Ring theoretic properties of quantum grassmannians

A C Kelly*, T H Lenagan and L Rigal†

Abstract

The $m \times n$ quantum grassmannian, $\mathcal{G}_q(m,n)$, with $m \leq n$, is the subalgebra of the algebra $\mathcal{O}_q(M_{mn})$ of quantum $m \times n$ matrices that is generated by the maximal $m \times m$ quantum minors. Several properties of $\mathcal{G}_q(m,n)$ are established. In particular, a k-basis of $\mathcal{G}_q(m,n)$ is obtained, and it is shown that $\mathcal{G}_q(m,n)$ is a noetherian domain of Gelfand-Kirillov dimension $m(n-m)+1$. The algebra $\mathcal{G}_q(m,n)$ is identified as the subalgebra of coinvariants of a natural left coaction of $\mathcal{O}_q(SL_m)$ on $\mathcal{O}_q(M_{mn})$ and it is shown that $\mathcal{G}_q(m,n)$ is a maximal order.

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Introduction

Fix a base field k, a nonzero scalar $q \in k$ and positive integers m, n with $m \leq n$. The *coordinate ring of quantum $m \times n$ matrices*, $\mathcal{O}_q(M_{mn})$, is the k-algebra generated by mn indeterminates X_{ij}, $1 \leq i \leq m$ and $1 \leq j \leq n$, subject to the following relations:

$$\begin{array}{rcl} X_{ij}X_{il} & = & qX_{il}X_{ij}, \\ X_{ij}X_{kj} & = & qX_{kj}X_{ij}, \\ X_{il}X_{kj} & = & X_{kj}X_{il}, \\ X_{ij}X_{kl} - X_{kl}X_{ij} & = & (q-q^{-1})X_{il}X_{kj}, \end{array} \tag{1}$$

for $1 \leq i < k \leq m$ and $1 \leq j < l \leq n$. It is well-known that $\mathcal{O}_q(M_{mn})$ can be presented as an iterated skew polynomial algebra over k with the generators added in lexicographic order. As a consequence of this presentation, it is easy to establish that $\mathcal{O}_q(M_{mn})$ is a noetherian domain of Gelfand-Kirillov dimension mn.

We will usually write $\mathcal{O}_q(M_n)$ for the algebra $\mathcal{O}_q(M_{nn})$. In this algebra the *quantum determinant*, $D_q = \det_q$ is defined by

$$D_q := \sum_{\sigma \in S_n} (-q)^{l(\sigma)} X_{1,\sigma(1)} \dots X_{n,\sigma(n)};$$

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from [13, Theorem 4.6.1], we know that D_q is in the centre of $\mathcal{O}_q(M_n)$.

Following [6], we use the notation $[I \mid J]$ to denote the quantum determinant of the quantum matrix subalgebra $\mathcal{O}_q(M_{I,J})$ of $\mathcal{O}_q(M_{mn})$ generated by the elements X_{ij} with $i \in I$ and $j \in J$, where I and J are index sets with $|I| = |J|$. The element $[I|J]$ is the *quantum minor* determined by the index sets I and J. If $I = \{i_1, \ldots, i_s\}$ and $J = \{j_1, \ldots, j_s\}$ where the indices are written in ascending order, then we will often denote $[I \mid J]$ by $[i_1 \ldots i_s \mid j_1 \ldots j_s]$.

In this paper we are interested in studying the ring theoretic properties of a certain subalgebra of $\mathcal{O}_q(M_{mn})$, the *quantum deformation of the homogeneous coordinate ring of the* $m \times n$ *grassmannian*, $\mathcal{G}_q(m, n)$. This is a deformation of the classical homogeneous coordinate ring of the grassmannian of m-dimensional k-subspaces of n-dimensional k-space and is generated by the maximal quantum minors of $\mathcal{O}_q(M_{mn})$; to be more specific, $\mathcal{G}_q(m, n)$ is the subalgebra of $\mathcal{O}_q(M_{mn})$ generated by the $m \times m$ quantum minors of $\mathcal{O}_q(M_{mn})$. In the quantum grassmannian $\mathcal{G}_q(m, n)$, any $m \times m$ quantum minor will involve rows $1, \ldots, m$ of the quantum matrix (X_{ij}) associated to $\mathcal{O}_q(M_{mn})$. Thus, to simplify notation, we may denote a quantum minor by its columns only; that is, the quantum minor given by the row set $\{1, \ldots, m\}$ and column set J will be denoted by $[J]$.

Example $\mathcal{G}_q(2, 4)$ is the k-algebra generated by the 2×2 minors of the 2×4 quantum matrix of $\mathcal{O}_q(M_{2,4})$: $[12], [13], [14], [23], [24]$ and $[34]$.
Using the relations for $\mathcal{O}_q(M_{mn})$ and [6, Lemma A.1] we can calculate the following commutation relations:

$$[12]\,[13] = q\,[13]\,[12]\,, \quad [12]\,[14] = q\,[14]\,[12]\,, \quad [12]\,[23] = q\,[23]\,[12]\,,$$

$$[12]\,[24] = q\,[24]\,[12]\,, \quad [12]\,[34] = q^2\,[34]\,[12]\,, \quad [13]\,[14] = q\,[14]\,[13]\,,$$

$$[13]\,[23] = q\,[23]\,[13]\,, \quad [13]\,[24] = [24]\,[13] + \left(q - q^{-1}\right)[14]\,[23]\,,$$

$$[13]\,[34] = q\,[34]\,[13]\,, \quad [14]\,[23] = [23]\,[14]\,, \quad [14]\,[24] = q\,[24]\,[14]\,,$$

$$[14]\,[34] = q\,[34]\,[14]\,, \quad [23]\,[24] = q\,[24]\,[23]\,, \quad [23]\,[34] = q\,[34]\,[23]\,,$$

$$[24]\,[34] = q\,[34]\,[24]\,,$$

and the Quantum Plücker relation

$$[12]\,[34] - q\,[13]\,[24] + q^2\,[14]\,[23] = 0.$$

Remark Quantum matrices and quantum grassmannians can be defined in an exactly similar manner over any commutative ring R with an invertible element $q \in R$. In the next section, we shall need to consider quantum grassmannians defined over a Laurent polynomial extension either of a field or of the integers.

1 Fioresi's commutation relations

In [3], Fioresi has developed useful commutation relations for the $m \times m$ quantum minors which generate $\mathcal{G}_q(m,n)$. However, Fioresi works in the following setting. The field k that she considers is required to be algebraically closed of characteristic zero, and the quantum matrix algebra that she considers is generated as an algebra over the ring $k[q,q^{-1}]$, where q is transcendental over k. The first thing that we need to do is to observe that these commutation relations hold over any field k and for any $0 \neq q \in k$. A couple of warnings about notation for readers comparing [3] with this paper. First, because of the choice of relations for $\mathcal{O}_q(M_{mn})$, it is necessary to replace q by q^{-1} in any relation taken from [3]. Secondly, Fioresi works with the quantum grassmannian defined by the maximal $m \times m$ minors of $\mathcal{O}_q(M_{nm})$; thus, in any maximal minor, she uses all of the m columns, and a generating quantum minor of the grassmannian is specified by choosing m rows. To deal with this second difference, we can think of both versions of the quantum grassmannian as being subalgebras in the quantum matrix algebra $\mathcal{O}_q(M_n)$ and observe that the transpose automorphism, τ, see [13, 3.7.1], transforms Fioresi's quantum grassmannian to our quantum grassmannian.

Recall the following total *lexicographic ordering* on quantum minors: $[j_1 j_2 \dots j_m] <_{\mathrm{lex}} [i_1 i_2 \dots i_m]$ if and only if there exists an index α such that $j_l = i_l$ for $l < \alpha$, but $j_\alpha < i_\alpha$.

Let $[I] = [i_1 \dots i_m]$ denote an $m \times m$ quantum minor. If $[I] \neq [1 \dots m]$, consider the least integer s such that $i_s > s$. Let $\sigma([I])$ be the quantum minor obtained from $[I]$ by replacing i_s by $i_s - 1$ and leaving the other indices unchanged. Obviously, $\sigma([I]) <_{\mathrm{lex}} [I]$. The *standard tower* of $[I]$ is the sequence of quantum minors $[I_N] >_{\mathrm{lex}} [I_{N-1}] >_{\mathrm{lex}} \dots >_{\mathrm{lex}} [I_1] >_{\mathrm{lex}} [I_0]$ where $[I_N] = [I]$, $[I_{l-1}] = \sigma([I_l])$, and $[I_0] = [1, \dots, r]$. If $[I] = [1 \dots r]$ then the standard tower is defined to be the single quantum minor $[I]$.

We will denote the version of the $m \times n$ quantum grassmannian constructed by Fioresi by $\mathcal{G}_h(m,n)$. Note also that the relations in [3] use h where we would use h^{-1}; thus we should interchange h and h^{-1}.

Proposition 1.1 *Let K be an algebraically closed field of characteristic zero, and let h be an indeterminate over K. Set $\mathcal{G}_h(m,n)$ to be the quantum grassmannian subalgebra of $\mathcal{O}_h(M(K[h,h^{-1}])_{mn})$. Let $I, J \subseteq \{1, \dots, n\}$ with $|I| = |J| = m$, and $[I] <_{\mathrm{lex}} [J]$. Set $s = m - |I \cap J|$. Then in $\mathcal{G}_h(m,n)$,*

$$[I]\,[J] = h^s\,[J]\,[I] + \sum_{[L] <_{\mathrm{lex}} [I]} \lambda_{[L]} \left(h - h^{-1}\right)^{i_{[L]}} (-h)^{j_{[L]}}\,[L]\,[L'] \,,$$

where $i_{[L]}, j_{[L]} \in \mathbb{N}$ and $\lambda_{[L]}$ is either 0 or 1, while L' is the set $(I \cap J) \cup ((I \cup J) \setminus L)$.

