Krull dimension of Generalized Weyl Algebras with non-commutative coefficients *

V V Bavula T H Lenagan

Department of Mathematics
Kiev University
Volodymyrs'ka Str, 64
Kiev 252617
Ukraine

sveta@kinr.kiev.ua

Department of Mathematics
University of Edinburgh
James Clerk Maxwell Building
King's Buildings
Mayfield Road
Edinburgh EH9 3JZ

tom@maths.ed.ac.uk

Abstract

We obtain formulae for the Krull dimension of the Generalized Weyl Algebra $T = R(\sigma, a)$, where R is a left noetherian ring.

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Introduction

Krull dimension, in the sense of Rentschler and Gabriel, is one of the most useful invariants in the study of noetherian rings. However, it is notoriously difficult to calculate its exact value in general. For example, until recently, the Krull dimension of the enveloping algebra of a finite dimensional simple Lie algebra was known only for $U(\mathrm{sl}(2))$, [16], and $U(\mathrm{sl}(3))$, [13]. In this paper, we consider the problem of calculating the Krull dimension of the Generalised Weyl Algebras, introduced by the first author in [1]. A start has already been made on this problem in [7] where the Krull dimension of Generalised Weyl Algebras over a commutative noetherian base ring is calculated. However, in order to make progress on iterated Generalised Weyl Algebras it is necessary first to consider the case of a Generalised Weyl Algebra over a noncommutative base ring, and this is the problem we answer in this paper.

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Generalised Weyl Algebras are defined in the following way. Let R be a ring, and let σ be an automorphism of the ring R. Let a be a central element of R. Then the Generalised Weyl Algebra $T = R(\sigma, a)$ is defined to be the ring generated over R by two indeterminates X, Y subject to the relations

$$X\alpha = \sigma(\alpha)X \quad Y\alpha = \sigma^{-1}(\alpha)Y \qquad YX = a \quad XY = \sigma(a),$$

for all $\alpha \in R$. The terminology, Generalised Weyl Algebra, is appropriate, since the Weyl algebras can be presented as iterated Generalised Weyl Algebras. The first author has used this fact to place the classification of the simple modules over the first Weyl algebra in a more general framework, see, for example, [2], [3] and [6].

In the case that $a = 1$, or more generally, a is a unit, then the Generalised Weyl Algebra T is a skew Laurent polynomial ring (with $Y = aX^{-1}$). In this case, Goodearl and the second author, [8], have obtained formulae for the Krull dimension of T. The methods of **clean** modules and **height** developed there are adapted here for use in the case of Generalised Weyl Algebras. However, when a is not a unit new phenomena appear which are not present in the skew Laurent case, mainly concerned with the problem that $aM = 0$ is possible for non-trivial modules. In order to deal with this problem, the methods of **stars** and **holes** introduced in [7] are adapted to the non-commutative setting.

Generalised Weyl Algebras are $\mathbb{Z}$-graded algebras: $T = \oplus v_i \otimes R$, where $v_i = X^i$, for $i \geq 0$, and $v_i = Y^{-i}$ for $i \leq 0$. Wherever possible, we exploit this graded structure of T.

A reference for most of the basic notions concerning Krull dimension that we need is [15, Chapter 6]. We denote the Krull dimension of a module M by $\mathcal{K}(M)$. If R is a left noetherian ring, and $T = R(\sigma, a)$ is a Generalised Weyl Algebra then it is known that

$$\mathcal{K}(R) \leq \mathcal{K}(T) \leq \mathcal{K}(R) + 1$$

see [7, Proposition 2.2], and the hard problem is to decide which of the two possibilities occur.

The strategy we adopt is as follows. The ring T, considered as a left module, is an induced module $T = T \otimes_R R$. This suggests studying the structure of induced modules. This is a strategy that has proved successful in earlier results on Krull dimension, see, for example, [9], [8] and [7], for the cases of differential operator rings, skew Laurent extensions, and Generalized Weyl Algebras over commutative base rings, respectively. The aim is to reach $T = T \otimes_R M$, via a chain of induced modules, starting from the induced modules $T(M) = T \otimes_R M$, where M runs through the simple modules of R. At each step, the induced modules are $\mathbb{Z}$-graded modules, and this leads us to suspect that the structure of $\mathbb{Z}$-graded modules should play a large rôle in the proof. Indeed, in the end, the result we obtain shows that the graded Krull dimension and the ordinary Krull dimension coincide.

However, we are not able to prove this directly, and then use this fact; it just emerges as a bonus at the end. It would be interesting to investigate which classes of graded rings can be shown to have the same graded and ordinary Krull dimensions.

The graded submodules that occur as subfactors of induced modules fall into two distinct classes: the graded modules with infinite support, and those with finite support. The former class can be dealt with by developing existing methods. There are two cases to consider: the graded modules which are infinite in both directions, and those which are either left or right restricted. It is the class of graded modules with finite support which causes most problems in the analysis; these modules are both left and right restricted. Specifically, we need to be able to deal with the extension theory for graded T-modules with finite support. In some of the earlier results on calculating Krull dimension progress was possible precisely because this subcategory of the module category was known to be semisimple, see, for example [16] and [11]. In the latter part of the proof of Theorem 3.5, we employ graded techniques to reduce the study of the extensions of the finite support modules that occur within certain factors of induced modules to the study of the modules over the ring $T/[m]$, where $[m]$ is the ideal generated by the powers X^m and Y^m, for a suitable m.

Here is a brief outline of the contents of the sections. In the first section, we develop the methods of leading coefficients which are needed to deal with the modules with infinite support. We find it necessary to introduce three types of modules to deal with this case: $T(M) = T \otimes_R M$, and $V_{\pm}(M)$, the latter being needed to deal with the one-sided restricted modules. In the second section, we investigate the structure of the R-module subfactors of induced modules, and introduce the ideas of **stars** and **holes** which are not needed in the skew Laurent case, but which are basic tools for modules over Generalized Weyl Algebras. One of the problems that arises is that the induced module $T(M)$ need not be a critical module, even if M is a critical R-module, see [8], for example. In the earlier studies, this problem is overcome by introducing **clean** modules, and then proving that there are enough clean modules in the sense that every module contains a clean submodule. Unfortunately, this is not the case here, because of the existence of restricted induced modules, and we find it necessary to have three kinds of cleanliness in order to have enough clean submodules.

Section three is the heart of the paper. Here we establish the basic inductive step, Theorem 3.5, from which all the formulae follow. It is here that we have to come to grips with the problems caused by the modules with finite support, and graded techniques are used to overcome these problems. After this, in section four, the required formulae follow in a (relatively) routine way. In section five, we specialize to the case of fully bounded noetherian base rings, and obtain the same theorem as that obtained for commutative base rings in [7]. Finally, in section six, we provide examples of each of the two possible kinds of behaviour for the Krull dimension of T. In a subsequent paper, [5], we exploit the results obtained here and in [7] to calculate the Krull dimension for many classes of algebras which can be presented as Generalised Weyl Algebras.

Throughout the paper, we use "module" to stand for "left module", and abbreviate "left Krull dimension" to "Krull dimension".

1 Induced modules

Let $T = R(\sigma, a)$ be a Generalised Weyl Algebra. The ring $T = \oplus_{n\in\mathbb{Z}} T_n$ is $\mathbb{Z}$-graded, where $T_n = Rv_n$ and

$$v_n = X^n \ (n > 0), \quad v_n = Y^{-n} \ (n < 0), \quad v_0 = 1.$$

Moreover,

$$v_n v_m = (n, m)v_{n+m}$$

for some $(n, m) \in R$. If $n > 0$ and $m > 0$, then

$$n \geq m : \qquad (n, -m) = \sigma^n(a) \dots \sigma^{n-m+1}(a), \quad (-n, m) = \sigma^{-n+1}(a) \dots \sigma^{-n+m}(a),$$

$$n \leq m : \qquad (n, -m) = \sigma^n(a) \dots \sigma(a), \quad (-n, m) = \sigma^{-n+1}(a) \dots a,$$

and $(n, m) = 1$ in all other cases.

The ring T contains the skew polynomial rings $T^+ = \oplus_{i\geq 0} T_i = R[X; \sigma]$ and $T^- = \oplus_{i\leq 0} T_i = R[Y; \sigma^{-1}]$.

The ring isomorphism

$$R(\sigma, a) \simeq R(\sigma^{-1}, \sigma(a)), \qquad X \leftrightarrow Y, \quad Y \leftrightarrow X, \quad r \leftrightarrow r, \quad r \in R,$$

is called the **$\pm$-symmetry** (or the left-right symmetry).

If the element $a \in Z(R)$ is not nilpotent, then the (multiplicative) submonoid S of $R\backslash 0$ generated by all $\sigma^i(a)$, $i \in \mathbb{Z}$, satisfies the (left and right) Ore condition in T. In other words, there exists the (left and right) localization $S^{-1}T = T_S$ of the ring T at S. Moreover, σ extends to R_S, and $T_S \simeq R_S[X, X^{-1}; \sigma]$ is the skew Laurent polynomial ring. A T-module M contains the S-**torsion** (or the a-**torsion**, for short) submodule $\mathrm{tor}(M) := \{m \in M \mid sm = 0 \text{ for some } s \in S\}$. A T-module M is called a-**torsion**, if $M = \mathrm{tor}(M)$; and a-**torsionfree**, if $\mathrm{tor}(M) = 0$. If a is a nilpotent element, then, by definition, any T-module is a-torsion.

Let M be an R-module and $\tau \in \mathrm{Aut}(R)$. The **twisted** module ${}^{\tau}M$ as an abelian group coincides with M and the action of R on M is given as follows: $rm := \tau(r)m$. We often write ${}_{\tau^i}M \equiv {}^{\tau^{-i}}M$, in order to avoid sign changes. Note that $v_i \otimes M \simeq {}_{\sigma^i}M$, as R-modules.

