1 \leq j < r \leq n-m$ and $1 \leq i \leq m$. Then $X_{ij}X_{ir} = qX_{ir}X_{ij}$. Let $t = n+1-i$ and, as in (1), think of the product

$$\left[j\ n-m+1 \ldots \widehat{t} \ldots n\right]\left[r\ n-m+1 \ldots \widehat{t} \ldots n\right]$$

as sitting inside $\mathcal{O}_q(M_{m+1})$ where the rows are indexed by $1,\ldots,m+1$ and the columns by $j, r, n-m+1, \ldots \widehat{t}, \ldots, n$. Then $\Gamma\circ\tau$ applied to the relation $X_{m+1,j}X_{m+1,r} = qX_{m+1,r}X_{m+1,j}$ in $\mathcal{O}_q(M_{m+1})$ gives us

$$\left[j\ n-m+1 \ldots \widehat{t} \ldots n\right]\left[r\ n-m+1 \ldots \widehat{t} \ldots n\right]$$
$$= q\left[r\ n-m+1 \ldots \widehat{t} \ldots n\right]\left[j\ n-m+1 \ldots \widehat{t} \ldots n\right].$$

Therefore, multiplying through this equation on the right by $[n-m+1,\ldots,n]^{-2}$ and using Lemma 1.5, we get

$$\{j\ n-m+1 \ldots \widehat{t} \ldots n\}\{r\ n-m+1 \ldots \widehat{t} \ldots n\}$$
$$= q\{r\ n-m+1 \ldots \widehat{t} \ldots n\}\{j \quad n-m+1 \ldots \widehat{t} \ldots n\};$$

that is, $\rho(X_{ij})\rho(X_{ir}) = q\rho(X_{ir})\rho(X_{ij})$

(3) Let $1 \le i < l \le m$, and $1 \le j < r \le n-m$. Then

$$X_{ij}X_{lr} = X_{lr}X_{ij} + \left(q - q^{-1}\right) X_{lj}X_{ir}.$$

Let $t = n+1-i$ and $s = n+1-l$. Note that $n-m+1 \le s < t \le n$, and that $j < r < s < t$. Consider the product

$$\left[j\ n-m+1 \ldots \widehat{t} \ldots n\right] \left[r\ n-m+1 \ldots \widehat{s} \ldots n\right]$$

as a product in $\mathcal{O}_q(M_{m+2})$, where the $m+2$ rows are indexed by $1, \ldots, m+2$ and the columns by $j, r, n-m+1, \ldots, n$.

The relation

$$[13]\,[24] = [24]\,[13] + \left(q - q^{-1}\right) [14]\,[23]$$

that we calculated earlier for $\mathcal{G}_q(2,4)$ shows that, in $\mathcal{O}_q(M_{m+2})$,

$$[I \mid js][I \mid rt] = [I \mid rt][I \mid js] + (q - q^{-1})[I \mid jt][I \mid rs]$$

where $I = \{m+1, m+2\}$, since $j < r < s < t$. By applying the anti-endomorphism $\Gamma \circ \tau$ to this relation, we obtain

$$\begin{aligned}
&\left[j\ n-m+1 \ldots \widehat{t} \ldots n\right] \left[r\ n-m+1 \ldots \widehat{s} \ldots n\right] \\
&\quad = \left[r\ n-m+1 \ldots \widehat{s} \ldots n\right] \left[j\ n-m+1 \ldots \widehat{t} \ldots n\right] \\
&\qquad + \left(q - q^{-1}\right) \left[j\ n-m+1 \ldots \widehat{s} \ldots n\right] \left[r\ n-m+1 \ldots \widehat{t} \ldots n\right]
\end{aligned}$$

in $\mathcal{G}_q(m,n)$. Multiplying through by $[n-m+1, \ldots, n]^{-2}$ and using Lemma 1.5 we get

$$\begin{aligned}
&\{j\ n-m+1 \ldots \widehat{t} \ldots n\}\{r\ n-m+1 \ldots \widehat{s} \ldots n\} \\
&\quad = \{r\ n-m+1 \ldots \widehat{s} \ldots n\}\{j\ n-m+1 \ldots \widehat{t} \ldots n\} \\
&\qquad + \left(q - q^{-1}\right) \{j\ n-m+1 \ldots \widehat{s} \ldots n\}\{r\ n-m+1 \ldots \widehat{t} \ldots n\};
\end{aligned}$$

that is, $\rho(X_{ij})\rho(X_{lr}) = \rho(X_{lr})\rho(X_{ij}) + (q - q^{-1})\rho(X_{lj})\rho(X_{ir})$, as required.