Proof In [3, Proposition 2.21 and Theorem 3.6], Fioresi obtains commutation relations of the above form, but with the products $[L]\,[L']$ on the right hand side of the equation above more carefully stated. In Proposition 2.21 she first obtains the result for the case that $I \cap J = \emptyset$. In this case, the quantum minors $[L]$ involved are members of the standard tower of $[I]$, and so $[L] <_{\mathrm{lex}} [I]$, as we require. The general case where $I \cap J \neq \emptyset$ is dealt with in Theorem 3.6. Set $[\widetilde{I}]$ to be the quantum minor obtained from columns $I\backslash(I\cap J)$, and similarly, define $[\widetilde{J}]$. Proposition 2.21 provides a commutation rule for $[\widetilde{I}][\widetilde{J}]$ with terms on the right hand side $[\widetilde{L}][\widetilde{L'}]$ where $[\widetilde{L}] <_{\text{lex}} [\widetilde{I}]$. In Theorem 3.6, a commutation rule with the same coefficients is then obtained for $[I]\,[J]$ by replacing each $[\widetilde{L}][\widetilde{L'}]$ by $[\widetilde{L}\cup(I\cap J)][\widetilde{L'}\cup(I\cap J)]$. Thus, all that needs to be done is to make the easy observation that if $[\widetilde{L}] <_{\text{lex}} [\widetilde{I}]$ then $[\widetilde{L}\cup(I\cap J)] <_{\text{lex}} [\widetilde{I}\cup(I\cap J)] = [I]$. ▪

Corollary 1.2 *Let k be any field and q any nonzero element of k. Set $\mathcal{G}_q(m,n)$ to be the quantum grassmannian subalgebra of $\mathcal{O}_q(M_{mn})$. Let $I, J \subseteq \{1,\ldots,n\}$ with $|I| = |J| = m$, and $[I] <_{\text{lex}} [J]$. Set $s = m - |I\cap J|$. Then in $\mathcal{G}_q(m,n)$,*

$$[I]\,[J] = q^s\,[J]\,[I] + \sum_{[L]<_{\text{lex}}[I]} \lambda_{[L]}\left(q-q^{-1}\right)^{i_{[L]}}(-q)^{j_{[L]}}\,[L]\,[L'],$$

where $\lambda_{[L]} \in k$, $i_{[L]}, j_{[L]} \in \mathbb{N}$ and $\lambda_{[L]}$ is either 0 or 1, while L' is the set $(I\cap J)\cup((I\cup J)\setminus L)$.

Proof Proposition 1.1 applies in the case that $K = \mathbb{C}$. In this case, observe that the coefficients of the monomials in the maximal minors are all in $\mathbb{Z}[h,h^{-1}]$; so that these relations hold in the quantum grassmannian over $\mathbb{Z}[h,h^{-1}]$. There is then a natural homomorphism from this quantum grassmannian to $\mathcal{G}_q(m,n)$, such that $z \mapsto z1_k$ for $z\in\mathbb{Z}$ and $h\mapsto q$, which produces the required relations. ▪

Recall that an element a of an algebra A is a *normal* element if $aA = Aa$. The next result follows immediately from the previous Corollary.

Corollary 1.3 *An $m\times m$ quantum minor $[I] \in \mathcal{G}_q(m,n)$ is normal modulo the ideal generated by the set $\{[J] \mid [J] <_{\text{lex}} [I]\}$.*

The algebra $\mathcal{O}_q(M_{mn})$ is a connected $\mathbb{N}$-graded algebra, graded by the total degree in the canonical generators. Since $\mathcal{G}_q(m,n)$ is a subalgebra generated by homogeneous elements of degree m with respect to this grading, $\mathcal{G}_q(m,n)$ inherits a connected $\mathbb{N}$-graded structure in which its canonical generators have degree one.

Theorem 1.4 *The quantum grassmannian $\mathcal{G}_q(m,n)$ is a noetherian domain.*

Proof The quantum grassmannian $\mathcal{G}_q(m,n)$ is generated by the $\binom{n}{m}$ quantum minors of size m in $\mathcal{O}_q(M_{mn})$. Denote these quantum minors by $u_1 <_{\text{lex}} u_2 <_{\text{lex}} \ldots <_{\text{lex}} u_{\binom{n}{m}}$. Then by Corollary 1.3, $\{u_1,\ldots,u_{\binom{n}{m}}\}$ is a normalising sequence of $\mathcal{G}_q(m,n)$; that is, u_1 is normal and u_l is normal modulo the ideal generated by $\{u_1,\ldots,u_{l-1}\}$, for $l>1$. The factor by the ideal generated by this normalising sequence is the base field; so the fact that $\mathcal{G}_q(m,n)$ is noetherian follows by repeated use of [1, Lemma 8.2].

Finally, $\mathcal{G}_q(m,n)$ is a domain since it is a subalgebra of $\mathcal{O}_q(M_{mn})$ which is a domain. ▪

Remark If A is a noetherian, connected $\mathbb{N}$-graded k-algebra such that every non-simple graded prime factor ring A/P contains a nonzero homogeneous normal element in $\oplus_{i\geq 1}(A/P)_i$ then we say that A has *enough normal elements* ([14]). Thus, the two previous results show that the quantum grassmannian has enough normal elements.

There is a useful isomorphism between $\mathcal{G}_q(m,n)$ and $\mathcal{G}_{q^{-1}}(m,n)$ which we now describe. Notice that, if $1 \leq i_1 < \cdots < i_m \leq n$, $\mathcal{G}_q(m,n)$ is isomorphic to the subalgebra of $\mathcal{O}_q(M_n)$ generated by the $m \times m$ minors that use rows $i_1, \ldots, i_m$, that is, the minors $[I|J]$ with $I = \{i_1, \ldots, i_m\}$ and $J \subseteq \{1, \ldots, n\}$, $|J| = m$. Let $A := \mathcal{O}_q(M_n)$ with generators X_{ij} and $A' := \mathcal{O}_{q^{-1}}(M_n)$ with generators X'_{ij}. Take a copy R of $\mathcal{G}_q(m,n)$ inside A generated by the $m \times m$ quantum minors that use the first m rows of A, and take a copy R' of $\mathcal{G}_{q^{-1}}(m,n)$ that uses the last m rows of A'. Following the proof of [7, Corollary 5.9], we see that there is an isomorphism $\delta : A \longrightarrow A'$ which takes $[I|J]$ to $[\omega_0 I|\omega_0 J]'$, where $[-|-]'$ denotes a quantum minor in $A' := \mathcal{O}_{q^{-1}}(M_n)$ and ω_0 is the longest element of the symmetric group S_n; that is, $\omega_0(i) = n - i + 1$. Note that the isomorphism δ restricted to R produces an isomorphism from R to R' that takes a generating minor $[I]$ to the minor $[\omega_0 I]'$. In particular, note that under this isomorphism, $[12\ldots m]$, the leftmost minor of $R = \mathcal{G}_q(m,n)$, is translated into the rightmost minor $[n-m+1\ldots n]'$ of the quantum grassmannian $R' = \mathcal{G}_{q^{-1}}(m,n)$. We denote this induced isomorphism from $\mathcal{G}_q(m,n)$ to $\mathcal{G}_{q^{-1}}(m,n)$ by δ also.

As an example of the use of the isomorphism δ, we record the following lemma which we need later.

Lemma 1.5 *Let $I \subseteq \{1, \ldots, n\}$ with $|I| = m$. Then*

$$[I]\,[n-m+1\ldots n] = q^s[n-m+1\ldots n]\,[I]$$

where $s = m - |I \cap \{n-m+1, \ldots, n\}|$, and thus $[n-m+1\ldots n]$ is normal in $\mathcal{G}_q(m,n)$.

Proof Note that $\omega_0\{n-m+1, \ldots, n\} = \{1, \ldots, m\}$. Note also that $|I \cap \{n-m+1, \ldots, n\}| = |\omega_0 I \cap \omega_0\{n-m+1, \ldots, n\}| = |\omega_0 I \cap \{1, \ldots, m\}|$.

By Corollary 1.2, $[1\ldots m][\omega_0 I] = q^s[\omega_0 I][1\ldots m]$. Applying δ to this equation gives $[n-m+1\ldots n]'[I]' = q^s[I]'[n-m+1\ldots n]'$ in $\mathcal{G}_{q^{-1}}(m,n)$. This can be rewritten as $[I]'[n-m+1\ldots n]' = q^{-s}[n-m+1\ldots n]'[I]'$ in $\mathcal{G}_{q^{-1}}(m,n)$. Finally, replacing q^{-1} by q, we obtain

$$[I][n-m+1\ldots n] = q^s[n-m+1\ldots n][I]$$

in $\mathcal{G}_q(m,n)$. ▪

2 A basis for $\mathcal{G}_q(m,n)$

In this section, we obtain a basis for $\mathcal{G}_q(m,n)$. This basis is a subset of the basis of preferred products of $\mathcal{O}_q(M_{mn})$ obtained in [6, Section 1]. First, we adapt the language used in that paper to the grassmannian subalgebra $\mathcal{G}_q(m,n)$. Recall from Section 1 that if J is an m-element subset of $\{1, \ldots, n\}$ then $[J]$ denotes the quantum minor $[1, \ldots, m \mid J]$

of $\mathcal{O}_q(M_{mn})$. Thus, let $m, n \in \mathbb{N}^*$ with $n \geq m$. We define a partial ordering on m-element subsets of $\{1, \ldots, n\}$.