Let M be an R-module. The induced T-module

$$T(M) := T \otimes {}_R M = \oplus_{i\in\mathbb{Z}}\, v_i \otimes M$$

is the direct sum of R-submodules ${}_R v_i \otimes M \simeq {}_{\sigma^i}M$. The T-module $T(M)$ is $\mathbb{Z}$-graded:

$$T(M) = \oplus_{i\in\mathbb{Z}}\, T(M)_i, \quad \text{where } T(M)_i = v_i \otimes M.$$

Alternatively, we can write

$$T \otimes {}_R M = \ldots (Y^2 \otimes M) \oplus (Y \otimes M) \oplus (1 \otimes M) \oplus (X \otimes M) \oplus (X^2 \otimes M) \ldots.$$

It is also useful to keep in mind the following kind of pictures to help see what is happening.

$$\cdots \overset{v_{-2}\otimes M}{\bullet} \;\text{———}\; \overset{v_{-1}\otimes M}{\bullet} \;\text{———}\; \overset{v_0\otimes M}{\bullet} \;\text{———}\; \overset{v_1\otimes M}{\bullet} \;\text{———}\; \overset{v_2\otimes M}{\bullet} \;\text{———}\; \overset{v_3\otimes M}{\bullet} \cdots$$

$$\cdots \overset{Y^2\otimes M}{\bullet} \;\text{———}\; \overset{Y^1\otimes M}{\bullet} \;\text{———}\; \overset{1\otimes M}{\bullet} \;\text{———}\; \overset{X^1\otimes M}{\bullet} \;\text{———}\; \overset{X^2\otimes M}{\bullet} \;\text{———}\; \overset{X^3\otimes M}{\bullet} \cdots$$

Looking at the pictures, we see that multiplying $T \otimes M$ by elements from R preserves the grading, while multiplying by X shifts to the right, and multiplying by Y shifts to the left. If we are in a position to the right of $1 \otimes M$ then $X\cdot$ acts as a monomorphism. However, if we are to the left of $1 \otimes M$ then the action of $X\cdot$ depends on the action of some $\sigma^i(a)\cdot$. For example, consider $X \cdot (Y \otimes m)$, for some $m \in M$. We have

$$X \cdot (Y \otimes m) = XY \otimes m = \sigma(a) \otimes m = 1 \otimes \sigma(a)m.$$

In a similar manner, $Y\cdot$ acts as a monomorphism to the left of $1 \otimes M$, but depends on $\sigma^i(a)\cdot$ to the right. Thus, points where $\sigma^i(a)M = 0$ are especially important: for example, if $aM = 0$, then $Y \cdot (X \otimes M) = a \otimes M = 1 \otimes aM = 0$, and so

$$\mathcal{L}(M)_+ := \sum\nolimits_{i\geq 1} v_i \otimes M = \sum\nolimits_{i\geq 1} X^i \otimes M$$

which at first sight is only a module over $T^+ := R[X;\sigma]$, is in fact a T-module.

Integers i such that $a \cdot_{\sigma^i} M = 0$, or, equivalently $a \cdot (v_i \otimes M) = 0$, are called the **stars** of M, while the set of integers $i+1$, where i is a star, are called the **holes** of M. The structure of the induced module $T \otimes M$ depends to a large extent on whether the set of stars of M is empty, finite or infinite.

Let M be an R-module with $aM = 0$. The induced module $T \otimes {}_R M$ contains the T-submodule

$$\mathcal{L}(M)_+ := \sum\nolimits_{i\geq 1} v_i \otimes M = \sum\nolimits_{i\geq 1} X^i \otimes M,$$

and we set

$$V_-(M) := T \otimes {}_R M/\mathcal{L}(M)_+ = \oplus_{i\leq 0}\, v_i \otimes M.$$

and draw the picture

$$\underbrace{\cdots \overset{Y^3\otimes M}{\bullet} \;\text{———}\; \overset{Y^2\otimes M}{\bullet} \;\text{———}\; \overset{Y\otimes M}{\bullet} \;\text{———}\; \overset{1\otimes M}{\star}}_{V_-(M)} \;]\cdots \overset{X^1\otimes M}{\circ} \;\cdots$$

to illustrate this T-module.

Similarly, let N be an R-module with $\sigma(a)N = 0$. The induced module $T \otimes {}_R N$ contains the T-submodule

$$\mathcal{L}(N)_- := \sum\nolimits_{i\leq -1} v_i \otimes N = \sum\nolimits_{i\geq 1} Y^i \otimes M,$$

and we have the factor module

$$V_+(N) := T \otimes {}_R N/\mathcal{L}(N)_- = \oplus_{i\geq 0}\, v_i \otimes N.$$

which is represented by the picture

$$\cdots \overset{Y\otimes M}{\star} \cdots [\; \underbrace{\overset{1\otimes M}{\circ} \;\text{———}\; \overset{X\otimes M}{\bullet} \;\text{———}\; \overset{X^2\otimes M}{\bullet} \;\text{———}\; \overset{X^3\otimes M}{\bullet} \;\cdots}_{V_+(M)}$$

The T-module $V = V_+(N)$ is uniquely characterized by the following properties:

- it is a $\mathbb{Z}_0^+$-graded T-module; that is, $V = \oplus_{i\geq 0} V_i$ is a $\mathbb{Z}$-graded T-module,
- $V_0 = N$, and $TV_0 = V$,
- the map $X_V : V \to V$, given by $v \mapsto Xv$, is injective.

Similarly, the T-module $V = V_-(N)$ is uniquely characterized by the following properties:

- it is a $\mathbb{Z}_0^-$-graded T-module; that is, $V = \oplus_{i\leq 0} V_i$ is a $\mathbb{Z}$-graded T-module,
- $V_0 = N$, and $TV_0 = V$,
- the map $Y_V : V \to V$, given by $v \mapsto Yv$, is injective.

Given an R-module M, the nonzero elements of the induced module $T \otimes_R M$ can be written uniquely in the form

$$u = v_m \otimes u_m + v_{m+1} \otimes u_{m+1} + \cdots + v_n \otimes u_n,$$

where all $u_i \in M$ and both of u_m and u_n are nonzero. The u_i are called the **coefficients** of u. The $(+)$-**leading** coefficient of u is u_n, and the $(-)$-**leading** coefficient of u is u_m. The integer n is the $(+)$-**degree** of u, denoted by $\deg_+(u)$, and m is the $(-)$-**degree**, denoted by $\deg_-(u)$. The non-negative integer $n-m$ is the **length** of u, denoted by $l(u)$. The element 0 is defined to have $\deg_\pm(0) = \mp\infty$, leading coefficients 0 and length $-\infty$.

There are doubly infinite filtrations

$$\ldots \subseteq U_{-1} \subseteq U_0 \subseteq U_1 \subseteq \ldots$$

and

$$\ldots \supseteq V_{-1} \supseteq V_0 \supseteq V_1 \supseteq \ldots$$

on $T \otimes_R M$ given by the R-submodules

$$U_n = \{u \in T \otimes_R M \mid \deg_+(u) \leq n\},$$

and

$$V_n = \{v \in T \otimes_R M \mid \deg_-(v) \geq n\},$$

for $n \in \mathbb{Z}$.

Also,

$$U_n/U_{n-1} \simeq v_n \otimes M \simeq {}^{\sigma^{-n}}M = {}_{\sigma^n}M \text{ and } V_n/V_{n+1} \simeq v_n \otimes M \simeq {}^{\sigma^{-n}}M = {}_{\sigma^n}M,$$

for all n.

For a T-submodule N of $T \otimes_R M$ denote by $\lambda_-(N)$ the set of $(-)$-leading coefficients of elements of N which have non-positive $(-)$-degree. Similarly, $\lambda_+(N)$ denotes the set of $(+)$-leading coefficients of elements of N which have non-negative $(+)$-degree. The sets $\lambda_-(N), \lambda_+(N)$ are R-submodules of M.

For $i > 0$, let N_i be the set of all $(+)$-leading coefficients of U_i together with 0. Similarly, let N_{-i} be the set of all $(-)$-leading coefficients together with 0. Then

$$\ldots \geq N_{-i} \geq \ldots \geq N_{-1}, \quad N_1 \leq \ldots \leq N_i \leq \ldots,$$

are chains of R-modules and

$$\lambda_\pm(N) = \cup_{i \geq 1} N_{\pm i}.$$

The most important case of an induced module is the ring T, where we have

$${}_T T = \oplus_{i \in \mathbb{Z}} v_i R \simeq T \otimes_R R = \oplus_{i \in \mathbb{Z}} v_i \otimes R, \qquad v_i r \leftrightarrow v_i \otimes r,\, r \in R.$$

Lemma 1.1 *1. If I is a left ideal of T, then $\lambda_{\pm}(I) = I_{\pm n}$ for some $n > 0$.*

2. If I, J are left ideals of T with $J \subseteq I$ and $\lambda_+(J) = \lambda_+(I)$, $\lambda_-(J) = \lambda_-(I)$, then ${}_R(I/J)$ is a finitely generated R-module.

Proof. Straightforward. ∎

Lemma 1.2 *1. Let M an R-module that is a-torsionfree and let N be a nonzero T-submodule of $T \otimes_R M$. Then there exists a nonzero R-submodule L of M such that $T \otimes_R L$ embeds in N.*

2. Let M an R-module such that $\sigma(a)M = 0$ and let N be a nonzero T-submodule of $V_+(M)$. Then there exists a nonzero R-submodule L of ${}_{\sigma^i}M$, for some $i \geq 0$, such that $V_+(L)$ embeds in N. Similarly, if, instead, $aM = 0$ and N is a nonzero T-submodule of $V_-(M)$, then there exists a nonzero R-submodule L of ${}_{\sigma^i}M$ for some $i \leq 0$ such that $V_-(L)$ embeds in N.

Proof. The first statement is exactly [7, Lemma 2.6]. In the second case, the $T^{\pm}$-module $V_{\pm}(M)$ is isomorphic to the induced module $T^{\pm} \otimes_R M$. Now we may repeat the arguments of [8, Lemma 1.1]. ∎

A module N is a **subfactor** of a module M if there exist submodules $V \leq U$ in M such that $N = U/V$. Also, if $0 \neq V$ then N is a **minor subfactor** of M.

Proposition 1.3 *([7, 2.7]) Let M be a Noetherian R-module, and let I, J be T-submodules of $T \otimes_R M$ such that $I < J$. Suppose that N is a nonzero Noetherian R-module such that*

1. *$T \otimes_R N$ is isomorphic to a T-submodule subfactor of J/I. Then there exists a nonzero subfactor L of $\lambda_{\pm}(J)/\lambda_{\pm}(I)$ such that $L \leq {}_{\sigma^{\pm i}}N$, for some $i \geq 0$.*

2. *$V_+(N)$ is isomorphic to a T-submodule subfactor of J/I. Then there exists a nonzero subfactor L of $\lambda_+(J)/\lambda_+(I)$ such that $L \leq {}_{\sigma^i}N$, for some $i \geq 0$.*

3. *$V_-(N)$ is isomorphic to a T-submodule subfactor of J/I. Then there exists a nonzero subfactor L of $\lambda_-(J)/\lambda_-(I)$ such that $L \leq {}_{\sigma^{-i}}N$, for some $i \geq 0$.* ∎

2 Submodules of induced modules

The results in the next Proposition are easy to establish, by using the graded structure of Generalised Weyl Algebras.