(4) Let $1 \le i < l \le m$ and $1 \le j < r \le n-m$. Then

$$X_{ir}X_{lj} = X_{lj}X_{ir}.$$

Let $t = n+1-i$ and $s = n+1-l$ so that $n-m+1 \le s < t \le n$ and $j < r < s < t$. Arguing as in (3), the relation $[23][14] = [14][23]$ in $\mathcal{G}_q(2,4)$ produces, in $\mathcal{O}_q(M_{m+2})$, the relation

$$[I \mid rs][I \mid jt] = [I \mid jt][I \mid rs].$$

Applying $\Gamma \circ \tau$ to this relation gives

$$\left[r\ n-m+1 \ldots \widehat{t} \ldots n\right]\left[j\ n-m+1 \ldots \widehat{s} \ldots n\right]$$
$$= \left[j\ n-m+1 \ldots \widehat{s} \ldots n\right]\left[r\ n-m+1 \ldots \widehat{t} \ldots n\right].$$

Multiplying through by $[n-m+1, \ldots, n]^{-2}$ we get

$$\{r\ n-m+1 \ldots \widehat{t} \ldots n\}\{j\ n-m+1 \ldots \widehat{s} \ldots n\}$$
$$= \{j\ n-m+1 \ldots \widehat{s} \ldots n\}\{r\ n-m+1 \ldots \widehat{t} \ldots n\};$$

that is, $\rho(X_{ir})\rho(X_{lj}) = \rho(X_{lj})\rho(X_{ir})$, as required.

Thus, ρ extends to a homomorphism. The images of the generators under ρ generate $\mathrm{Dhom}(\mathcal{G}_q(m,n), [n-m+1, \ldots, n])$, by Lemma 4.2; so ρ is surjective. We show that ρ is injective by comparing Gelfand-Kirillov dimensions. If ρ was not injective, then $\mathrm{GKdim}(\mathrm{Dhom}(\mathcal{G}_q(m,n), [n-m+1, \ldots, n]) < \mathrm{GKdim}(\mathcal{O}_q(M_{m,n-m})) = m(n-m)$, since $\mathcal{O}_q(M_{m,n-m})$ is a domain. However, by Corollary 3.3 and Proposition 2.9, we know that $\mathrm{GKdim}(\mathrm{Dhom}(\mathcal{G}_q(m,n), [n-m+1, \ldots, n]) = \mathrm{GKdim}(\mathcal{G}_q(m,n)) - 1 = m(n-m)+1-1 = m(n-m)$. Thus, ρ is injective and hence ρ is an isomorphism. ▪

Corollary 4.4 *Let ϕ be the automorphism of $\mathcal{O}_q(M_{m,n-m})$ defined by $\phi(X_{ij}) = q^{-1}X_{ij}$, for $1 \le i \le m$ and $1 \le j \le n-m$. Then*

$$\mathcal{O}_q(M_{m,n-m})[y, y^{-1}; \phi] \longrightarrow \mathcal{G}_q(m,n)\left[[n-m+1, \ldots, n]^{-1}\right]$$

defined by $X_{ij} \mapsto \{j\, n-m+1 \ldots \widehat{n+1-i} \ldots n\}$ and $y \mapsto [n-m+1, \ldots, n]$ is an isomorphism of algebras.