Definition 2.1 *Let $A, B \subseteq \{1, \ldots, n\}$ with $|A| = m = |B|$. We define a partial ordering, denoted by $\leq_*$. Write A and B in ascending order:*

$$A = \{a_1 < a_2 < \cdots < a_m\} \qquad \textit{and} \qquad B = \{b_1 < b_2 < \cdots < b_m\}.$$

Define $A \leq_* B$ to mean that $a_i \leq b_i$ for $i = 1, \ldots, m$.

This naturally defines a partial ordering on the generators of $\mathcal{G}_q(m, n)$.

Definition 2.2 *Let $[I]$ and $[J]$ belong to the generating set of $\mathcal{G}_q(m, n)$. Then we write that $[I] \leq_c [J]$ if and only if $I \leq_* J$.*

For example, Figure 1 shows the ordering on generators of $\mathcal{G}_q(3, 6)$.

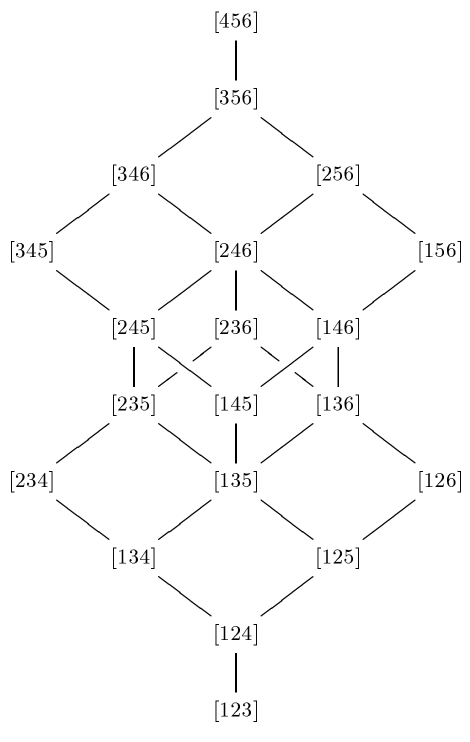

Figure 1: The partial ordering $\leq_c$ on $\mathcal{G}_q(3, 6)$

Recall that a *tableau* is a Young diagram with entries in each box. If each row of a tableau T has length m then we will say that T is an *m-tableau*. Here, we consider tableaux with entries from $\{1, \ldots, n\}$ and no repetitions in each row. An *allowable m-tableau* T is an

m-tableau with strictly increasing rows. If an allowable m-tableau T has rows $J_1, \ldots, J_s$, then T is *preferred* if and only if $J_1 \leq_* J_2 \leq_* \ldots \leq_* J_s$.

Let $I = \{m, m-1, \ldots, 1\}$ and let S be an m-tableau which has the same number of rows as T and such that each row of S is I. Then T is an allowable (preferred) m-tableau if and only if the bitableau $(S \mid T)$ is allowable (preferred) in the sense of [6]. With this in mind, we define the following ordering on allowable m-tableau. Let

$$T = \begin{pmatrix} J_1 \\ J_2 \\ \vdots \\ J_t \end{pmatrix}, \qquad S = \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_s \end{pmatrix}.$$

Then $T \prec S$ if $t > s$, or if $s = t$ and

$$\{J_1, \ldots, J_t\} <_{\text{lex}} \{L_1, \ldots, L_s\};$$

that is, there exists an index i such that $J_\alpha = L_\alpha$ for $\alpha < i$, but $J_i <_* L_i$.

Any allowable m-tableau determines a product of quantum minors in the quantum grassmannian as follows.

Definition 2.3 *For any (allowable) m-tableau*

$$T = \begin{pmatrix} J_1 \\ J_2 \\ \vdots \\ J_s \end{pmatrix},$$

define $[T] = [J_1][J_2] \ldots [J_s]$.

Definition 2.4 *The* content *of an m-tableau T is the multiset $\{1^{t_1}, 2^{t_2}, \cdots, n^{t_n}\}$, where t_i is the number of times i appears in T.*

We will use the content of a tableau to define a natural $\mathbb{Z}^n$-grading on the $m \times n$ quantum grassmannian. There is a $\mathbb{Z}^n$-grading on $\mathcal{O}_q(M_{mn})$ defined by assigning degree ε_j to X_{ij}, where ε_j for $j = 1, \ldots, n$ form the natural basis of $\mathbb{Z}^n$. Since the maximal minors of $\mathcal{O}_q(M_{mn})$ are homogeneous with respect to this basis, there is an induced $\mathbb{Z}^n$-grading on $\mathcal{G}_q(m,n)$: consider a product of minors $[T]$ in $\mathcal{G}_q(m,n)$, if the tableau T has content $\{1^{t_1}, 2^{t_2}, \ldots, n^{t_n}\}$, then $[T]$ is homogeneous of degree $(t_1, t_2, \ldots, t_n)$. Thus, the degree of a product is dependent on the number of times each column of the $m \times n$ quantum matrix appears in it.

Theorem 2.5 (Generalised Quantum Plücker Relations for Quantum grassmannians) *Let $J_1, J_2, K \subseteq \{1, 2, \ldots, n\}$ be such that $|J_1|, |J_2| \leq m$ and $|K| = 2m - |J_1| - |J_2| > m$. Then*

$$\sum_{K' \sqcup K'' = K} (-q)^{\ell(J_1;K') + \ell(K';K'') + \ell(K'';J_2)} [J_1 \sqcup K'][K'' \sqcup J_2] = 0,$$

where $\ell(I; J) = |\{(i,j) \in I \times J : i > j\}|$.

Proof We work in the algebra $\mathcal{O}_q(M_n)$ and apply [6, Proposition B2(a)] with $I_1 = I_2 = \{1, \ldots, m\} =: I$. Thus,

$$\sum_{K' \sqcup K'' = K} (-q)^{\ell(J_1;K')+\ell(K';K'')+\ell(K'';J_2)} [I|J_1 \sqcup K'][I|K'' \sqcup J_2] = 0,$$

since $|K| > m = |I_1 \cup I_2|$, see [6, B3]. This is the desired relation. ∎

Lemma 2.6 *Let T be an m-tableau with content γ and suppose that T is not preferred. Then*

(a) *T is not minimal with respect to $\prec$ among m-tableaux with content γ;*

(b) *$[T]$ can be expressed as a linear combination of products $[S]$, where each S is an m-tableau with content γ such that $S \prec T$.*

Proof Follow the proof of [6, Lemma 1.7]. Note that in the proof the only place where the shape of a bitableau might change is near the end of the proof where the right-hand side of the Exchange Formula is considered. In our situation, the right-hand side is zero, as noted in Theorem 2.5. ∎

Note that fixing the content of an m-tableau fixes its shape and thus fixes the number of rows in the m-tableau.

Let $\partial = (c_1, \ldots, c_n) \in \mathbb{N}^n$. Let V be the homogeneous component of degree ∂ in $\mathcal{G}_q(m,n)$. Note that V might be zero, and that this is the case if and only if there is no product $[T]$ where T is an m-tableau of content $(1^{c_1} \ldots n^{c_n})$. Also, an element of $\mathcal{G}_q(m,n)$ belongs to V if and only if it is a linear combination of products $[T]$, where T runs over all m-tableau with content $(1^{c_1} \ldots n^{c_n})$; that is, the products $[T]$, where T runs over all m-tableau with content $(1^{c_1} \ldots n^{c_n})$ span V.

Theorem 2.7 *Let $\partial = (c_1, \ldots, c_n) \in \mathbb{N}^n$, let V be the homogeneous component of $\mathcal{G}_q(m,n)$ with degree ∂, and set $\gamma = (1^{c_1} 2^{c_2} \cdots n^{c_n})$. The products $[T]$, as T runs over all preferred m-tableau with content γ, form a basis for V.*

Proof It is enough to prove that for any m-tableau T with content γ the product $[T]$ is a linear combination of products $[S]$ where S is a preferred m-tableau with content γ. Let $\mathcal{E}$ be the set of m-tableau with content γ; clearly, $\mathcal{E}$ is a finite set and we order it by $\prec$. We use induction on $\prec$ to show the result. Let $U \in \mathcal{E}$. If U is minimal, then it is preferred, by part (a) of the previous result. Otherwise, by part (b) of the previous result, $[U]$ is a linear combination of products $[S]$, where $S \in \mathcal{E}$ and $S \prec U$. Thus, by an induction argument applied to S, we may conclude that $[U]$ is a linear combination of products $[S]$ where S is a preferred m-tableau with content γ.

Recall that $\mathcal{G}_q(m,n)$ is a subalgebra of $\mathcal{O}_q(M_{mn})$ and notice that the products $[T]$, as T runs over all preferred m-tableaux of content γ, form a subset of the basis of $\mathcal{O}_q(M_{mn})$ constructed in [6]. Therefore, they are linearly independent and we have the result. ∎

Corollary 2.8 *The products $[T]$, as T runs over all preferred m-tableaux, form a basis for $\mathcal{G}_q(m,n)$.*

This basis can be used to calculate the Gelfand-Kirillov dimension of the $m \times n$ quantum grassmannian.