Proposition 2.1 *Let $T = R(\sigma, a)$ be a generalized Weyl Algebra.*

1. *([3]) If R is left (right) Noetherian, then so is T.*
2. *([7, 2.2]) Let R be a left Noetherian ring. Then $\mathcal{K}(R) \leq \mathcal{K}(T) \leq \mathcal{K}(R) + 1$.* ∎

In order to study the Krull dimension of an induced module $T(M) = T \otimes_R M$, we find it necessary to deal with the R-module structure of certain T-module factors $T(M)/C$. The next result gives the details necessary to do this.

Proposition 2.2 *Let R be a ring and*

$$A \geq B_1 \geq B_2 \geq \ldots \geq B > 0$$

be a descending chain of R-submodules with $B \neq 0$. Suppose that α is an ordinal, and let $\ldots \leq A_n \leq A_{n+1} \leq \ldots \leq A$ be an ascending chain of R-submodules, with $\mathcal{K}(A_n) \leq \alpha$, for each n, and such that $A = \cup A_n$. Suppose that, for sufficiently large n, each factor A_{n+1}/A_n is an α-critical R-module and $B \cap A_{n-1} \neq B \cap A_n$. Then there exists an integer p such that for any integer $j \geq p$, all finitely generated R-module subfactors of B_j/B_{j+1} have Krull dimension less than α.

Proof. Choose an integer t such that, for all $n \geq t$, each factor A_n/A_{n+1} is α-critical and $B \cap A_{n-1} \neq B \cap A_n$. Note that $(A_n \cap B + A_{n-1})/A_{n-1} \simeq (A_n \cap B)/(A_{n-1} \cap B) \neq 0$, for all $n \geq t$.

Since

$$(B + A_n)/(B + A_{n-1}) \simeq A_n/(A_n \cap B + A_{n-1}) \simeq (A_n/A_{n-1})/((A_n \cap B + A_{n-1})/A_{n-1})$$

for $n \geq t$, the R-module $(B + A_n)/(B + A_{n-1})$ is a proper epimorphic image of A_n/A_{n-1}, so $\mathcal{K}((B + A_n)/(B + A_{n-1})) < \alpha$, for each $n \geq t$, by the choice of t. As $A/(B + A_t)$ is the union of the submodules

$$(B + A_{t+1})/(B + A_t) \leq (B + A_{t+2})/(B + A_t) \leq \ldots$$

it follows that all finitely generated R-module subfactors of $A/(B + A_t)$ have Krull dimension less than α. In the R-module A/B, consider the following two chains of submodules

$$A/B \geq (B + A_t)/B \geq 0, \quad A/B \geq B_1/B \geq B_2/B \geq \ldots \geq 0.$$

On applying the Schreier Refinement Theorem to the two chains above we obtain a common refinement of them, cf. [8, 3.2(b)]. Namely, there exist R-submodules

$$A \geq V_{01} \geq V_{02} \geq \ldots \geq B + A_t \geq V_{11} \geq V_{12} \geq \ldots \geq B,$$

$$B_j = W_{0j} \geq W_{1j} \geq W_{2j} = B_{j+1}, \quad j = 1, 2, \ldots$$

such that $V_{ij}/V_{i,j+1} \simeq W_{ij}/W_{i+1,j}$ for all i, j. Now

$$\mathcal{K}((B + A_t)/B) \leq \mathcal{K}(A_t) = \alpha.$$

Hence, there exists an integer p such that $\mathcal{K}(V_{1j}/V_{1,j+1}) < \alpha$, for all $j \geq p$. Consequently, for $j \geq p$: $\mathcal{K}(W_{1j}/W_{2,j}) < \alpha$. Since $W_{0j}/W_{1,j}$ is isomorphic to $V_{0j}/V_{0,j+1}$ which is a subfactor of $A/(B+A_t)$, all finitely generated subfactors of $W_{0j}/W_{1,j}$ have Krull dimension less than α. Therefore for $j \geq p$, every finitely generated subfactor of B_j/B_{j+1} has Krull dimension less than α. ▪

Let M be a nonzero R-module and let N be a nonzero T-submodule of $T \otimes_R M$. We say that N has **type** $(-)$ if $\lambda_-(N) \neq 0$, but $\lambda_+(N) = 0$. Similarly, N has **type** $(+)$ if $\lambda_+(N) \neq 0$, while $\lambda_-(N) = 0$; and **type** $(-,+)$ if $\lambda_-(N) \neq 0$ and $\lambda_+(N) \neq 0$. Finally, N has type $(0,0)$ if both $\lambda_\pm(N) = 0$. In this latter case, N is contained in a finite sum $\oplus v_i \otimes M$; so N is finitely generated as an R-module, provided that M is a noetherian R-module, by Lemma 1.1.(2). Let $N_1 \geq N_2 \geq \ldots$ be a chain of nonzero submodules of $T \otimes_R M$; then, for sufficiently large i, each of the submodules N_i has the same type, say t, and we say that the chain above has type t.

Corollary 2.3 *Let M be an α-critical noetherian R-module, for some ordinal α. Let*

$$T \otimes_R M \geq N_1 \geq N_2 \geq \ldots \geq N > 0$$

be a chain of nonzero T-submodules of $T \otimes_R M$ which is of the the same type t as N, with $t \neq (0,0)$. Then there is a positive integer n such that, for any $i \geq n$, all finitely generated R-module subfactors of N_i/N_{i+1} have Krull dimension less than α.

Proof. Without loss of generality, we may assume that $n = 1$. Suppose that the T-module N has one of the types (i) $(-, +)$, (ii) $(+)$, or (iii) $(-)$. In cases (ii) and (iii), there exists $k \in \mathbb{Z}$ such that N_1 is contained in $A^+ := \oplus_{i \geq k} T(M)_i$, or $A^- := \oplus_{i \leq k} T(M)_i$, respectively. The T-module $T \otimes_R M = \oplus_{i \in \mathbb{Z}} T(M)_i$ is $\mathbb{Z}$-graded where $T(M)_i = v_i \otimes M \simeq {}_{\sigma^i}M$, as an R-module. Consider an ascending chain of R-modules: $\ldots \leq A_i \leq A_{i+1} \leq \ldots$, where, in case (i) $A_{-1} = 0$, $A_{2n} = T(M)_{-n} \oplus \cdots \oplus T(M)_n$ and $A_{2n+1} = T(M)_{-n} \oplus \cdots \oplus T(M)_{n+1}$, for $n \geq 1$; while in cases (ii) or (iii), set $A_i = \oplus_{k \leq j \leq i} T(M)_j$, or $A_i = \oplus_{k \geq j \geq -i} T(M)_j$, respectively. Observe that the conditions of Proposition 2.2 hold and the result follows. In more detail, we set $A = T \otimes_R M$, A^+ and A^-, in cases (i), (ii) and (iii), respectively. ▪

For an R-module M define the sets

$$St(M) = \{i \in \mathbb{Z} \mid a\, {}_{\sigma^i}M = \sigma^{-i}(a)M = 0\}, \;\; Ho(M) = \{i + 1 \mid i \in St(M)\}.$$

The elements of $St(M)$ and $Ho(M)$ are the **stars** and **holes** respectively. Set $St_-(M) = \{i \in St(M) \mid i < 0\}$ and $Ho_+(M) = \{j \in Ho(M) \mid j > 0\}$. Denote by $s_-(M)$ the largest element of $St_-(M)$, and by $h_+(M)$ the smallest element of $Ho_+(M)$ (if they exist). The importance of stars and holes is as follows: if i is a positive hole, then $Y \cdot (X^i \otimes M) = 0$; while if i is a negative star then $X \cdot (Y^{-i} \otimes M) = 0$.

An R-module M is called a-**monic** if for every $i \in \mathbb{Z}$ the map $\sigma^i(a)_M : M \to M$, given by $m \to \sigma^i(a)m$ is either injective or zero. It is obvious that a submodule of an a-monic module is a-monic, and easy to check that any critical R-module is a-monic. Let M be an a-monic R-module. Then

$$\ker X_{T(M)} \;=\; \oplus_{i \in St_-(M)} T(M)_i, \tag{1}$$

$$\ker Y_{T(M)} = \oplus_{j \in Ho_+(M)} T(M)_j. \tag{2}$$

If M is a-monic then either M is a-torsionfree, or $St(M) \neq \emptyset$. In the latter case, there is either a negative star, or a positive hole (or both), and so, either $s_-(M)$ or $h_+(M)$ exists (they may both exist).

The T-submodule of $T(M)$

$$\mathcal{L}(M) := \sum\nolimits_{i \leq s_-(M)} T(M)_i \oplus \sum\nolimits_{j \geq h_+(M)} T(M)_j$$

is the largest homogeneous submodule of $T(M)$ which has zero intersection with $1 \otimes M$. Denote by $\mathcal{L}(M)_-$ and $\mathcal{L}(M)_+$ the first and the second summand, respectively. If $St_-(M) = \emptyset$ or $Ho_+(M) = \emptyset$, we set $\mathcal{L}(M)_- = 0$ and $\mathcal{L}(M)_+ = 0$ respectively. The $\mathcal{L}(M)_\pm$ are T-submodules of $T(M)$. If $\mathcal{L}(M)_- \neq 0$, then set

$$\mathcal{V}_+(M) = T(M)/\mathcal{L}(M)_- = \oplus_{i > s_-(M)} T(M)_i,$$

pictured as

$$\cdots \overset{s_-(M)}{\star} \cdots [\underbrace{\circ \text{———} \cdots \text{———} \overset{1\otimes M}{\bullet} \text{———} \cdots}_{\mathcal{V}_+(M)} \quad ,$$

and if $\mathcal{L}(M)_+ \neq 0$, then set

$$\mathcal{V}_-(M) = T(M)/\mathcal{L}(M)_+ = \oplus_{i < h_+(M)} T(M)_i,$$

pictured as

$$\underbrace{\cdots \text{———} \overset{1\otimes M}{\bullet} \text{———} \cdots \text{———} \star}_{\mathcal{V}_-(M)}] \cdots \overset{h_+(M)}{\circ} \cdots \quad .$$

Finally, if both $\mathcal{L}(M)_- \neq 0$ and $\mathcal{L}(M)_+ \neq 0$ then set

$$L(M) := T(M)/\mathcal{L}(M) = \oplus_{s_-(M)<i<h_+(M)}\, v_i \otimes M,$$

$$\dots \overset{s_-(M)}{\star} \cdots [\underbrace{\circ \text{———} \dots \text{———} \cdots \text{———} \star}_{L(M)}] \cdots \overset{h_+(M)}{\circ} \dots \quad .$$

If $s_-(M) = -1$ then $\mathcal{V}_+(M) = V_+(M)$, while if $h_+(M) = 1$ then $\mathcal{V}_-(M) = V_-(M)$.