Proof Recall from Lemma 3.2 that there is an isomorphism

$$\theta : \mathrm{Dhom}(\mathcal{G}_q(m,n), [n-m+1, \ldots, n])[y, y^{-1}; \sigma] \longrightarrow \mathcal{G}_q(m,n)\left[[n-m+1, \ldots, n]^{-1}\right]$$

given by $y \mapsto [n-m+1, \ldots, n]$ and $\{j\, n-m+1 \ldots \widehat{t} \ldots n\} \mapsto \{j\, n-m+1 \ldots \widehat{t} \ldots n\}$, where σ is the automorphism of $\mathrm{Dhom}(\mathcal{G}_q(m,n), [n-m+1, \ldots, n])$ given by conjugation by the quantum minor $[n-m+1, \ldots, n]$. On the other hand, by Theorem 4.3, there is an isomorphism $\rho : \mathcal{O}_q(M_{m,n-m}) \longrightarrow \mathrm{Dhom}(\mathcal{G}_q(m,n), [n-m+1, \ldots, n])$, and it is easy to see, by using Lemma 1.5, that the automorphism induced in $\mathcal{O}_q(M_{m,n-m})$ by σ via ρ is ϕ. Thus, ρ extends to an isomorphism

$$\overline{\rho} : \mathcal{O}_q(M_{m,n-m})[y, y^{-1}; \phi] \longrightarrow \mathrm{Dhom}(\mathcal{G}_q(m,n), [n-m+1, \ldots, n])[y, y^{-1}; \sigma]$$

such that $\overline{\rho}(y) = y$. Clearly, $\theta \circ \overline{\rho}$ is the desired isomorphism. ▪

Note that in [4] Fioresi proves a restricted version of Theorem 4.3. More specifically, operating over the ring $K[q, q^{-1}]$, where K is algebraically closed of characteristic zero and q is transcendental over K, she shows that $\mathcal{O}_q(M_n)$ is isomorphic to the subalgebra of $\mathcal{G}_q(n, 2n)[[n+1\dots 2n]^{-1}]$ generated by the elements $\{j \quad n+1\dots\widehat{i}\dots 2n\}$, but does not show that this subalgebra is the dehomogenisation of $\mathcal{G}_q(n, 2n)$ at $[n+1\dots 2n]$.

Example Let $S = \mathcal{G}_q(2,4)[[34]^{-1}]$. Then $\mathrm{Dhom}(\mathcal{G}_q(2,4), [34]) = S_0$ and S_0 is generated by the elements

$$[12]\,[34]^{-1}, \quad [13]\,[34]^{-1}, \quad [14]\,[34]^{-1}, \quad [23]\,[34]^{-1}, \quad [24]\,[34]^{-1}.$$

Recall that $\{ij\} = [ij]\,[34]^{-1}$. From the commutation relations for $\mathcal{G}_q(2,4)$ given in the introduction, we can calculate the following commutation relations:

$$\{13\}\{23\} = q\{23\}\{13\}; \quad \{13\}\{14\} = q\{14\}\{13\};$$

$$\{13\}\{24\} = \{24\}\{13\} + \left(q - q^{-1}\right)\{23\}\{14\};$$

$$\{14\}\{23\} = \{23\}\{14\}; \quad \{14\}\{24\} = q\{24\}\{14\}; \quad \{23\}\{24\} = q\{24\}\{23\}$$

and from the Quantum Plücker relation;

$$\{12\} = \{13\}\{24\} - q\{23\}\{14\}.$$

We can immediately see the correspondence (or we can use ρ to find the correspondence):

$$\begin{array}{ccc} \mathcal{O}_q(M(2)) & \longleftrightarrow & S_0 \\ X_{11} & \longleftrightarrow & \{13\} \\ X_{12} & \longleftrightarrow & \{23\} \\ X_{21} & \longleftrightarrow & \{14\} \\ X_{22} & \longleftrightarrow & \{24\} \\ D_q & \longleftrightarrow & \{12\}, \end{array}$$

and from Theorem 4.3

$$\mathrm{Dhom}(\mathcal{G}_q(2,4), [34]) \cong \mathcal{O}_q(M(2)).$$

5 $\mathcal{G}_q(m, n)$ as coinvariants of $\mathcal{O}_q(SL_m)$

Recall that the $m \times m$ *quantum special linear group*, $\mathcal{O}_q(SL_m)$, is defined by $\mathcal{O}_q(SL_m) := \mathcal{O}_q(M_m)/\langle D_q - 1\rangle$.