Consider the partial ordering $\leq_c$ on the generating minors of $\mathcal{G}_q(m,n)$. A *saturated path* between two minors $a <_c b$ will be an 'upwards path' $a = a_1 <_c a_2 <_c \ldots <_c a_l = b$ of minors such that no additional terms can be added; that is, for any index i there is no minor d such that $a_i <_c d <_c a_{i+1}$. The *length* of such a saturated path is defined to be l. For example, a saturated path between the minors $[134]$ and $[256]$ in $\mathcal{G}_q(3,6)$ is

$$[134] <_c [234] <_c [235] <_c [236] <_c [246] <_c [256].$$

The length of this saturated path is 6.

A *maximal path* is a saturated path between the two minors $[1 \ldots m]$ and $[n-m+1 \ldots n]$. It is easy to check that any maximal path has length $m(n-m)+1$.

Proposition 2.9 *Let $G = \mathcal{G}_q(m,n)$ and let α be the length of a maximal path in G. Then*

$$\operatorname{GKdim}(\mathcal{G}_q(m,n)) = \alpha = m(n-m)+1.$$

Proof Let V be the k-subspace of G spanned by the $m \times m$ minors which generate G. Then $\operatorname{GKdim}(G) = \overline{\lim} \log_n d_V(n)$ where $d_V(n) = \dim_k\left(\sum_{i=0}^{n} V^i\right)$. Let $a_1, a_2, \ldots, a_\alpha$ be a maximal path in G. Then $a_1^{s_1} a_2^{s_2} \ldots a_\alpha^{s_\alpha} \in V^{n+1}$ whenever $\sum_{i=1}^{\alpha} s_i = n+1$. The set $\{a_1^{s_1} a_2^{s_2} \ldots a_\alpha^{s_\alpha} \mid \Sigma s_i = n+1\}$ is linearly independent. Therefore

$$\dim_k\left(V^{n+1}\right) \geq |\{a_1^{s_1} a_2^{s_2} \ldots a_\alpha^{s_\alpha} \mid \Sigma s_i = n+1\}| = \binom{n+\alpha}{\alpha-1}$$

which is a polynomial in n of degree $\alpha - 1$. It follows that $\operatorname{GKdim}(G) \geq \alpha$.

Let $a_{i_1} \ldots a_{i_n} \in V^n$. By Theorem 2.7, $a_{i_1} \ldots a_{i_n}$ may be rewritten as a linear combination of preferred products from V^n.

There are finitely many maximal paths in $\mathcal{G}_q(m,n)$. Suppose there are c such paths and index them $1, \ldots, c$. Let $a_1 <_c a_2 <_c \ldots <_c a_\alpha$ be the ith maximal path and let $W_i^{(n)}$ denote the subspace generated by monomials $a_1^{s_1} \ldots a_\alpha^{s_\alpha}$ such that $\Sigma s_j = n$. The above observation shows that $V^n \subseteq \sum_{i=1}^{c} W_i^{(n)}$. Consider $\dim(W_i^{(n)})$. The products $a_1^{s_1} \ldots a_\alpha^{s_\alpha}$ such that $\Sigma s_j = n$ are linearly independent. Therefore

$$\dim(W_i^{(n)}) \;=\; |\{a_1^{s_1} \ldots a_\alpha^{s_\alpha} \mid \sum s_i = n\}| = |\{(s_1, \ldots, s_\alpha) \in \mathbb{N}^\alpha \mid \sum s_i = n\}|.$$

Therefore $\dim(W_i^{(n)}) = \dim(W_j^{(n)})$ for all $i, j \in \{1, \ldots, c\}$. Thus

$$\dim(V^n) \leq \dim\left(\sum_{i=1}^{c} W_i^{(n)}\right) \leq c \dim\left(W_1^{(n)}\right) = c\binom{n+\alpha-1}{\alpha-1}$$

and $d_V(n) \leq c \sum_{i=0}^{n} \binom{i+\alpha-1}{\alpha-1}$, a polynomial of degree α. It follows that $\operatorname{GKdim}(G) \leq \alpha$. Hence, $\operatorname{GKdim}(G) = \alpha = m(n-m)+1$. ∎

For example, $\operatorname{GKdim}(\mathcal{G}_q(2,4)) = 2(4-2)+1 = 5$.

3 Noncommutative Dehomogenisation

If R is a commutative $\mathbb{N}$-graded algebra, and x is a homogeneous nonzerodivisor in degree one, then the *dehomogenisation* of R at x is usually defined to be the factor algebra $R/(x-1)R$, [2, Appendix 16.D]. This definition is unsuitable in a noncommutative algebra if the element x is merely normal rather than central: in this case, the factor algebra is often too small to be useful. For example, let R be the quantum plane $k_q[x,y]$ with $xy = qyx$ and $q \neq 1$. Setting $x = 1$ forces $y = 0$; so that the factor algebra $R/\langle x-1\rangle$ is isomorphic to k rather than being an algebra of Gelfand-Kirillov dimension 1, as one might hope. However, in the commutative case, an alternative approach is to observe that the localised algebra $S := R[x^{-1}]$ is $\mathbb{Z}$-graded, $S = \oplus_{i\in\mathbb{Z}} S_i$, and that $S_0 \cong R/(x-1)R$. If x is a normal nonzerodivisor of degree one in a noncommutative $\mathbb{N}$-graded algebra $R = \oplus_{i\in\mathbb{N}} R_i$, then one can form the Ore localisation $R[x^{-1}] =: S$, and then this second approach does yield a useful algebra in the noncommutative case. Indeed, for $i, j \in \mathbb{N}$ denote by $R_i x^{-j}$ the k-subspace of elements of S that can be written as rx^{-j} with $r \in R_i$; clearly, $R_i x^{-j} \subseteq R_{i+1}x^{-(j+1)}$. For $l \in \mathbb{Z}$, set $S_l = \sum_{t\geq 0} R_{l+t}x^{-t} = \cup_{t\geq 0} R_{l+t}x^{-t}$. Then S is a $\mathbb{Z}$-graded algebra with $S = \oplus_{l\in\mathbb{Z}} S_l$.

Definition 3.1 *Let $R = \oplus R_i$ be an $\mathbb{N}$-graded k-algebra and let x be a regular homogeneous normal element of R of degree one. Then the* dehomogenisation *of R at x, written $\mathrm{Dhom}(R,x)$, is defined to be the zero degree subalgebra S_0 of the $\mathbb{Z}$-graded algebra $S := R[x^{-1}]$.*

It is easy to check that $\mathrm{Dhom}(R,x) = \sum_{i=0}^{\infty} R_i x^{-i} = \cup_{i=0}^{\infty} R_i x^{-i}$. In particular, if $R = k[R_1]$ then $\mathrm{Dhom}(R,x) = \sum_{i=0}^{\infty} (R_1 x^{-1})^i = \cup_{i=0}^{\infty} (R_1 x^{-1})^i$, and further, if $R_1 = ka_1 + \cdots + ka_s$ then $\mathrm{Dhom}(R,x) = k[a_1x^{-1}, \ldots, a_sx^{-1}]$.

Denote by σ the automorphism of S given by $\sigma(s) = xsx^{-1}$ for $s \in S$. Note that σ induces an automorphism of S_0, also denoted by σ.

Lemma 3.2 *Let R be an $\mathbb{N}$-graded algebra and let x be a regular normal homogeneous element of degree 1. Then there is an isomorphism*

$$\theta : \mathrm{Dhom}(R,x)[y, y^{-1}; \sigma] \longrightarrow R[x^{-1}]$$

which is the identity on $\mathrm{Dhom}(R,x)$ and sends y to x.

Proof The existence of θ is clear from the universal property of skew-Laurent extensions. It is easy to check that θ is an isomorphism. ▪

Some properties of dehomogenisation follow in an elementary way from this result.

Corollary 3.3 *Let $R = \oplus_{i\geq 0} R_i$ be an $\mathbb{N}$-graded algebra and let x be a regular homogeneous normal element of degree one.*
(i) R is a domain if and only if $\mathrm{Dhom}(R,x)$ is a domain.
(ii) If R is noetherian then $\mathrm{Dhom}(R,x)$ is noetherian.
(iii) If R is locally finite (that is, $\dim(R_i) < \infty$ for all $i \in \mathbb{N}$) then $\mathrm{GKdim}(R) = \mathrm{GKdim}(\mathrm{Dhom}(R,x)) + 1$.

Proof Point (i) follows at once from the isomorphism in Lemma 3.2.
(ii) If R is noetherian then so is $R[x^{-1}]$ and thus $\mathrm{Dhom}(R,x)[y,y^{-1};\sigma]$ is noetherian, by Lemma 3.2. As is well-known, since σ is an automorphism of $\mathrm{Dhom}(R,x)$, this implies that $\mathrm{Dhom}(R,x)$ is noetherian.
(iii) Let σ be the automorphism of R induced by conjugation by x. It is clear that σ is a graded automorphism; and so from the local finiteness of R, we see that the elements x^i, for $i \geq 1$, are local normal elements in the sense of [9, p168]. By using [9, 12.4.4], it follows that $\mathrm{GKdim}(R[x^{-1}]) = \mathrm{GKdim}(R)$. On the other hand, the automorphism σ induced on S_0 by conjugation by x in S is locally algebraic in the sense of [9, p164]. Indeed, $S_0 = \cup_{t\geq 0} R_t x^{-t}$ and for all $t \in \mathbb{N}$ the k-subspace $R_t x^{-t}$ is a finite dimensional σ-stable subspace of S_0. It follows from [9, p164] that $\mathrm{GKdim}(S_0[y,y^{-1};\sigma]) = \mathrm{GKdim}(S_0)+1$. The conclusion follows from Lemma 3.2. ∎

4 Dehomogenisation of $\mathcal{G}_q(m,n)$

In the classical commutative theory it is a well-known and basic result that the dehomogenisation of the homogeneous coordinate ring of the $m \times n$ grassmannian at the minor $[n-m+1,\dots,n]$ is isomorphic to the coordinate ring of $m \times (n-m)$ matrices; that is,

$$\frac{\mathcal{O}(\mathcal{G}(m,n))}{\langle [n-m+1,\dots,n]-1\rangle} \cong \mathcal{O}(M_{m,n-m}(k)).$$

In this section, we show that the corresponding result holds for $\mathcal{G}_q(m,n)$ when we use the noncommutative dehomogenisation defined in the previous section. Recall from Lemma 1.5 that $[n-m+1,\dots,n]$ is a normal element of $\mathcal{G}_q(m,n)$: in fact, it q-commutes with the other maximal minors, and this will be important in calculations.