Let

$$St_-(M) = \{\dots < i_2 < i_1 = s_-(M)\}, \qquad Ho_+(M) = \{h_+(M) = j_1 < j_2, \dots\}. \quad (3)$$

Then there are two strictly descending chains of T-submodules of $T(M)$:

$$\dots < V_-(i_n) < \dots < V_-(i_1) < T(M), \quad (4)$$

$$T(M) > V_+(j_1) > \dots > V_+(j_m) > \dots, \quad (5)$$

where

$$V_-(i_n) := \oplus_{i \leq i_n}\, T(M)_i \simeq V_-({}_{\sigma^{i_n}}M),$$

and

$$V_+(j_m) = \oplus_{j \geq j_m}\, T(M)_j \simeq V_+({}_{\sigma^{j_m}}M).$$

Define

$$L_-(i_{n+1}, i_n] \equiv L_-({}_{\sigma^{i_{n+1}}}M, {}_{\sigma^{i_n}}M] \ := \ V_-(i_n)/V_-(i_{n+1}) \simeq L({}_{\sigma^{i_n}}M), \quad (6)$$

and

$$L_+(j_m - 1, j_{m+1} - 1] \equiv L_+({}_{\sigma^{j_m - 1}}M, {}_{\sigma^{j_{m+1}-1}}M] \ := \ V_+(j_m)/V_+(j_{m+1}) \simeq L({}_{\sigma^{j_m}}M). \quad (7)$$

The T-module $V = \mathcal{V}_+(M)$ is uniquely characterized by the following properties:

- the T-module $V = \oplus_{i \in \mathbb{Z}} V_i$ is $\mathbb{Z}$-graded and left restricted (that is, $V_i = 0$, for all $i \ll 0$);

- $V = TV_j$, for some j;

• the map $X_V : V \to V$, given by $v \mapsto Xv$, is injective.

Clearly, the T-module $\mathcal{V}_+(M)$ has the properties listed above, with $V_j = M$.

Conversely, suppose that the T-module V satisfies the conditions above. It follows from the first two conditions that V is isomorphic to a T-module $T(V_j)/N$, where N is a homogeneous submodule of $T(V_j)$. Further, N contains $\mathcal{L}_-(V_j)$ follows from the third condition. Therefore, V is isomorphic to the factor module $\mathcal{V}_+(V_j)/L$, where L is homogeneous submodule of $\mathcal{V}_+(V_j)$. Now, $L \subseteq \oplus_{i\geq 1} X^i \otimes V_j$, since the map X_V is injective and $(1 \otimes V_j) \cap L = 0$. If $L \neq 0$ then there exists a nonzero element $X^i \otimes l \in L$. Thus, $X^i(1 \otimes l) \in L$, and so $0 = 1 \otimes l$, by the third condition. However, $0 \neq 1 \otimes l \in L$, a contradiction. Thus $L = 0$, and $V \simeq \mathcal{V}_+(V_j)$.

Similarly, T-module $V = \mathcal{V}_-(M)$ is uniquely characterized by the following properties:

• the T-module $V = \oplus_{i\in\mathbb{Z}} V_i$ is $\mathbb{Z}$-graded and right restricted (that is, $V_i = 0$, for all $i \gg 0$);

• $V = TV_j$, for some j;

• the map $Y_V : V \to V$, given by $v \mapsto Yv$, is injective.

Lemma 2.4 *Let M be an a-monic R-module.*

1. *If $s_-(M) = h-1$ exists, then the T-submodule $T\,v_h \otimes M$ of $\mathcal{V}_+(M)$ is isomorphic to $V_+({}_{\sigma^h}M)$; and the T-submodule $T\,v_{-h} \otimes {}_{\sigma^h}M$ of $V_+({}_{\sigma^h}M)$ is isomorphic to $\mathcal{V}_+(M)$. So, we have*

$$\mathcal{V}_+(M) \geq V_+({}_{\sigma^h}M) \geq \mathcal{V}_+(M) \text{ and } \mathcal{K}(\mathcal{V}_+(M)) = \mathcal{K}(V_+({}_{\sigma^h}M)).$$

2. *If $h_+(M) = s+1$ exists, then the T-submodule $T\,v_s \otimes M$ of $\mathcal{V}_-(M)$ is isomorphic to $V_-({}_{\sigma^s}M)$; and the T-submodule $T\,v_{-s} \otimes {}_{\sigma^s}M$ of $V_-({}_{\sigma^s}M)$ is isomorphic to $\mathcal{V}_-(M)$. So, we have*

$$\mathcal{V}_-(M) \geq V_-({}_{\sigma^s}M) \geq \mathcal{V}_-(M) \text{ and } \mathcal{K}(\mathcal{V}_-(M)) = \mathcal{K}(V_-({}_{\sigma^s}M)).$$

Proof. This follows immediately from the characterization of the modules $\mathcal{V}_\pm(M)$ and $V_\pm(M)$. ▪

Definition A critical R-module M is called T-**clean** if M is a-torsionfree and $T(M)$ is a critical T-module. In a similar manner, a critical R-module M is called $(T,+)$-**clean** if $s_-(M)$ exists and $\mathcal{V}_+(M)$ is a critical T-module, and M is $(T,-)$-**clean** if $h_+(M)$ exists and $\mathcal{V}_-(M)$ is a critical T-module.

In view of Lemma 2.4, if M is a critical R-module, then the following hold:

• M is $(T,+)$-clean if and only if the T-module $V_+({}_{\sigma^h}M)$ is critical, where $h = s_-(M)+1$;

• M is $(T,-)$-clean if and only if the T-module $V_-({}_{\sigma^s}M)$ is critical, where $s = h_+(M)-1$.

It follows from Lemma 2.4, and (4), (5) that if M is $(T,+)$-clean then so is ${}_{\sigma^i}M$, for all $i \geq s_-(M)$, and if M is $(T,-)$-clean, then so is ${}_{\sigma^i}M$, for all $i \leq h_+(M)$.

The next result shows that there are enough $(T,\bullet)$-clean modules.

Lemma 2.5 *Let M be a noetherian critical R-module.*

1. *If M is a-torsionfree, then M contains a T-clean submodule.*
2. *If $s_-(M)$ exists, then for some $i \geq 0$ the R-module ${}_{\sigma^i}M$ contains a $(T,+)$-clean submodule.*
3. *Similarly, if $h_+(M)$ exists, then for some $i \leq 0$ the R-module ${}_{\sigma^i}M$ contains a $(T,-)$-clean submodule.*

Proof. The T-module $T(M)$ is noetherian, and so are $\mathcal{V}_\pm(M)$, whenever they exist, since the R-module M is noetherian.

1. Assume that M is a-torsionfree. Fix a critical T-submodule, N say, of $T(M)$. By Lemma 1.2.(1), there exists a nonzero R-submodule L of M such that $T(L)$ embeds in N. Let U be a critical R-submodule of L. Since T is a flat right R-module, $T(U)$ embeds in $T(L)$, and so embeds in the critical module N. Hence, the T-module $T(U)$ is critical; and so U is T-clean.

2. Assume that $s_-(M)$ exists. By Lemma 2.4, the module $\mathcal{V}_+(M)$ contains the submodule $V_+(M' := {}_{\sigma^h}M)$, where $h = s_-(M)+1$. Fix any critical T-submodule, N say, of $V_+(M')$. By Proposition 1.3.(2), there exists a nonzero R-submodule L of $\sigma^i M'$, for some $i \geq 0$ such that $V_+(L)$ embeds in N. Let U be a critical R-submodule of L. Since T is a flat right R-module, $V_+(U)$ embeds in $V_+(L)$, and so embeds in the critical module N. Hence, the T-module $V_+(U)$ is critical; and so U is $(T,+)$-clean. Unfortunately, if $i < |h|$, then $\sigma^i M'$ is still to the left of M, so we need to shift to the right, and get the submodule $\sigma^{|h|}U$ of $\sigma^{i+|h|}M' = \sigma^i M$, which is also $(T,+)$-clean by the comment immediately before the lemma.

3. The case where $h_+(M)$ exists is similar to the previous case. ∎

Let $M = \oplus_{i\in\mathbb{Z}} M_i$ be a $\mathbb{Z}$ -graded module. Given a submodule N of M, then

$$N^{hom} := \oplus_{i\in\mathbb{Z}} N \cap M_i$$

is the largest homogeneous submodule of N.

The next lemma establishes various easy properties of the Krull dimension of the various kinds of T-modules that have just been introduced.

Lemma 2.6 *Let M be an α-critical noetherian R-module.*

1. $\alpha \leq \mathcal{K}(T(M)) \leq \alpha + 1$ *and if the set $St(M)$ is infinite, then* $\mathcal{K}(T(M)) = \alpha + 1$.
2. $\mathcal{K}(L_\pm({}_{\sigma^i}M, {}_{\sigma^j}M]) = \alpha$.
3. *If M is a-torsion, then $\alpha \leq \mathcal{K}(\mathcal{V}_\pm(M)) \leq \alpha+1$. If the set $Ho_+(M)$ is infinite, then $\mathcal{K}(\mathcal{V}_-(M))$ equals $\alpha+1$. Similarly, if $St_-(M)$ is infinite, then $\mathcal{K}(\mathcal{V}_+(M))$ equals $\alpha + 1$.*
4. *Let V be one of the following T-modules: (i) $\mathcal{V}_-(M)$ and $St_-(M) = \emptyset$; (ii) $\mathcal{V}_+(M)$ and $Ho_+(M) = \emptyset$; (iii) $L(M)$ and both $s_-(M)$ and $h_+(M)$ exists. If N is a non-zero T-submodule of V, then $N^{hom} \neq 0$.*
5. *If $\mathcal{V}_+(M)$ exists, then $\mathcal{K}(\mathcal{V}_+(M)) = \mathcal{K}(\mathcal{V}_+({}_{\sigma^i}M))$, for $i > s_-(M)$, while if $\mathcal{V}_-(M)$ exists, then $\mathcal{K}(\mathcal{V}_-(M)) = \mathcal{K}(\mathcal{V}_-({}_{\sigma^i}M))$, for $i < h_+(M)$.*

Proof. 2. As a left R-module $L_\pm({}_{\sigma^i}M, {}_{\sigma^j}M] = \oplus_{i<k\leq j}\, {}_{\sigma^k}M$ and the result follows.

1 and 3. Follow from (4) and (5) and statement 2.