In this section we show that $\mathcal{G}_q(m, n)$ is the algebra of coinvariants of a natural left coaction of $\mathcal{O}_q(SL_m)$ on $\mathcal{O}_q(M_{mn})$. There is a natural epimorphism $\pi : \mathcal{O}_q(GL_m) \longrightarrow \mathcal{O}_q(SL_m)$ which sends D_q to 1. In order to distinguish generators in the various algebras, we will often denote the canonical generators in $\mathcal{O}_q(M_n)$ by X_{ij}, in $\mathcal{O}_q(M_{nm})$ by Y_{ij}, in $\mathcal{O}_q(M_{mn})$ by Z_{ij} and in $\mathcal{O}_q(GL_m)$ by T_{ij}. Further, set $U_{ij} := \pi(T_{ij}) \in \mathcal{O}_q(SL_m)$. Note that both $\mathcal{O}_q(GL_m)$ and $\mathcal{O}_q(SL_m)$ are Hopf algebras.

It is easy to check that one can define a morphism of algebras satisfying the following rule:

$$\lambda : \mathcal{O}_q(M_{mn}) \longrightarrow \mathcal{O}_q(GL_m) \otimes \mathcal{O}_q(M_{mn}), \quad Z_{ij} \mapsto \sum_{k=1}^{m} T_{ik} \otimes Z_{kj}$$

and that this induces a morphism of algebras

$$\Lambda : \mathcal{O}_q(M_{mn}) \longrightarrow \mathcal{O}_q(SL_m) \otimes \mathcal{O}_q(M_{mn}), \quad Z_{ij} \mapsto \sum_{k=1}^{m} U_{ik} \otimes Z_{kj}$$

where $\Lambda := (\pi \otimes \mathrm{id}) \circ \lambda$.

The morphisms λ and Λ endow $\mathcal{O}_q(M_{mn})$ with left comodule algebra structures over $\mathcal{O}_q(GL_m)$ and $\mathcal{O}_q(SL_m)$, respectively. Recall that if H is a Hopf algebra and M is a left H-comodule via the coaction $\gamma : M \longrightarrow H \otimes M$ then $m \in M$ is a *coinvariant* if $\gamma(m) = 1 \otimes m$. In this section we show that $\mathcal{G}_q(m,n)$ is the set of coinvariants of the $\mathcal{O}_q(SL_m)$-comodule $\mathcal{O}_q(M_{mn})$ under the comodule map Λ. In fact, this result is an easy consequence of [8, Theorem 6.6], once we have described the set-up of that paper.

The map $Y_{ij} \mapsto \sum_{k=1}^{m} Y_{ik} \otimes T_{kj}$ induces a morphism of algebras $\rho : \mathcal{O}_q(M_{nm}) \longrightarrow \mathcal{O}_q(M_{nm}) \otimes \mathcal{O}_q(GL_m)$ which endows $\mathcal{O}_q(M_{nm})$ with a right comodule algebra stucture over $\mathcal{O}_q(GL_m)$. Let $\mathcal{O}_q(V)$ denote the algebra $\mathcal{O}_q(M_{nm}) \otimes \mathcal{O}_q(M_{mn})$. The coactions λ and ρ defined above can be combined to give a left comodule structure on $\mathcal{O}_q(V)$ which we denote by γ. To be precise,

$$\gamma : \mathcal{O}_q(V) \longrightarrow \mathcal{O}_q(GL_m) \otimes \mathcal{O}_q(V)$$

is given by the rule

$$\gamma(a \otimes b) := \sum_{(a),(b)} S(a_1)b_{-1} \otimes a_0 \otimes b_0$$

for $a \in \mathcal{O}_q(M_{nm})$ and $b \in \mathcal{O}_q(M_{mn})$, where $\lambda(b) = \sum_{(b)} b_{-1} \otimes b_0$ and $\rho(a) = \sum_{(a)} a_0 \otimes a_1$. Here, we are using the Sweedler notation and S is the antipode of $\mathcal{O}_q(GL_m)$. In turn, this coaction induces a coaction $\Gamma : \mathcal{O}_q(V) \longrightarrow \mathcal{O}_q(SL_m) \otimes \mathcal{O}_q(V)$ given by $\Gamma := (\pi \otimes \mathrm{id}) \circ \gamma$; so that

$$\Gamma(a \otimes b) := \sum_{(a),(b)} \pi(S(a_1)b_{-1}) \otimes a_0 \otimes b_0.$$