Recall that we may consider $\mathcal{G}_q(m,n)$ to be a $\mathbb{N}$-graded algebra with each $m \times m$ quantum minor given degree 1. Set $x = [n-m+1,\dots,n]$ and $S := \mathcal{G}_q(m,n)[x^{-1}]$, and note that $\mathrm{Dhom}(\mathcal{G}_q(m,n),[n-m+1,\dots,n]) = S_0$ is generated by elements of the form $\{I\} := [I]\,[n-m+1,\dots,n]^{-1}$ with $I \subseteq \{1,\dots,n\}$ and $|I| = m$, see Section 3.

Now let u be a positive integer and consider $\mathcal{O}_q(M_u)$. If $I \subseteq \{1,\dots,u\}$ then $\widetilde{I} := \{1,\dots,u\}\backslash I$. In an exponent I denotes the sum of the indices occuring in the index set I.

Let D_q be the quantum determinant of $\mathcal{O}_q(M_u)$. Since D_q is a central element, we can invert it to form the $u \times u$ *quantum general linear group* $\mathcal{O}_q(GL_u) := \mathcal{O}_q(M_u)[D_q^{-1}]$. The algebra $\mathcal{O}_q(GL_u)$ is a Hopf algebra, with antipode S, and counit ε.

There is a useful antiendomorphism $\Gamma : \mathcal{O}_q(M_u) \longrightarrow \mathcal{O}_q(M_u)$ defined on generators by $\Gamma(X_{ij}) = (-q)^{i-j}[\widetilde{\{j\}}|\widetilde{\{i\}}]$, see [13, Corollary 5.2.2]. We need to know the effect of Γ on quantum minors. This is given in the following lemma, which is presumably well-known but we give a proof since we have been unable to find a clear exposition. Recall that $\Delta([I|J]) = \sum_{|K|=|I|}[I|K]\otimes[K|J]$, where Δ is the comultiplication map on $\mathcal{O}_q(M_u)$, by [12, (1.9)]. Recall also that $\varepsilon([I|J])$ equals 1 if $I = J$ and 0 otherwise.

Lemma 4.1 *Let $[I|J]$ be an $r \times r$ quantum minor in $\mathcal{O}_q(M_u)$. Then,*
(i) $S([I|J]) = (-q)^{I-J}[\widetilde{J}|\widetilde{I}]D_q^{-1}$
(ii) $\Gamma([I|J]) = (-q)^{I-J}[\widetilde{J}|\widetilde{I}]D_q^{r-1}$

Proof We establish the first claim by calculating the expression

$$\sum_{K,L}(-q)^{L-J}S([I|K])[K|L][\widetilde{J}|\widetilde{L}]D_q^{-1}$$

in two different ways.

First,

$$\begin{aligned}\sum_{K,L}(-q)^{L-J}S([I|K])[K|L][\widetilde{J}|\widetilde{L}]D_q^{-1} &= \sum_{K} S([I|K])\left\{\sum_{L}(-q)^{L-J}[K|L][\widetilde{J}|\widetilde{L}]D_q^{-1}\right\} \\ &= \sum_{K} S([I|K])\varepsilon([K|J])1 = S([I|J]),\end{aligned}$$

by using the first equality of [13, 4.4.3].

Secondly,

$$\begin{aligned}\sum_{K,L}(-q)^{L-J}S([I|K])[K|L][\widetilde{J}|\widetilde{L}]D_q^{-1} &= \sum_{L}\left\{\sum_{K} S([I|K])[K|L]\right\}(-q)^{L-J}[\widetilde{J}|\widetilde{L}]D_q^{-1} \\ &= \sum_{L}\varepsilon([I|L])(-q)^{L-J}[\widetilde{J}|\widetilde{L}]D_q^{-1} \\ &= (-q)^{I-J}[\widetilde{J}|\widetilde{I}]D_q^{-1},\end{aligned}$$

by using the defining property of the antipode.

The second claim follows easily from the first, since $S([I|J]) = \Gamma([I|J])D_q^{-r}$ for $r \times r$ quantum minors $[I|J]$. This is easily established from the fact that it holds on the generators X_{ij} and that S and Γ are anti-endomorphisms. ▪

We will need the anti-endomorphism $\Gamma\circ\tau : \mathcal{O}_q(M_u) \longrightarrow \mathcal{O}_q(M_u)$ defined by $\Gamma\circ\tau(X_{ij}) = (-q)^{j-i}[\widetilde{\{i\}}|\widetilde{\{j\}}]$ for $1 \le i, j \le u$. Here, τ is the transposition automorphism given in [13, Proposition 3.7.1(1)]. Note that, by Lemma 4.1, the effect of $\Gamma\circ\tau$ on the $r \times r$ quantum minor $[I \mid J]$ is given by $\Gamma\circ\tau([I \mid J]) = (-q)^{J-I}[\widetilde{I} \mid \widetilde{J}]D_q^{r-1}$.

Given $I = \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$ the set $I\backslash\{i_k\}$ is denoted by $\{i_1, \ldots, \widehat{i_k}, \ldots, i_m\}$. Given two sets $I, J \subseteq \{1, \ldots, n\}$ recall that

$$\ell\,(I; J) := |\{(i,j) \in I \times J : i > j\}|.$$

In the next proof, and throughout the paper, $(-q)^\bullet$ denotes a power of $-q$ that is not necessary to keep track of explicitly.

Lemma 4.2 *The k-algebra* $\mathrm{Dhom}(\mathcal{G}_q(m,n),[n-m+1,\dots,n]) = S_0$ *is generated as an algebra by the elements* $\{j \quad n-m+1\dots\widehat{i}\dots n\}$ *for* $1 \le j \le n-m < i \le n$.

Proof We know that S_0 is generated by the elements $\{I\} := [I]\,[n-m+1,\dots,n]^{-1}$, where $I \subseteq \{1,\dots,n\}$ and $|I| = m$. We show that each such element can be expressed as a k-linear combination of products of elements of the form $\{j \quad n-m+1\dots\widehat{i}\dots n\}$, where $1 \le j \le n-m < i \le n$. Denote by A the subalgebra of S_0 generated by the elements $\{j \quad n-m+1\dots\widehat{i}\dots n\}$.

Let $I = \{i_1 \le \dots \le i_m\} \ne \{n-m+1,\dots,n\}$ be an ordered subset of $\{1,\dots,n\}$ and let $2 \le t \le m+1$ be such that $i_t \ge n-m+1$ but $i_{t-1} < n-m+1$; that is, $I \cap \{1,\dots,n-m\} = \{i_1,\dots i_{t-1}\}$. We will use induction on t to show that $\{I\} \in A$.

If $t = 2$, then I is of the form $\{j \quad n-m+1\dots\widehat{i}\dots n\}$ and so $\{I\} \in A$. Consider a fixed $t \in \{3,\dots,m+1\}$ and suppose that the result is true for $t-1$. Now consider $[I] = [i_1\dots i_m]$ with $I \cap \{1,\dots,n-m\} = \{i_1,\dots i_{t-1}\}$. We use the generalised Quantum Plücker relations (Theorem 2.5) to rewrite the product $[n-m+1,\dots,n]\,[i_1\dots i_m]$.

Let $K = \{i_1, n-m+1,\dots,n\}$, $J_1 = \emptyset$ and $J_2 = \{i_2,\dots,i_m\}$. Then

$$\sum_{K' \sqcup K'' = K} (-q)^{\ell(K';K'')+\ell(K'';J_2)}\,[K']\,[K'' \sqcup J_2] = 0$$

where either

$$K' = \{n-m+1,\dots,n\} \ \text{ and } \ K'' = \{i_1\},$$

or

$$K' = \{i_1\} \cup \{n-m+1,\dots,\widehat{l},\dots,n\} \ \text{ and } \ K'' = \{l\}$$

where $n-m+1 \le l \le n$ and $l \notin \{i_2,\dots,i_m\}$. Let $S = \{n-m+1,\dots,n\} \setminus \{i_2,\dots,i_m\}$. By re-arranging the above equation, we obtain

$$[n-m+1,\dots,n]\,[i_1\dots i_m] = -\sum_{l \in S} (-q)^{\bullet} \left[i_1 \; n-m+1\dots\widehat{l}\dots n\right] [l \; i_2\dots i_m]\,.$$

Multiplying through by $[n-m+1,\dots,n]^{-2}$ from the right, and using Lemma 1.5 gives

$$\{i_1\dots i_m\} = \sum_{l \in S} \pm (-q)^{\bullet}\,\{i_1 \; n-m+1\dots\widehat{l}\dots n\}\{l \; i_2\dots i_m\}.$$

Now $\{l, i_2,\dots,i_m\} \cap \{1,\dots,n-m\} = \{i_2,\dots,i_{t-1}\}$ and so, by the inductive hypothesis, $\{l \; i_2\dots i_m\} \in A$. Clearly $\{i_1 \; n-m+1\dots\widehat{l}\dots n\} \in A$, therefore $\{i_1\dots i_m\} \in A$. This completes the inductive step and the result follows. ∎

Theorem 4.3 *There is an isomorphism*

$$\rho : \mathcal{O}_q(M_{m,n-m}) \longrightarrow \mathrm{Dhom}(\mathcal{G}_q(m,n),[n-m+1,\dots,n])$$

which is defined on generators by $\rho(X_{ij}) = \{j \; n-m+1\dots\widehat{n-i+1}\dots n\}$*, for* $1 \le i \le m$ *and* $1 \le j \le n-m$.