4. Evident.

5. By Lemma 2.4.(1), $\mathcal{K}(\mathcal{V}_+(M)) = \mathcal{K}(V_+({}_{\sigma^h}M))$, $h = s_-(M)+1$ and $\mathcal{K}(\mathcal{V}_+({}_{\sigma^i}M)) = \mathcal{K}(V_+({}_{\sigma^n}M))$ for some $n \in Ho_+({}_{\sigma^i}M) \subseteq Ho_+(M)$. Since $V_+({}_{\sigma^h}M) \geq V_+({}_{\sigma^n}M)$ and, by 2, $\mathcal{K}(V_+({}_{\sigma^h}M)/V_+({}_{\sigma^n}M)) = \mathcal{K}(L_+({}_{\sigma^{h-1}}M, {}_{\sigma^{n-1}}M]) = \mathcal{K}(M)$, the result follows from 3. The case $(-)$ is symmetrical to $(+)$. ∎

In the following lemma, we establish the existence of clean subfactors in induced modules, which have the same Krull dimension as the modules, and see the need for the various types of clean modules that we have introduced.

Lemma 2.7 *Let M be a nonzero noetherian R-module.*

1. *If $\sigma(a)M = 0$ then, for some $i \geq 0$, there exists a $(T,+)$-clean subfactor N of ${}_{\sigma^i}M$ such that $\sigma(a)N = 0$ and $\mathcal{K}(V_+(M)) = \mathcal{K}(V_+(N))$, while if $aM = 0$, then, for some $i \leq 0$, there exists a $(T,-)$-clean subfactor N of ${}_{\sigma^i}M$, such that $aN = 0$ and $\mathcal{K}(V_-(M)) = \mathcal{K}(V_-(N))$.*

2. *There exists either*
(i) a T-clean subfactor L of M such that $\mathcal{K}(T \otimes_R M) = \mathcal{K}(T \otimes_R L)$, or
(ii) a $(T,+)$-clean subfactor N of ${}_{\sigma^i}M$, for some $i \geq 0$, such that $\mathcal{K}(T \otimes_R M) = \mathcal{K}(V_+(N))$, or
(iii) a $(T,-)$-clean subfactor N of ${}_{\sigma^i}M$, for some $i \leq 0$, such that $\mathcal{K}(T \otimes_R M) = \mathcal{K}(V_-(N))$.

Proof. 1. Let us consider the first case. Without loss of generality we may assume that M is critical. In fact, the R-module M is noetherian, thus we can choose a chain of submodules $M = M_1 > M_2 > \cdots > M_n = 0$ with critical factors M_i/M_{i+1}. Observe $V_+(M) = T^+ \otimes_R M$ where $T^+ = R[X;\sigma]$ is the skew polynomial ring. The right R-module T^+ is faithfully flat, so we have the chain of T-submodules:

$$V_+(M) = V_+(M_1) > V_+(M_2) > \cdots > V_+(M_n) = 0$$

with factors $V_+(M_i)/V_+(M_{i+1}) \simeq V_+(M_i/M_{i+1})$. So

$$\mathcal{K}(V_+(M)) = \max\{\mathcal{K}(V_+(M_i/M_{i+1}), \ \ i = 1, \ldots, n-1\}.$$

By replacing M by a suitable factor M_i/M_{i+1}, we may assume that M is critical. Recall that, in this case, M is also a-monic. By Lemma 2.5.(2), there exists $0 \leq j = j_1 \in Ho_+(M)$ such that ${}_{\sigma^j}M$ contains a $(T,+)$-clean submodule L_1. By Lemma 2.6.(5), $n := \mathcal{K}(V_+(M)) = \mathcal{K}(V_+({}_{\sigma^j}M))$. If $\mathcal{K}(V_+(L_1)) = n$, then $L = L_1$ and we are done. If not, that is, if $\mathcal{K}(V_+(L_1)) < n$, then $n = \mathcal{K}(V_+({}_{\sigma^j}M)/V_+(L_1)) = \mathcal{K}(V_+({}_{\sigma^j}M/L_1))$. Using the same sort of arguments, we find an integer $j_2 \geq j_1$ and R-submodules L_2, N_2 such that ${}_{\sigma^{j_2}}M \geq L_2 > N_2 > {}_{\sigma^{j_2-j_1}}L_1$, L_2/N_2 is $(T,+)$-clean and $V_+(L_2/N_2)$ exists. If $n = \mathcal{K}(V_+(L_2/N_2))$, then set $L = L_2/N_2$. If not, we find L_3 and so on. Since ${}_RM$ is noetherian and σ is automorphism this process must stop.

2. The module T_R is faithfully flat. Applying $T \otimes_R -$ to the above chain $M = M_1 > M_2 > \cdots > M_n = 0$ we obtain the chain of T-submodules

$$T(M) = T(M_1) > T(M_2) > \cdots > T(M_n) = 0$$

with i'th factor isomorphic to $T(M_i/M_{i+1})$, hence

$$\mathcal{K}(T(M)) = \max\{\mathcal{K}(T(M_i/M_{i+1}), \ \ i = 1, \ldots, n-1\}.$$

Thus, without loss of generality, we may assume that M is critical.

If $St(M) \neq \emptyset$, then either $i_1 = s_-(M)$ or $j_1 = h_+(M)$ exists (or both). By (4), (5), we have at least one of the following exact sequence of T-modules:

$$0 \to V_-(i_1) \to T(M) \to \mathcal{V}_+(M) \to 0 \quad \text{or} \quad 0 \to V_+(j_1) \to T(M) \to \mathcal{V}_-(M) \to 0.$$

Observe that $\mathcal{K}(\mathcal{V}_+(M)) = \mathcal{K}(V_+({}_{\sigma^{i_1+1}}M))$ and $\mathcal{K}(\mathcal{V}_-(M)) = \mathcal{K}(V_-({}_{\sigma^{j_1-1}}M))$, by Lemma 2.4, so that $\mathcal{K}(T(M))$ is either equal to $\max\{\mathcal{K}(V_-(i_1)), \mathcal{K}(V_+({}_{\sigma^{i_1+1}}M))\}$, or to $\max\{\mathcal{K}(V_+(j_1)), \mathcal{K}(V_-({}_{\sigma^{j_1-1}}M))\}$ and the result follows from statement 1.

If $St(M) = \emptyset$; then M is a critical a-torsionfree R-module. By Lemma 2.5, M contains a T-clean submodule $L_1 \leq M$. If $n := \mathcal{K}(T(M)) = \mathcal{K}(T(L_1))$, there is nothing to prove. Otherwise, if $n > \mathcal{K}(T(L_1))$, then the Krull dimensions of the modules $T(M)$ and $T(M/L_1)$ coincide. Applying the same argument to the factor module M/L_1 we either reduce the problem to the case considered in statement 1 or find a nonzero subfactor $\overline{L}_2$ of M/L_1 (i.e. submodules $L_2 > N_2 \geq L_1$, $\overline{L}_2 = L_2/N_2$) which has the same property as L_1. The module M is noetherian, so this process must stop. ▪

The next result is part of the inductive step in proving a formula, Theorem 4.1, for the Krull dimension of an induced module $T \otimes M$, in the case that M is is an a-torsion, critical noetherian R-module. This is the easy part of the inductive step. The most difficult part is the case where M is a-torsionfree, see Theorem 3.5, which requires a refined study of the submodule structure of induced modules. This is done in the remainder of the present section, see Lemma 2.13.

Proposition 2.8 *Let M be a noetherian non-simple R-module of finite Krull dimension.*

1. *If M is $(T,+)$-clean, then*

$$\mathcal{K}(\mathcal{V}_+(M)) = \max\{\mathcal{K}(V_+(N)) \mid N \in \mathbf{a}_+\} + 1,$$

   where $\mathbf{a}_+$ is the family of $(T,+)$-clean minor subfactors of ${}_{\sigma^i}M$, $i > s_-(M)$.

2. *If M is $(T,-)$-clean, then*

$$\mathcal{K}(\mathcal{V}_-(M)) = \max\{\mathcal{K}(V_-(N)) \mid N \in \mathbf{a}_-\} + 1,$$

   where $\mathbf{a}_-$ is the family of $(T,-)$-clean minor subfactors of ${}_{\sigma^i}M$, $i < h_+(M)$.

Proof. 1. First, suppose that $Ho_+(M)$ is a finite set (it could be empty). Note that M has a negative star, and so, M has a non-positive hole. Let i be the maximal hole of M. By Lemma 2.4, $\mathcal{K}(\mathcal{V}_+(M)) = \mathcal{K}(V_+({}_{\sigma^i}M))$. So, without loss of generality, we may replace M by ${}_{\sigma^i}M$, which is also $(T,+)$-clean, and assume that $i = 0$; that is, $\mathcal{V}_+(M) = V_+(M)$ and $Ho_+(M) = \emptyset$.

The R-module M is noetherian, so $V_+(M)$ is a noetherian T-module. Set $n := \mathcal{K}(V_+(M))$, so that $1 \leq \mathcal{K}(M) \leq n = \mathcal{K}(V_+(M)) \leq \mathcal{K}(M) + 1$. Let N be a nonzero submodule of $V_+(M)$ such that $\mathcal{K}(V_+(M)/N) = n - 1$. The homogeneous component $N_0 := N \cap M \neq 0$, by Lemma 2.6.(4)(ii), since $Ho_+(M) = \emptyset$, and so $0 \neq V_+(N_0) \leq N \leq V_+(M)$.

The T-module $V_+(M)$ is critical, since M is assumed to be $(T,+)$-clean; so $n-1 \geq \mathcal{K}(V_+(M)/V_+(N_0)) = \mathcal{K}(V_+(M/N_0))$. On the other hand, $n-1 = \mathcal{K}(V_+(M)/N) \leq \mathcal{K}(V_+(M)/V_+(N_0))$, so that $\mathcal{K}(V_+(M/N_0)) = n-1$. By Lemma 2.7.(1), for some $i \geq 0$ there exists a $(T,+)$-clean subfactor L of ${}_{\sigma^i}(M/N_0)$ such that $\mathcal{K}(V_+(L)) = n-1$. The module N_0 is nonzero, so $L \in \mathbf{a}_+$.

Next, suppose that $Ho_+(M)$ is infinite. As above, we may assume that $\sigma(a)M = 0$; that is, $\mathcal{V}_+(M) = V_+(M)$. The R-module M is noetherian of finite Krull dimension, say m, and $m \geq 1$, since M is not simple. We can choose a minor subfactor C of M such that C is critical of Krull dimension $m-1$. For some $i \geq 0$, the R-module ${}_{\sigma^i}C$ contains a $(T,+)$-clean submodule, say C', by Lemma 2.5.(2). By going to the next hole bigger than i, we may assume that $\sigma(a)C' = 0$. The R-module ${}_{\sigma^i}C$ is critical, since C is critical, and $\mathcal{K}(C') = \mathcal{K}({}_{\sigma^i}C) = \mathcal{K}(C) = m-1$. The sets $Ho_+(M)$ and $Ho_+(C')$ are infinite, so $\mathcal{K}(V_+(C')) = \mathcal{K}(C') + 1 = m$ and $\mathcal{K}(V_+(M)) = \mathcal{K}(M) + 1 = m+1$, by Lemma 2.6.(3). Finally, observe that $C' \in \mathbf{a}_+$.