The main results of [8] identify the coinvariants of the coactions γ and Γ. In particular, Theorem 6.6 of [8] identifies the coinvariants of the coaction Γ in the following way. There is a morphism of algebras $\mu : \mathcal{O}_q(M_n) \longrightarrow \mathcal{O}_q(V) = \mathcal{O}_q(M_{nm}) \otimes \mathcal{O}_q(M_{mn})$ given by $X_{ij} \mapsto \sum_{k=1}^{m} Y_{ik} \otimes Z_{kj}$. Let R denote $\mu(\mathcal{O}_q(M_n))$. It is proved in [6] that $R \cong \mathcal{O}_q(M_n)/I$, where I is the ideal generated by the $(m+1) \times (m+1)$ quantum minors of $\mathcal{O}_q(M_n)$. We have the following theorem.

Theorem 5.1 *[8, Theorem 6.6] Let G_1 and G_2 denote the respective grassmannian subalgebras of $\mathcal{O}_q(M_{nm})$ and $\mathcal{O}_q(M_{mn})$ generated by all the $m \times m$ quantum minors. The set of*

Γ*-coinvariants in* $\mathcal{O}_q(V) = \mathcal{O}_q(M_{nm}) \otimes \mathcal{O}_q(M_{mn})$ *is the subalgebra generated by* $G_1 \otimes G_2$ *and* R*. More precisely,*

$$(\mathcal{O}_q(M_{nm}) \otimes \mathcal{O}_q(M_{mn}))^{\mathrm{co}\,\mathcal{O}_q(SL_m)} = (G_1 \otimes G_2) \cdot R.$$

The result we are aiming for follows easily from this.

Theorem 5.2

$$(\mathcal{O}_q(M_{mn}))^{\mathrm{co}\,\mathcal{O}_q(SL_m)} = \mathcal{G}_q(m,n).$$

Proof It is easily seen that there is a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_q(M_{mn}) & \xrightarrow{i} & \mathcal{O}_q(M_{nm}) \otimes \mathcal{O}_q(M_{mn}) \\ \Lambda \downarrow & & \Gamma \downarrow \\ \mathcal{O}_q(SL_m) \otimes \mathcal{O}_q(M_{mn}) & \xrightarrow{\mathrm{id}\otimes i} & \mathcal{O}_q(SL_m) \otimes \mathcal{O}_q(M_{nm}) \otimes \mathcal{O}_q(M_{mn}) \end{array}$$

where i is the canonical injection. Moreover, let $j : \mathcal{O}_q(M_{nm}) \otimes \mathcal{O}_q(M_{mn}) \longrightarrow \mathcal{O}_q(M_{mn})$ be the canonical projection; that is,

$$j : \mathcal{O}_q(M_{nm}) \otimes \mathcal{O}_q(M_{mn}) \xrightarrow{p\otimes \mathrm{id}} k \otimes \mathcal{O}_q(M_{mn}) \cong \mathcal{O}_q(M_{mn})$$

where p is the projection modulo the irrelevant ideal of $\mathcal{O}_q(M_{nm})$. Clearly, we have that $j \circ i = \mathrm{id}$. We see from the above commutative diagram that, if $b \in \mathcal{O}_q(M_{mn})$ is a Λ-coinvariant, then $i(b) = 1 \otimes b$ is a Γ-coinvariant. Thus, it follows from Theorem 5.1 that $1 \otimes b \in (G_1 \otimes G_2).R$. Hence, $b = j(1 \otimes b) \in j(G_1 \otimes G_2)j(R)$. Clearly, $j(R) \subseteq k$ and $j((G_1 \otimes G_2)) \subseteq G_2$; and so $b \in G_2 = \mathcal{G}_q(m,n)$. This shows that $\mathcal{O}_q(M_{mn})^{\mathrm{co}\mathcal{O}_q(SL_m)} \subseteq \mathcal{G}_q(m,n)$. Since it is clear that an $m \times m$ quantum minor of $\mathcal{O}_q(M_{mn})$ is a Λ-coinvariant, the converse inclusion follows from the fact that Λ is a morphism of algebras. ∎

Note that Fioresi and Hacon, [5], have a version of this result, with the usual restrictions as described earlier in this paper.