Proof In order to show that ρ is a homomorphism we have to show that the images of the X_{ij} under ρ still obey the relevant commutation relations. We will make repeated use of the anti-endomorphism $\Gamma \circ \tau$ defined before Lemma 4.2. There are four types of products to consider.

(1) Let $1 \leq i < l \leq m$ and $1 \leq j \leq n-m$. Then $X_{ij}X_{lj} = qX_{lj}X_{ij}$, and so we must show that $\rho(X_{ij})\rho(X_{lj}) = q\rho(X_{lj})\rho(X_{ij})$. Let $t = n+1-i$ and $s = n+1-l$. Note that $s < t$, and consider the product

$$\left[j \quad n-m+1\ldots\widehat{t}\ldots n\right]\left[j \quad n-m+1\ldots\widehat{s}\ldots n\right]$$

in $\mathcal{G}_q(m,n)$. We can think of this as a product in $\mathcal{O}_q(M_{m+1})$ where the rows are indexed by $1,\ldots,m+1$ and the columns by $j, n-m+1,\ldots,n$. Apply the anti-endomorphism $\Gamma\circ\tau$ to the commutation relation $X_{m+1,s}X_{m+1,t} = qX_{m+1,t}X_{m+1,s}$ we obtain:

$$\begin{aligned}&\left[j\ n-m+1\ldots\widehat{t}\ldots n\right]\left[j\ n-m+1\ldots\widehat{s}\ldots n\right]\\ &\qquad = q\left[j\ n-m+1\ldots\widehat{s}\ldots n\right]\left[j\ n-m+1\ldots\widehat{t}\ldots n\right].\end{aligned}$$

Multiplying through this equation on the right by $[n-m+1,\ldots,n]^{-2}$ on each side and using Lemma 1.5 gives

$$\begin{aligned}&\{j\ n-m+1\ldots\widehat{t}\ldots n\}\{j\ n-m+1\ldots\widehat{s}\ldots n\}\\ &\qquad = q\{j\ n-m+1\ldots\widehat{s}\ldots n\}\{j\ n-m+1\ldots\widehat{t}\ldots n\};\end{aligned}$$

that is, $\rho(X_{ij})\rho(X_{lj}) = q\rho(X_{lj})\rho(X_{ij})$.

(2) Let $1 \leq j < r \leq n-m$ and $1 \leq i \leq m$. Then $X_{ij}X_{ir} = qX_{ir}X_{ij}$. Let $t = n+1-i$ and, as in (1), think of the product

$$\left[j\ n-m+1\ldots\widehat{t}\ldots n\right]\left[r\ n-m+1\ldots\widehat{t}\ldots n\right]$$

as sitting inside $\mathcal{O}_q(M_{m+1})$ where the rows are indexed by $1,\ldots,m+1$ and the columns by $j, r, n-m+1,\ldots\widehat{t},\ldots,n$. Then $\Gamma\circ\tau$ applied to the relation $X_{m+1,j}X_{m+1,r} = qX_{m+1,r}X_{m+1,j}$ in $\mathcal{O}_q(M_{m+1})$ gives us

$$\begin{aligned}&\left[j\ n-m+1\ldots\widehat{t}\ldots n\right]\left[r\ n-m+1\ldots\widehat{t}\ldots n\right]\\ &\qquad = q\left[r\ n-m+1\ldots\widehat{t}\ldots n\right]\left[j\ n-m+1\ldots\widehat{t}\ldots n\right].\end{aligned}$$

Therefore, multiplying through this equation on the right by $[n-m+1,\ldots,n]^{-2}$ and using Lemma 1.5, we get

$$\begin{aligned}&\{j\ n-m+1\ldots\widehat{t}\ldots n\}\{r\ n-m+1\ldots\widehat{t}\ldots n\}\\ &\qquad = q\{r\ n-m+1\ldots\widehat{t}\ldots n\}\{j \quad n-m+1\ldots\widehat{t}\ldots n\};\end{aligned}$$

that is, $\rho(X_{ij})\rho(X_{ir}) = q\rho(X_{ir})\rho(X_{ij})$

(3) Let $1 \le i < l \le m$, and $1 \le j < r \le n-m$. Then

$$X_{ij}X_{lr} = X_{lr}X_{ij} + \left(q - q^{-1}\right) X_{lj}X_{ir}.$$

Let $t = n+1-i$ and $s = n+1-l$. Note that $n-m+1 \le s < t \le n$, and that $j < r < s < t$. Consider the product

$$\left[j\ n-m+1 \dots \widehat{t} \dots n\right] \left[r\ n-m+1 \dots \widehat{s} \dots n\right]$$

as a product in $\mathcal{O}_q(M_{m+2})$, where the $m+2$ rows are indexed by $1, \dots, m+2$ and the columns by $j, r, n-m+1, \dots, n$.

The relation

$$[13]\,[24] = [24]\,[13] + \left(q - q^{-1}\right) [14]\,[23]$$

that we calculated earlier for $\mathcal{G}_q(2,4)$ shows that, in $\mathcal{O}_q(M_{m+2})$,

$$[I \mid js][I \mid rt] = [I \mid rt][I \mid js] + (q - q^{-1})[I \mid jt][I \mid rs]$$

where $I = \{m+1, m+2\}$, since $j < r < s < t$. By applying the anti-endomorphism $\Gamma \circ \tau$ to this relation, we obtain

$$\begin{aligned}
&\left[j\ n-m+1 \dots \widehat{t} \dots n\right] \left[r\ n-m+1 \dots \widehat{s} \dots n\right] \\
&\quad = \left[r\ n-m+1 \dots \widehat{s} \dots n\right] \left[j\ n-m+1 \dots \widehat{t} \dots n\right] \\
&\qquad + \left(q - q^{-1}\right) \left[j\ n-m+1 \dots \widehat{s} \dots n\right] \left[r\ n-m+1 \dots \widehat{t} \dots n\right]
\end{aligned}$$

in $\mathcal{G}_q(m,n)$. Multiplying through by $[n-m+1, \dots, n]^{-2}$ and using Lemma 1.5 we get

$$\begin{aligned}
&\{j\ n-m+1 \dots \widehat{t} \dots n\}\{r\ n-m+1 \dots \widehat{s} \dots n\} \\
&\quad = \{r\ n-m+1 \dots \widehat{s} \dots n\}\{j\ n-m+1 \dots \widehat{t} \dots n\} \\
&\qquad + \left(q - q^{-1}\right) \{j\ n-m+1 \dots \widehat{s} \dots n\}\{r\ n-m+1 \dots \widehat{t} \dots n\};
\end{aligned}$$

that is, $\rho(X_{ij})\rho(X_{lr}) = \rho(X_{lr})\rho(X_{ij}) + (q - q^{-1})\rho(X_{lj})\rho(X_{ir})$, as required.

(4) Let $1 \le i < l \le m$ and $1 \le j < r \le n-m$. Then

$$X_{ir}X_{lj} = X_{lj}X_{ir}.$$

Let $t = n+1-i$ and $s = n+1-l$ so that $n-m+1 \le s < t \le n$ and $j < r < s < t$. Arguing as in (3), the relation $[23][14] = [14][23]$ in $\mathcal{G}_q(2,4)$ produces, in $\mathcal{O}_q(M_{m+2})$, the relation

$$[I \mid rs][I \mid jt] = [I \mid jt][I \mid rs].$$

Applying $\Gamma \circ \tau$ to this relation gives

$$\begin{aligned}\left[r\; n-m+1 \ldots \widehat{t} \ldots n\right] & [j\; n-m+1 \ldots \widehat{s} \ldots n] \\ & = [j\; n-m+1 \ldots \widehat{s} \ldots n] \left[r\; n-m+1 \ldots \widehat{t} \ldots n\right].\end{aligned}$$

Multiplying through by $[n-m+1, \ldots, n]^{-2}$ we get

$$\begin{aligned}\{r\; n-m+1 \ldots \widehat{t} \ldots n\} & \{j\; n-m+1 \ldots \widehat{s} \ldots n\} \\ & = \{j\; n-m+1 \ldots \widehat{s} \ldots n\}\{r\; n-m+1 \ldots \widehat{t} \ldots n\};\end{aligned}$$

that is, $\rho(X_{ir})\rho(X_{lj}) = \rho(X_{lj})\rho(X_{ir})$, as required.