2. The proof of the second part is similar to 1, or one can invoke the $\pm$-symmetry. . ∎

Definition. Let M and N be T-clean noetherian R-modules. Define $h(N : M) = 1$ if N is isomorphic to a minor subfactor of M, but no nonzero submodule of N is isomorphic to a minor subfactor of a T-clean minor subfactor of M.

Note that if $h(N : M) = 1$ then $h({}_{\sigma^i}N' : {}_{\sigma^i}M) = 1$, for all nonzero submodules N' of N and for all $i \in \mathbb{Z}$.

Definition. Let M be a noetherian R-module that is either T-clean or $(T,+)$-clean, and let N be a $(T,+)$-clean noetherian R-module. Define $h_+(N : M) = 1$ if N is isomorphic to a minor subfactor of M, but, for any $i \geq 0$, no nonzero submodule of ${}_{\sigma^i}N$ is isomorphic to a minor subfactor of a T-clean or $(T,+)$-clean minor subfactor of ${}_{\sigma^i}M$. Similarly, if M is either T-clean or $(T,-)$-clean, and N is a $(T,-)$-clean noetherian R-module then $h_-(N : M) = 1$ if N is isomorphic to a minor subfactor of M, but, for any $i \leq 0$, no nonzero submodule of ${}_{\sigma^i}N$ is isomorphic to a minor subfactor of a T-clean or $(T,-)$-clean minor subfactor of ${}_{\sigma^i}M$.

Again, note the following.

- If $h_\pm(N : M) = 1$ then $h_\pm({}_{\sigma^{\pm i}}N' : {}_{\sigma^{\pm i}}M) = 1$, for all nonzero $(T,\pm)$-clean submodules N' of N and for all $i \geq 0$.

- Let M and ${}_{\sigma^i}M$ be noetherian R-modules which are either T-clean or $(T,\pm)$-clean, and let N and ${}_{\sigma^i}N$ be $(T,\pm)$-clean noetherian R-modules, for some $i \in \mathbb{Z}$. Then $h_\pm(N : M) = 1$ if and only if $h_\pm({}_{\sigma^i}N : {}_{\sigma^i}M) = 1$.

Lemma 2.9 *Let M be a noetherian R-module and let*

$$M \geq C_1 \geq C_2 \geq \ldots \geq C > 0$$

be R-submodules of M. If M is $(T,+)$-clean then there exists an integer $m \geq 1$ such that each module ${}_{\sigma^j}(C_i/C_{i+1})$, for $i \geq m$ and $j \geq 0$, has no $(T,+)$-clean submodules N for which $h_+(N : {}_{\sigma^j}M) = 1$. Similarly, if M is $(T,-)$-clean there exists an integer $m \geq 1$ such that each module ${}_{\sigma^j}(C_i/C_{i+1})$, for $i \geq m$ and $j \leq 0$, has no $(T,-)$-clean submodules N for which $h_-(N : {}_{\sigma^j}M) = 1$.

Proof. Suppose that M is $(T,+)$-clean. The module M is noetherian, so we can choose C to be maximal among those submodules of M contained in a descending chain for which the conclusion fails. Then, for each submodule ${}_{\sigma^i}C$ of ${}_{\sigma^i}M$, for $i \geq 0$, the conclusion fails as well. Thus, C can be chosen in such a way that all ${}_{\sigma^i}C$, for $i \geq 0$, are maximal with failure.

Refine the chain, so that each of the modules C_i/C_{i+1} is a critical module. By Lemma 2.5.(2), there exists an integer $n \geq 0$ and a submodule V of ${}_{\sigma^n}M$ such that ${}_{\sigma^n}C \subseteq V$ and $V/{}_{\sigma^n}C$ is $(T,+)$-clean.

By the choice of C, there exist two sequences of non-negative integers, $1 < i_1 < i_2 < \ldots$ and $j_1, j_2, \ldots$ such that each of the factor modules ${}_{\sigma^{j_\nu}}(C_{i_\nu}/C_{i_\nu+1})$, for $\nu \geq 1$, contains a $(T,+)$-clean submodule, N_ν, say, for which $h_+(N_\nu : {}_{\sigma^{j_\nu}}M) = 1$.

Without loss of generality, we may assume that $N_\nu = {}_{\sigma^{j_\nu}}(C_{i_\nu}/C_{i_\nu+1})$, and that $j_1 \leq j_2 \leq \ldots$, since if $h_+(N : M) = 1$ then $h_+({}_{\sigma^i}N : {}_{\sigma^i}M) = 1$, for all $i \geq 0$. Fix $j = j_\nu$, for any ν, and consider the two chains of modules

$${}_{\sigma^{n+j}}M \supseteq {}_{\sigma^j}V \supseteq {}_{\sigma^{n+j}}C$$

and

$${}_{\sigma^{n+j}}M \geq {}_{\sigma^{n+j}}C_1 \geq {}_{\sigma^{n+j}}C_2 \geq \ldots \geq {}_{\sigma^{n+j}}C.$$

Comparing these chains, as in Proposition 2.2, we obtain chains of modules

$${}_{\sigma^{n+j}}M \geq V_{01} \geq V_{02} \geq \ldots \geq {}_{\sigma^j}V \geq V_{11} \geq V_{12} \geq \ldots \geq {}_{\sigma^{n+j}}C$$

and

$${}_{\sigma^{n+j}}(C_l) = W_{0l} \geq W_{1l} \geq W_{2l} = {}_{\sigma^{n+j}}(C_{l+1}), \quad l = 1, 2, \ldots$$

such that $V_{kl}/V_{k,l+1} \simeq W_{kl}/W_{k+1,l}$ for all k, l.

Of course, the modules V_{0l}, V_{1l} and W_{1l} depend on j; that is, $V_{kl} = V_{kl}(j)$, etc. For two integers $j_\nu \leq j_\mu$, it follows from the Schrier Refinement Theorem, that

$$V_{kl}(j_\mu) = {}_{\sigma^p}V_{kl}(j_\nu), \qquad W_{kl}(j_\mu) = {}_{\sigma^p}W_{kl}(j_\nu), \tag{8}$$

where $p = j_\mu - j_\nu$.

Now, $h_+({}_{\sigma^n}N_\nu : {}_{\sigma^{n+j}}M) = 1$, since $h_+(N_\nu : {}_{\sigma^j}M) = 1$.

The module ${}_{\sigma^{n+j}}C$ is nonzero, and the module

$$\frac{{}_{\sigma^j}V}{{}_{\sigma^{n+j}}C} \simeq {}_{\sigma^j}\!\left(\frac{V}{{}_{\sigma^n}C}\right)$$

is a $(T,+)$-clean minor subfactor of ${}_{\sigma^{n+j}}M$.

It follows that $W_{1j\nu}(j_\nu) = W_{2j\nu}(j_\nu)$, for all j_ν, since $h_+({}_{\sigma^n}N_\nu : {}_{\sigma^{n+j}}M) = 1$. By (8), $W_{1j_\mu}(j_\nu) = W_{2j_\mu}(j_\nu)$, for all μ, ν. Now,

$$\frac{V_{0l}(j)}{V_{0,l+1}(j)} \simeq \frac{{}_{\sigma^{n+j}}C_l}{{}_{\sigma^{n+j}}C_{l+1}},$$

for all $l = j_1, j_2, \ldots$. Thus, we obtain the chain

$${}_{\sigma^{n+j}}M \geq V_{0,1} \geq V_{0,2} \geq \ldots \geq {}_{\sigma^j}V,$$

for which the conclusion of the lemma fails. However, ${}_{\sigma^j}V > {}_{\sigma^{n+j}}C$, a contradiction. ∎

Lemma 2.10 *Let M be a T-clean noetherian R-module and let*

$$M \geq C_1 \geq C_2 \geq \ldots \geq C > 0$$

be R-submodules of M. Then, there exists an integer $n \geq 1$ such that the following conditions hold.

1. *Each module C_i/C_{i+1}, for $i \geq n$, has no T-clean submodule N for which $h(N : M) = 1$.*
2. *Each module ${}_{\sigma^{\pm j}}(C_i/C_{i+1})$, for $i \geq n$ and $j \geq 0$, has no $(T,\pm)$-clean submodule N such that $h_\pm(N : {}_{\sigma^{\pm j}}M) = 1$.*

Proof. Assume that the result is not true, and suppose that C is maximal among submodules of M contained in a chain for which the result fails. Thus, the result also fails above ${}_{\sigma^i}C$, for each $i \in \mathbb{Z}$, and each of these submodules is maximal for failure also.

Choose a submodule M' of M, containing C, such that M'/C is critical; this is possible since M is noetherian. If M'/C is a-torsionfree then there exists a submodule V with $C \subseteq V \subseteq M'$ such that V/C is T-clean. We now use the argument of [9, Lemma 4.3] and Lemma 2.9 to obtain a contradiction.