6 $\mathcal{G}_q(m,n)$ is a maximal order

Let R be a noetherian domain with division ring of fractions Q. Then R is said to be a **maximal order** in Q if the following condition is satisfied: if T is a ring such that $R \subseteq T \subseteq Q$ and such that there exist nonzero elements $a, b \in R$ with $aTb \subseteq R$, then $T = R$. This condition is the natural noncommutative analogue of normality for commutative domains, see, for example, [11, Section 5.1].

Recall that an element d in a ring R is said to be *left regular* if $rd = 0$ implies that $r = 0$ for $r \in R$. The following is a general result that we will be able to apply to show that the quantum grassmannian $\mathcal{G}_q(m,n)$ is a maximal order.

Proposition 6.1 *Suppose that R is a noetherian domain with division ring of fractions Q. Suppose that $a, b \in R$ are nonzero normal elements such that $R[a^{-1}]$ and $R[b^{-1}]$ are both maximal orders, that b is left regular modulo aR and that $ab = \lambda ba$ for some central unit $\lambda \in R$. Then R is a maximal order.*

Proof First, we show that $R[a^{-1}] \cap R[b^{-1}] = R$. Suppose that this is not the case, and choose $q \in R[a^{-1}] \cap R[b^{-1}] \setminus R$. Write $q = ra^{-d} = sb^{-e}$ with $d, e \geq 1$ and $r \in R \setminus Ra$, $s \in R \setminus Rb$. Cross multiply to get $rb^e = \lambda^\bullet sa^d$ (remember that $ab = \lambda ba$). Since b is left regular modulo aR, this gives $r \in Ra$, a contradiction. Thus, $R[a^{-1}] \cap R[b^{-1}] = R$.

Now, to show that R is a maximal order, it is enough to show that if J is a nonzero ideal of R and $q \in Q$ with either $qJ \subseteq J$ or $Jq \subseteq J$ then $q \in R$, [11, Proposition 5.1.4]. Suppose, without loss of generality, that $qJ \subseteq J$. By assumption, $S := R[a^{-1}]$ and $T := R[b^{-1}]$ are maximal orders. Also, $SJ = JS$ is an ideal of S and $TJ = JT$ is an ideal of T. We have $qJS \subseteq JS$ and so $q \in S$. Similarly, $q \in T$. Thus, $q \in S \cap T = R$; and so R is a maximal order. ▪

Theorem 6.2 *$\mathcal{G}_q(m, n)$ is a maximal order.*

Proof We will apply the previous result to $R := \mathcal{G}_q(m, n)$ with $a := [1, \ldots, m]$ and $b := [n - m + 1, \ldots, n]$. Observe that b is normal by Lemma 1.5 and that a is normal by Corollary 1.2. Note that $ab = (-q)^\bullet ba$, by Lemma 1.5. First we observe that b is left regular modulo aR. The reason is that since a is the minimal minor in the preferred ordering, a basis for aR is given by preferred products that start with a. If $r \in R$ is such that $rb \in aR$, then when we write r as a linear combination of preferred products then multiplying each preferred product that occurs by b on the right still gives a preferred product, since b is the maximal element with respect to the preferred order. Thus, since $rb \in aR$ each of these preferred products must begin with a, and so the original ones also begin with a, hence $r \in aR$.

In Corollary 4.4, we have shown that $R[b^{-1}] \cong \mathcal{O}_q(M_{m,n-m})[y, y^{-1}; \phi]$ and so $R[b^{-1}]$ is a maximal order ([10, V. Proposition 2.5, IV. Proposition 2.1]). Also $R[a^{-1}]$ is a maximal order by using the isomorphism δ introduced in Section 1 and the fact that $R[b^{-1}]$ is a maximal order.

Thus, the hypotheses of Proposition 6.1 are satisfied, and we deduce that $\mathcal{G}_q(m, n)$ is a maximal order. ▪

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