Thus, ρ extends to a homomorphism. The images of the generators under ρ generate $\mathrm{Dhom}(\mathcal{G}_q(m,n), [n-m+1, \ldots, n])$, by Lemma 4.2; so ρ is surjective. We show that ρ is injective by comparing Gelfand-Kirillov dimensions. If ρ was not injective, then $\mathrm{GKdim}(\mathrm{Dhom}(\mathcal{G}_q(m,n), [n-m+1, \ldots, n]) < \mathrm{GKdim}(\mathcal{O}_q(M_{m,n-m})) = m(n-m)$, since $\mathcal{O}_q(M_{m,n-m})$ is a domain. However, by Corollary 3.3 and Proposition 2.9, we know that $\mathrm{GKdim}(\mathrm{Dhom}(\mathcal{G}_q(m,n), [n-m+1, \ldots, n]) = \mathrm{GKdim}(\mathcal{G}_q(m,n)) - 1 = m(n-m)+1-1 = m(n-m)$. Thus, ρ is injective and hence ρ is an isomorphism. ▪

Corollary 4.4 *Let ϕ be the automorphism of $\mathcal{O}_q(M_{m,n-m})$ defined by $\phi(X_{ij}) = q^{-1}X_{ij}$, for $1 \le i \le m$ and $1 \le j \le n-m$. Then*

$$\mathcal{O}_q(M_{m,n-m})[y, y^{-1}; \phi] \longrightarrow \mathcal{G}_q(m,n)\left[[n-m+1, \ldots, n]^{-1}\right]$$

defined by $X_{ij} \mapsto \{j\, n-m+1 \ldots \widehat{n+1-i} \ldots n\}$ and $y \mapsto [n-m+1, \ldots, n]$ is an isomorphism of algebras.

Proof Recall from Lemma 3.2 that there is an isomorphism

$$\theta : \mathrm{Dhom}(\mathcal{G}_q(m,n), [n-m+1, \ldots, n])[y, y^{-1}; \sigma] \longrightarrow \mathcal{G}_q(m,n)\left[[n-m+1, \ldots, n]^{-1}\right]$$

given by $y \mapsto [n-m+1, \ldots, n]$ and $\{j\, n-m+1 \ldots \widehat{t} \ldots n\} \mapsto \{j\, n-m+1 \ldots \widehat{t} \ldots n\}$, where σ is the automorphism of $\mathrm{Dhom}(\mathcal{G}_q(m,n), [n-m+1, \ldots, n])$ given by conjugation by the quantum minor $[n-m+1, \ldots, n]$. On the other hand, by Theorem 4.3, there is an isomorphism $\rho : \mathcal{O}_q(M_{m,n-m}) \longrightarrow \mathrm{Dhom}(\mathcal{G}_q(m,n), [n-m+1, \ldots, n])$, and it is easy to see, by using Lemma 1.5, that the automorphism induced in $\mathcal{O}_q(M_{m,n-m})$ by σ via ρ is ϕ. Thus, ρ extends to an isomorphism

$$\overline{\rho} : \mathcal{O}_q(M_{m,n-m})[y, y^{-1}; \phi] \longrightarrow \mathrm{Dhom}(\mathcal{G}_q(m,n), [n-m+1, \ldots, n])[y, y^{-1}; \sigma]$$

such that $\overline{\rho}(y) = y$. Clearly, $\theta \circ \overline{\rho}$ is the desired isomorphism. ▪

Note that in [4] Fioresi proves a restricted version of Theorem 4.3. More specifically, operating over the ring $K[q, q^{-1}]$, where K is algebraically closed of characteristic zero and q is transcendental over K, she shows that $\mathcal{O}_q(M_n)$ is isomorphic to the subalgebra of $\mathcal{G}_q(n, 2n)[[n+1 \dots 2n]^{-1}]$ generated by the elements $\{j \quad n+1 \dots \widehat{i} \dots 2n\}$, but does not show that this subalgebra is the dehomogenisation of $\mathcal{G}_q(n, 2n)$ at $[n+1 \dots 2n]$.

Example Let $S = \mathcal{G}_q(2,4)[[34]^{-1}]$. Then $\mathrm{Dhom}(\mathcal{G}_q(2,4), [34]) = S_0$ and S_0 is generated by the elements

$$[12]\,[34]^{-1}, \quad [13]\,[34]^{-1}, \quad [14]\,[34]^{-1}, \quad [23]\,[34]^{-1}, \quad [24]\,[34]^{-1}.$$

Recall that $\{ij\} = [ij]\,[34]^{-1}$. From the commutation relations for $\mathcal{G}_q(2,4)$ given in the introduction, we can calculate the following commutation relations:

$$\{13\}\{23\} = q\{23\}\{13\}; \quad \{13\}\{14\} = q\{14\}\{13\};$$

$$\{13\}\{24\} = \{24\}\{13\} + \left(q - q^{-1}\right)\{23\}\{14\};$$

$$\{14\}\{23\} = \{23\}\{14\}; \quad \{14\}\{24\} = q\{24\}\{14\}; \quad \{23\}\{24\} = q\{24\}\{23\}$$

and from the Quantum Plücker relation;

$$\{12\} = \{13\}\{24\} - q\{23\}\{14\}.$$

We can immediately see the correspondence (or we can use ρ to find the correspondence):

$$\begin{array}{ccc}
\mathcal{O}_q(M(2)) & \longleftrightarrow & S_0 \\
X_{11} & \longleftrightarrow & \{13\} \\
X_{12} & \longleftrightarrow & \{23\} \\
X_{21} & \longleftrightarrow & \{14\} \\
X_{22} & \longleftrightarrow & \{24\} \\
D_q & \longleftrightarrow & \{12\},
\end{array}$$

and from Theorem 4.3

$$\mathrm{Dhom}(\mathcal{G}_q(2,4), [34]) \cong \mathcal{O}_q(M(2)).$$

5 $\mathcal{G}_q(m,n)$ as coinvariants of $\mathcal{O}_q(SL_m)$

Recall that the $m \times m$ *quantum special linear group*, $\mathcal{O}_q(SL_m)$, is defined by $\mathcal{O}_q(SL_m) := \mathcal{O}_q(M_m)/\langle D_q - 1\rangle$.

In this section we show that $\mathcal{G}_q(m,n)$ is the algebra of coinvariants of a natural left coaction of $\mathcal{O}_q(SL_m)$ on $\mathcal{O}_q(M_{mn})$. There is a natural epimorphism $\pi : \mathcal{O}_q(GL_m) \longrightarrow \mathcal{O}_q(SL_m)$ which sends D_q to 1. In order to distinguish generators in the various algebras, we will often denote the canonical generators in $\mathcal{O}_q(M_n)$ by X_{ij}, in $\mathcal{O}_q(M_{nm})$ by Y_{ij}, in $\mathcal{O}_q(M_{mn})$ by Z_{ij} and in $\mathcal{O}_q(GL_m)$ by T_{ij}. Further, set $U_{ij} := \pi(T_{ij}) \in \mathcal{O}_q(SL_m)$. Note that both $\mathcal{O}_q(GL_m)$ and $\mathcal{O}_q(SL_m)$ are Hopf algebras.

It is easy to check that one can define a morphism of algebras satisfying the following rule:

$$\lambda : \mathcal{O}_q(M_{mn}) \longrightarrow \mathcal{O}_q(GL_m) \otimes \mathcal{O}_q(M_{mn}), \quad Z_{ij} \mapsto \sum_{k=1}^{m} T_{ik} \otimes Z_{kj}$$

and that this induces a morphism of algebras

$$\Lambda : \mathcal{O}_q(M_{mn}) \longrightarrow \mathcal{O}_q(SL_m) \otimes \mathcal{O}_q(M_{mn}), \quad Z_{ij} \mapsto \sum_{k=1}^{m} U_{ik} \otimes Z_{kj}$$

where $\Lambda := (\pi \otimes \mathrm{id}) \circ \lambda$.

The morphisms λ and Λ endow $\mathcal{O}_q(M_{mn})$ with left comodule algebra structures over $\mathcal{O}_q(GL_m)$ and $\mathcal{O}_q(SL_m)$, respectively. Recall that if H is a Hopf algebra and M is a left H-comodule via the coaction $\gamma : M \longrightarrow H \otimes M$ then $m \in M$ is a *coinvariant* if $\gamma(m) = 1 \otimes m$. In this section we show that $\mathcal{G}_q(m,n)$ is the set of coinvariants of the $\mathcal{O}_q(SL_m)$-comodule $\mathcal{O}_q(M_{mn})$ under the comodule map Λ. In fact, this result is an easy consequence of [8, Theorem 6.6], once we have described the set-up of that paper.