If M'/C is a-torsion then, for some $i \in \mathbb{Z}$, there exists a submodule V with ${}_{\sigma^i}C \subseteq V \subseteq {}_{\sigma^i}M'$ such that $V/{}_{\sigma^i}C$ is $(T,+)$-clean, or $(T,-)$-clean, by Lemma 2.5.(2, 3). In this case, we finish the proof as in Lemma 2.9 ∎

Lemma 2.11 *Let R be any ring, and let M be an R-module which contains an ascending chain of submodules $0 = M_1 \leq M_2 \leq \ldots \leq M$ such that $\cup_{i=1}^{\infty} M_i = M$. Let N be a nonzero R-module subfactor of M. Then there exists a nonzero submodule N' of N such that N' is isomorphic to a subfactor of some M_k/M_{k-1}.*

Proof. Let $N = E/F$, for some R-submodules $F < E \leq M$. We use [9, Proposition 3.2(a)] to obtain a common refinement of the two chains

$$0 \leq F < E \leq M, \qquad 0 = M_1 \leq M_2 \leq \ldots \leq M.$$

Namely, we can choose R-submodules

$$0 = V_{01} \leq V_{02} \leq \ldots \leq F = V_{11} \leq V_{12} \leq \ldots \leq E = V_{21} \leq V_{22} \leq \ldots \leq M$$

$$M_i = W_{0i} \leq W_{1i} \leq W_{2i} \leq W_{3i} = M_{i+1}, \quad \text{for } i = 1, 2, \ldots$$

such that

$$V_{i,j+1}/V_{ij} \simeq W_{i+1,j}/W_{ij},$$

for all i, j, and $E = \cup_{i=1}^{\infty} V_{1i}$. Let k be the least positive integer such that $V_{1k} > F$. Set $N' := V_{1k}/F$. Then N' is a nonzero submodule of N and $N' \simeq V_{1k}/V_{1,k-1} \simeq W_{2,k-1}/W_{1,k-1}$. Hence, N' is isomorphic to a subfactor of M_k/M_{k-1}. ∎

Corollary 2.12 *Let R be a ring and let M be an R-module. Suppose that $\mathcal{M}$ is one of the following T-modules: (i) $T(M)$; (ii) $\mathcal{V}_+(M)$; (iii) $\mathcal{V}_-(M)$. Let N be a nonzero R-module subfactor of $\mathcal{M}$. Then there exists a nonzero submodule N' of N such that N' is isomorphic to a subfactor of ${}_{\sigma^i}M$, where, in the first case, $i \in \mathbb{Z}$, in the second case $i > s_-(M)$, and in the third case $i < h_+(M)$.*

Proof. Observe that $\mathcal{M}$ is isomorphic to $\oplus_{\sigma^i} M$, as an R-module, and apply the previous lemma. ∎

Proposition 2.13 *Let M be a noetherian R-module, and let $\mathcal{M}$ be a nonzero subfactor of the T-module $T(M)$. Then, there exists a chain of T-submodules*

$$\mathcal{M} = \mathcal{M}_0 > \mathcal{M}_1 > \ldots > \mathcal{M}_{n-1} > \mathcal{M}_n = 0$$

such that to each i there corresponds a cyclic critical R-module $V = V_i$, which is a subfactor of some ${}_{\sigma^{n(i)}}M$, and each factor module $\mathcal{M}_i/\mathcal{M}_{i+1}$ is critical and is of one of the following seven types.

1. *$T(V)$, where V is T-clean.*

2. *$V_-(V)$, where V is $(T,-)$-clean.*

3. *$L_-({}_{\sigma^s}V, V]$, where $s = s_-(V)$.*

4. *$V_+(V)$, where V is $(T,+)$-clean.*

5. *$T(V)/N$, where V is T-clean and N is a nonzero T-submodule of $T(V)$, and $N^{hom} = 0$.*

6. *$V_-(V)/N$, where V is $(T,-)$-clean with $|St_-(V)| = \infty$ and N is a nonzero T-submodule of $V_-(V)$ such that $N^{hom} = 0$.*

7. *$V_+(V)/N$, where V is $(T,+)$-clean with $|Ho_+(V)| = \infty$ and N is a nonzero T-submodule of $V_+(V)$ such that $N^{hom} = 0$.*

Remark. If V is a critical R-module then $L = L_-({}_{\sigma^s}V, V]$, where $s = s_-(V)$, is a critical T-module.

Proof. The R-module M is noetherian, hence the T-modules $T(M)$ and $\mathcal{M}$ are also noetherian. It is enough to show that $\mathcal{M}$ has a submodule in one of the seven categories. Without loss of generality, we may assume that $\mathcal{M}$ is a critical T-module, in which case, any nonzero submodule of $\mathcal{M}$ is also critical. If follows from the decomposition $T(M) = \oplus_{\sigma^i} M$ and the noetherianity of ${}_RM$ that any finitely generated R-submodule of $T(M)$ or $\mathcal{M}$ is noetherian.

Suppose that the proposition is true without the following assumption on the module $V = V_i$

(subfac): "which is a subfactor of some ${}_{\sigma^{n(i)}}M$."

Then V is a subfactor of an R-module $N := {}_{\sigma^{i_1}}M \oplus \ldots \oplus {}_{\sigma^{i_s}}M$. The module N has the ascending chain of submodules

$$0 < {}_{\sigma^{i_1}}M < {}_{\sigma^{i_1}}M \oplus {}_{\sigma^{i_2}}M < \ldots < N$$

with subfactors $\{{}_{\sigma^{i_j}}M, j = 1, \ldots, s\}$. By Lemma 2.11, there exists a nonzero cyclic submodule V' of V such that V' is isomophic to subfactor of some ${}_{\sigma^{i_j}}M$. Let $A(V)$ be one of the modules from categories 1-4. Then the T-module $A(V)$ naturally contains the submodule $A(V')$ and the inclusion $A(V') \subseteq A(V)$ respects the grading. Moreover, the transition from $A(V)$ to $A(V')$ preserves all of the properties of $A(V)$. Let $A(V)$ be one of the modules from the remaining three cases. Repeating the same argument, we conclude that the transition from $A(V)$ to $A(V')$ preserves all of the properties of $A(V)$ except possibly the property $N \neq 0$. However, in this case $A(V')$ is in one of the categories 1,2 or 4, respectively.

Thus, we need only prove the proposition without the assumption (subfac).

Choose a cyclic, critical, noetherian R-submodule V of $\mathcal{M}$, with least possible Krull dimension among R-submodules of $\mathcal{M}$. There is no nonzero R-submodule of $\mathcal{M}$ that is a proper epimorphic image of the R-module ${}_{\sigma^i}V$, for any $i \in \mathbb{Z}$.

The R-module inclusion $V \to \mathcal{M}$ induces a T-module map $f : T(V) \to \mathcal{M}$. Suppose that f is injective. If $St(V) = \emptyset$, then $\mathrm{Im}(f)$ belongs to category 1. If $St(V) \neq \emptyset$, then, by considering the chains (4) and (5) we see that the T-module $T(V) \simeq \mathrm{Im}(f)$ contains either $V_-(U)$ or $V_+(W)$, hence the submodules $f(V_-(U))$ and $f(V_+(W))$ belong to the categories 2 and 4, respectively.

Suppose that $\ker(f) \neq 0$. If $St(V) = \emptyset$ then by the choice of V, we have $(\ker(f))^{hom} = 0$, for otherwise, if $(\ker(f))^{hom} \neq 0$ then $V \cap \ker(f) \neq 0$, a contradiction. So, $\mathrm{Im}(f)$ belongs to category 5, provided that V is T-clean. If not, the module V contains a T-clean submodule V', by Lemma 2.5.(1). Hence, the image of the homogeneous T-submodule $T(V')$ of $T(V)$ under the map f belongs either to category 1 or 5.

It remains to consider the case when $St(V) \neq \emptyset$. In this case, either (i) $h = h_+(V)$ exists, or (ii) $s = s_-(V)$ exists, or both exist.

If h exists, then there is an exact sequence of T-submodules

$$0 \to V_+ \to T(V) \to \mathcal{V}_-(V) \to 0,$$

where $V_+ = \oplus_{i \geq h} T(V)_i = V_+({}_{\sigma^h}V)$. If s exists, then there is an exact sequence of T-submodules

$$0 \to V_- \to T(V) \to \mathcal{V}_+(V) \to 0,$$

where $V_- = \oplus_{i \leq s} T(V)_i = V_+({}_{\sigma^s}V)$.

Consider the first case. We subdivide this into two. Either (I) $V_+ \not\leq \ker(f)$, or (II) $V_+ \leq \ker(f)$. In case (I), denote by g the restriction of f to V_+. This is a nonzero map, let $N = \ker(g)$. The nonzero T-module V_+/N is isomorphic to $\mathrm{Im}(g)$, a submodule of $\mathcal{M}$. If $N = 0$, then $V_+ \simeq \mathrm{Im}(g)$ belongs to category 4. Suppose that $N \neq 0$. Then $Ho_+({}_{\sigma^h}V) \neq \emptyset$. For, if $Ho_+({}_{\sigma^h}V) = \emptyset$, then it follows from Lemma 2.6.(4) that $N^{hom} \neq 0$, since $N \neq 0$. Hence, ${}_{\sigma^h}V \cap N^{hom} \neq 0$, which contradicts the choice of V.

Suppose next that $Ho_+({}_{\sigma^h}V) = \{h_1 < h_2 < \ldots < h_k\}$ is a finite nonempty set. We will show that

$$N = N^{hom} = \oplus_{i \geq h_j} (X^i \otimes {}_{\sigma^h}V),$$

for some j. Observe that the map $X_{V_+} : V_+ \to V_+$, given by $v \mapsto Xv$ is injective and the map $Y_{V_+} : V_+ \to V_+$, given by $v \mapsto Yv$, has $\ker(Y_{V_+}) = \oplus(X^j \otimes {}_{\sigma^h}V)$, where j runs through $Ho_+({}_{\sigma^h}V) \cup \{h\}$. Using this observation, we conclude that $N \cap (X^{h_k} \otimes {}_{\sigma^h}V) \neq 0$, since $N \neq 0$. Thus $N \supseteq X^{h_k} \otimes {}_{\sigma^h}V$ by the choice of V, and so $N \supseteq T\,X^{h_k} \otimes {}_{\sigma^h}V = \oplus_{i \geq h_k}(X^i \otimes {}_{\sigma^h}V)$.

Set $h_0 = h$. Let h_j be the minimal positive hole such that $N \geq N' = T\,X^{h_j} \otimes {}_{\sigma^h}V = \oplus_{i \geq h_j}(X^i \otimes {}_{\sigma^h}V)$. Then $j > 0$, since $V_+/N \neq 0$. We aim to show that $N = N'$, so that $N = N' = N^{hom}$. The factor module V_+/N' contains the homogeneous T-submodule $L = \oplus_{h_{j-1} \leq i < h_j} X^i \otimes {}_{\sigma^h}V$. If $N \neq N'$ then we see that $(X^{h_j - 1} \otimes {}_{\sigma^h}V) \cap N \neq 0$, by the observation above. By the choice of h_j, we see that $L \nleq N$, so the restriction of the map g to $X^{h_j-1} \otimes {}_{\sigma^h}V$ is nonzero, and is not injective, which contradicts the choice of V, since ${}_R(X^{h_j-1} \otimes {}_{\sigma^h}V) \simeq {}_{\sigma^{h+h_j-1}}V$.

Hence,

$$N = N^{hom} = \oplus_{i \geq h_j}(X^i \otimes {}_{\sigma^h}V),$$

for some j. Thus $V_+/N' \simeq \mathrm{Im}(g)$ is a T-submodule of $\mathcal{M}$ which contains the submodule L. The T-submodule L generated by $X^{h_j-1} \otimes {}_{\sigma^h}V$ is isomorphic to $L_-({}_{\sigma^{h+h_{j-1}-1}}V, {}_{\sigma^{h+h_j-1}}V]$, which belongs to category 3.

Suppose now that $Ho_+({}_{\sigma^h}V)$ is an infinite set, so that $|Ho_+(V)| = \infty$.