The map $Y_{ij} \mapsto \sum_{k=1}^{m} Y_{ik} \otimes T_{kj}$ induces a morphism of algebras $\rho : \mathcal{O}_q(M_{nm}) \longrightarrow \mathcal{O}_q(M_{nm}) \otimes \mathcal{O}_q(GL_m)$ which endows $\mathcal{O}_q(M_{nm})$ with a right comodule algebra stucture over $\mathcal{O}_q(GL_m)$. Let $\mathcal{O}_q(V)$ denote the algebra $\mathcal{O}_q(M_{nm}) \otimes \mathcal{O}_q(M_{mn})$. The coactions λ and ρ defined above can be combined to give a left comodule structure on $\mathcal{O}_q(V)$ which we denote by γ. To be precise,

$$\gamma : \mathcal{O}_q(V) \longrightarrow \mathcal{O}_q(GL_m) \otimes \mathcal{O}_q(V)$$

is given by the rule

$$\gamma(a \otimes b) := \sum_{(a),(b)} S(a_1)b_{-1} \otimes a_0 \otimes b_0$$

for $a \in \mathcal{O}_q(M_{nm})$ and $b \in \mathcal{O}_q(M_{mn})$, where $\lambda(b) = \sum_{(b)} b_{-1} \otimes b_0$ and $\rho(a) = \sum_{(a)} a_0 \otimes a_1$. Here, we are using the Sweedler notation and S is the antipode of $\mathcal{O}_q(GL_m)$. In turn, this coaction induces a coaction $\Gamma : \mathcal{O}_q(V) \longrightarrow \mathcal{O}_q(SL_m) \otimes \mathcal{O}_q(V)$ given by $\Gamma := (\pi \otimes \mathrm{id}) \circ \gamma$; so that

$$\Gamma(a \otimes b) := \sum_{(a),(b)} \pi(S(a_1)b_{-1}) \otimes a_0 \otimes b_0.$$

The main results of [8] identify the coinvariants of the coactions γ and Γ. In particular, Theorem 6.6 of [8] identifies the coinvariants of the coaction Γ in the following way. There is a morphism of algebras $\mu : \mathcal{O}_q(M_n) \longrightarrow \mathcal{O}_q(V) = \mathcal{O}_q(M_{nm}) \otimes \mathcal{O}_q(M_{mn})$ given by $X_{ij} \mapsto \sum_{k=1}^{m} Y_{ik} \otimes Z_{kj}$. Let R denote $\mu(\mathcal{O}_q(M_n))$. It is proved in [6] that $R \cong \mathcal{O}_q(M_n)/I$, where I is the ideal generated by the $(m+1) \times (m+1)$ quantum minors of $\mathcal{O}_q(M_n)$. We have the following theorem.

Theorem 5.1 *[8, Theorem 6.6] Let G_1 and G_2 denote the respective grassmannian subalgebras of $\mathcal{O}_q(M_{nm})$ and $\mathcal{O}_q(M_{mn})$ generated by all the $m \times m$ quantum minors. The set of*

Γ-coinvariants in $\mathcal{O}_q(V) = \mathcal{O}_q(M_{nm}) \otimes \mathcal{O}_q(M_{mn})$ is the subalgebra generated by $G_1 \otimes G_2$ and R. More precisely,

$$(\mathcal{O}_q(M_{nm}) \otimes \mathcal{O}_q(M_{mn}))^{\mathrm{co}\,\mathcal{O}_q(SL_m)} = (G_1 \otimes G_2) \cdot R.$$

The result we are aiming for follows easily from this.

Theorem 5.2

$$(\mathcal{O}_q(M_{mn}))^{\mathrm{co}\,\mathcal{O}_q(SL_m)} = \mathcal{G}_q(m,n).$$

Proof It is easily seen that there is a commutative diagram

$$\begin{array}{ccc}
\mathcal{O}_q(M_{mn}) & \stackrel{i}{\longrightarrow} & \mathcal{O}_q(M_{nm}) \otimes \mathcal{O}_q(M_{mn}) \\
\Lambda \downarrow & & \Gamma \downarrow \\
\mathcal{O}_q(SL_m) \otimes \mathcal{O}_q(M_{mn}) & \stackrel{\mathrm{id}\otimes i}{\longrightarrow} & \mathcal{O}_q(SL_m) \otimes \mathcal{O}_q(M_{nm}) \otimes \mathcal{O}_q(M_{mn})
\end{array}$$

where i is the canonical injection. Moreover, let $j : \mathcal{O}_q(M_{nm}) \otimes \mathcal{O}_q(M_{mn}) \longrightarrow \mathcal{O}_q(M_{mn})$ be the canonical projection; that is,

$$j : \mathcal{O}_q(M_{nm}) \otimes \mathcal{O}_q(M_{mn}) \stackrel{p\otimes \mathrm{id}}{\longrightarrow} k \otimes \mathcal{O}_q(M_{mn}) \cong \mathcal{O}_q(M_{mn})$$

where p is the projection modulo the irrelevant ideal of $\mathcal{O}_q(M_{nm})$. Clearly, we have that $j \circ i = \mathrm{id}$. We see from the above commutative diagram that, if $b \in \mathcal{O}_q(M_{mn})$ is a Λ-coinvariant, then $i(b) = 1 \otimes b$ is a Γ-coinvariant. Thus, it follows from Theorem 5.1 that $1 \otimes b \in (G_1 \otimes G_2).R$. Hence, $b = j(1 \otimes b) \in j(G_1 \otimes G_2)j(R)$. Clearly, $j(R) \subseteq k$ and $j((G_1 \otimes G_2)) \subseteq G_2$; and so $b \in G_2 = \mathcal{G}_q(m,n)$. This shows that $\mathcal{O}_q(M_{mn})^{\mathrm{co}\mathcal{O}_q(SL_m)} \subseteq \mathcal{G}_q(m,n)$. Since it is clear that an $m \times m$ quantum minor of $\mathcal{O}_q(M_{mn})$ is a Λ-coinvariant, the converse inclusion follows from the fact that Λ is a morphism of algebras. ▪

Note that Fioresi and Hacon, [5], have a version of this result, with the usual restrictions as described earlier in this paper.

6 $\mathcal{G}_q(m,n)$ is a maximal order

Let R be a noetherian domain with division ring of fractions Q. Then R is said to be a **maximal order** in Q if the following condition is satisfied: if T is a ring such that $R \subseteq T \subseteq Q$ and such that there exist nonzero elements $a, b \in R$ with $aTb \subseteq R$, then $T = R$. This condition is the natural noncommutative analogue of normality for commutative domains, see, for example, [11, Section 5.1].

Recall that an element d in a ring R is said to be *left regular* if $rd = 0$ implies that $r = 0$ for $r \in R$. The following is a general result that we will be able to apply to show that the quantum grassmannian $\mathcal{G}_q(m,n)$ is a maximal order.

Proposition 6.1 *Suppose that R is a noetherian domain with division ring of fractions Q. Suppose that $a, b \in R$ are nonzero normal elements such that $R[a^{-1}]$ and $R[b^{-1}]$ are both maximal orders, that b is left regular modulo aR and that $ab = \lambda ba$ for some central unit $\lambda \in R$. Then R is a maximal order.*

Proof First, we show that $R[a^{-1}] \cap R[b^{-1}] = R$. Suppose that this is not the case, and choose $q \in R[a^{-1}] \cap R[b^{-1}] \setminus R$. Write $q = ra^{-d} = sb^{-e}$ with $d, e \geq 1$ and $r \in R \backslash Ra$, $s \in R \backslash Rb$. Cross multiply to get $rb^e = \lambda^\bullet sa^d$ (remember that $ab = \lambda ba$). Since b is left regular modulo aR, this gives $r \in Ra$, a contradiction. Thus, $R[a^{-1}] \cap R[b^{-1}] = R$.

Now, to show that R is a maximal order, it is enough to show that if J is a nonzero ideal of R and $q \in Q$ with either $qJ \subseteq J$ or $Jq \subseteq J$ then $q \in R$, [11, Proposition 5.1.4]. Suppose, without loss of generality, that $qJ \subseteq J$. By assumption, $S := R[a^{-1}]$ and $T := R[b^{-1}]$ are maximal orders. Also, $SJ = JS$ is an ideal of S and $TJ = JT$ is an ideal of T. We have $qJS \subseteq JS$ and so $q \in S$. Similarly, $q \in T$. Thus, $q \in S \cap T = R$; and so R is a maximal order. ▪

Theorem 6.2 *$\mathcal{G}_q(m, n)$ is a maximal order.*

Proof We will apply the previous result to $R := \mathcal{G}_q(m, n)$ with $a := [1, \ldots, m]$ and $b := [n - m + 1, \ldots, n]$. Observe that b is normal by Lemma 1.5 and that a is normal by Corollary 1.2. Note that $ab = (-q)^\bullet ba$, by Lemma 1.5. First we observe that b is left regular modulo aR. The reason is that since a is the minimal minor in the preferred ordering, a basis for aR is given by preferred products that start with a. If $r \in R$ is such that $rb \in aR$, then when we write r as a linear combination of preferred products then multiplying each preferred product that occurs by b on the right still gives a preferred product, since b is the maximal element with respect to the preferred order. Thus, since $rb \in aR$ each of these preferred products must begin with a, and so the original ones also begin with a, hence $r \in aR$.

In Corollary 4.4, we have shown that $R[b^{-1}] \cong \mathcal{O}_q(M_{m,n-m})[y, y^{-1}; \phi]$ and so $R[b^{-1}]$ is a maximal order ([10, V. Proposition 2.5, IV. Proposition 2.1]). Also $R[a^{-1}]$ is a maximal order by using the isomorphism δ introduced in Section 1 and the fact that $R[b^{-1}]$ is a maximal order.

Thus, the hypotheses of Proposition 6.1 are satisfied, and we deduce that $\mathcal{G}_q(m, n)$ is a maximal order. ▪

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A. C. Kelly, T. H. Lenagan:
School of Mathematics, University of Edinburgh,
James Clerk Maxwell Building, King's Buildings, Mayfield Road,
Edinburgh EH9 3JZ, Scotland
E-mail: ACK@bosinternet.com, tom@maths.ed.ac.uk

L. Rigal:
Université Jean Monnet (Saint-Étienne), Faculté des Sciences et

Techniques, Département de Mathématiques
23 rue du Docteur Paul Michelon
42023 Saint-Étienne Cédex 2,
France
E-mail: Laurent.Rigal@univ-st-etienne.fr