If $N^{hom} = 0$, then $\mathrm{Im}(g) \simeq V_+/N$ belongs to category 7, if ${}_{\sigma^h}V$ is $(T,+)$-clean. If ${}_{\sigma^h}V$ is not $(T,+)$-clean then, by Lemma 2.5.(2), there exists a $(T,+)$-clean submodule U of ${}_{\sigma^i}V$, for some $i \in Ho_+(V)$. Then $V_+(U)$ is a homogeneous T-submodule of V_+. If $N_1 := N \cap V_+(U) = 0$ then $V_+(U)/N$ belongs to category 4, while if $N_1 \neq 0$ then $V_+(U)/N$ belongs to category 7, since $N_1^{hom} \leq N^{hom} = 0$.

If $N^{hom} \neq 0$, then, repeating the same argument as in the case when $Ho_+({}_{\sigma^h}V)$ is finite, we have that $N = N^{hom} = \oplus_{i \geq h_j}(X^i \otimes {}_{\sigma^h}V)$ holds. Applying the same arguments as in the case above to the module V_+/N^{hom} we find a submodule of $\mathcal{M}$ from category 3.

Consider the case (II) where $V_+ \leq \ker(f)$. The T-submodule of $\mathcal{V}_-(V)$ generated by $X^{h-1} \otimes V$ is $V_-({}_{\sigma^{h-1}}V)$. By the choice of V, we conclude that $V_-({}_{\sigma^{h-1}}V) \nleq \ker(f)$. This situation is dual to case (I).

The remaining case, when s exists is dual to the case when h exists. Observe that the dual cases are dealt with immediately, by applying the $\pm$ symmetry. ∎

3 Krull dimension of induced modules: the inductive step

Definition. Let U, W be critical noetherian R-modules. We say that U and W are **subisomorphic** if there exist submodules U' of U and W' of W such that $U' \simeq W'$. The relation, $\sim$, of subisomorphism is an equivalence relation on the set $\mathcal{C}$ of noetherian critical R-modules. For $U \in \mathcal{C}$, we denote by $\{U\}$ the $\sim$ equivalence class of U.

Let M be a noetherian module with $\mathcal{K}(M) = d$. Following [10] and [12], we construct a chain of submodules in the following way. Let M_1 be a submodule maximal such that $\mathcal{K}(M/M_1) = d$. Then M/M_1 is a d-critical module. Continue in this way to construct a chain

$$M = M_0 > M_1 > M_2 > \ldots > M_k$$

such that each factor is d-critical and $\mathcal{K}(M_k) < d$. Note the procedure stops, or we contradict $\mathcal{K}(M) = d$. Let V be any d-critical subfactor of M. Then V is subisomorphic to one of the factors M_i/M_{i+1} in the above chain.

Proposition 3.1 *Let M be an a-torsionfree noetherian R-module and suppose that the ideal Ra of R has the Artin-Rees property. Suppose that $\mathcal{K}(M) = \beta$, and that $\mathcal{K}(M/aM) = \gamma < \beta$. Let V be a γ-critical subfactor of M such that $aV = 0$. Then V is subisomorphic to a subfactor of M/aM. In particular, if V is γ-critical then V is subisomorphic to one of the (finitely many up to subisomorphism) γ-critical subfactors of M/aM.*

Proof. Suppose that $V = E/F$, for some submodules $F < E$ of M. The ideal $I = Ra$ has the Artin-Rees property. Apply the AR property to the module M/F, to find an n such that $a^n M \cap E \leq F$. It follows that $E/F \simeq (E + a^n M)/(F + a^n M)$, so without loss of generality, we may assume that $a^n M \leq F$. By comparing the chain $a^n M \leq F < E \leq M$ with the chain $a^n M \leq a^{n-1} M \leq \ldots \leq aM \leq M$, we see that some submodule V' of V is isomorphic to a subfactor of some $a^i M/a^{i+1} M$. However, M is a-torsionfree, so $a^i M/a^{i+1} M \simeq M/aM$. Thus V' is isomorphic to a subfactor of M/aM. Now, construct a chain for M/aM as above and V' is isomorphic to a subfactor of one of the finitely many γ-critical factors of this chain. In fact, since each of these factors is γ-critical and $\mathcal{K}(V') = \gamma$, the submodule V' is isomorphic to a submodule of one of these factors. ∎

If $V \sim U$, then $St(V) = St(U)$ and ${}^{\sigma^i}V \sim {}^{\sigma^i}U$, for all $i \in \mathbb{Z}$. The subgroup G of $\mathrm{Aut}(R)$ generated by the automorphism σ acts on the set $\mathcal{C}/\sim$ of subisomorphism classes via $\sigma^i\{V\} := \{{}^{\sigma^i}V\}$. Denote by $\mathcal{O}\{V\}$ the orbit $\{{}^{\sigma^i}V \mid i \in \mathbb{Z}\}$ of $\{V\}$. The elements in the set $St(\mathcal{O})$ of $\mathcal{O} = \mathcal{O}\{V\}$ annihilated by a are called **stars**. Clearly,

$$St(\mathcal{O}) = \{\{{}_{\sigma^i}V\} \mid i \in St(V)\}.$$

An orbit containing a star is called **degenerate**, otherwise it is **non-degenerate**. Note that $\mathcal{O}\{V\}$ is degenerate if and only if $St(V) \neq \emptyset$.

Any non-degenerate orbit $\mathcal{O}\{V\}$ is equal to $\mathcal{O}\{V'\}$ for some T-clean submodule of V; similarly, any degenerate orbit $\mathcal{O}\{V\}$ is equal to $\mathcal{O}\{V'_{\pm}\}$ for some $(T,\pm)$-clean submodule of ${}_{\sigma^{\pm i}}V$, for some $i \geq 0$, by Lemma 2.5.

Lemma 3.2 *Let M be a nonzero R-module and let $\mathcal{M} = T(M)$, $V_+(V)$ or $V_-(V)$.*

1. *If there exists a nonzero T-submodule N f $\mathcal{M}$ with $N^{hom} = 0$, then there exists nonzero elements a and b of M such that $\mathrm{ann}_R\, b = \sigma^l(\mathrm{ann}_R\, a)$ for some natural $l \geq 1$.*
2. *Hence, the map*

$$Ra \simeq R/\mathrm{ann}_R\, a \to {}^{\sigma^l}(R/\mathrm{ann}_R\, b) \simeq {}^{\sigma^l}(Rb), \quad r + \mathrm{ann}_R\, a \to \sigma^l(r) + \mathrm{ann}_R\, b,$$

is an isomorphism of R-modules. If, in addition, M is a critical R-module, then the orbit $\mathcal{O}\{M\}$ is finite.

Proof. 1. Let $u = v_m \otimes u_m + \cdots + v_n \otimes u_n$ be a nonzero element in N of minimal length $l = n - m$ where $u_i \in M$, $u_n \neq 0$, $u_m \neq 0$, $n < \cdots < m$. Observe that $l \geq 1$ since $N^{hom} = 0$. Denote by I and J the annihilators in R of the elements u_m and u_n respectively. Then $\sigma^m(I)$ and $\sigma^n(J)$ are the annihilators of $v_m \otimes u_m$ and $v_n \otimes u_n$. By the minimality of l we conclude that $\sigma^m(I) = \sigma^n(J)$ (otherwise, for any nonzero $x \in \sigma^m(I)\backslash\sigma^n(J) \cup \sigma^m(J)\backslash\sigma^n(I)$ the element xu is nonzero and has length less than l). So, $I = \sigma^l(J)$.

2. Evident. ∎

Lemma 3.3 *Let $0 < C \leq B < A$ be a chain of essential T-submodules such that ${}_RC = \oplus_{j \in J} C_j$ is a direct sum of noetherian critical R-modules C_j and A/B is an epimorphic image of a T-module D which, as R-module, is a direct sum $\oplus_{k \in K} D_k$ of noetherian critical R-modules D_k such that $\mathcal{O}\{D_k\} \neq \mathcal{O}\{C_j\}$ and $\mathcal{K}(D_k) \leq \mathcal{K}(C_j)$ for all k, j. Then every homogeneous ideal in T which annihilates B also annihilates A.*

Proof. Let $\mathbf{a} = \oplus_{i \in \mathbb{Z}}\, v_i \otimes \mathbf{a}_i$ be a nonzero homogeneous ideal in T with $\mathbf{a}B = 0$. Let $f : D \to A/B$ be the epimorphism as above. Since $\mathbf{a}B = 0$, we have an R-module epimorphism

$$g : \mathbf{a} \otimes_R D \to \mathbf{a}A, \quad a \otimes d \to af(d).$$

As an R-module $\mathbf{a} \otimes_R D$ is the direct sum $\oplus_{i,k} D_{i,k}$ of R-submodules $D_{i,k} = v_i \otimes \mathbf{a}_i D_k \simeq {}_{\sigma^i}(\mathbf{a}_i D_k)$. The $D_{i,k}$ are critical provided they are nonzero and then $\mathcal{O}(D_{i,k}) = \mathcal{O}(D_k)$. Suppose $\mathbf{a}A \neq 0$, then $\mathbf{a}A \cap C \neq 0$, since C is an essential submodule of A. Choose $0 \neq u \in \mathbf{a}A \cap C$, then Ru is a submodule of $C_{j_1} \oplus \cdots \oplus C_{j_s}$ where s is chosen as small as possible. Clearly, $\mathcal{K}(Ru) = n := \max\{\mathcal{K}(C_{j_\alpha}) \mid \alpha = 1, \dots, s\} = \mathcal{K}(C_{j_\beta})$ for some β. The projection W of Ru on the summand C_{j_β} is a nonzero (by the choise of s) n-critical subfactor of Ru with $\mathcal{O}(W) = \mathcal{O}(C_{j_\beta})$. Let $\tilde{u} \in \mathbf{a} \otimes_R D$ satisfy $g(\tilde{u}) = u$. Thus there is a natural R-module epimorphism $R\tilde{u} \to Ru$, given by $r\tilde{u} \to ru$. The R-submodule $R\tilde{u}$ of $\mathbf{a} \otimes_R D$ is a submodule of some direct sum $D_{i_1k_1} \oplus \cdots \oplus D_{i_tk_t}$ with k as small as possible. Since $\mathcal{K}(D_{i_\alpha k_\alpha}) \leq n$, for all α, we conclude that $\mathcal{K}(R\tilde{u}) = n$. Hence $\mathcal{O}(W) = \mathcal{O}(D_{i_\gamma k_\gamma})$ for some γ; that is, $\mathcal{O}(C_{j_\beta}) = \mathcal{O}(D_{k_\gamma})$, a contradiction. ∎

For any natural number i denote by $[i]$ the ideal of T generated by X^i and Y^i. Then $[i] = \oplus_{j\in\mathbb{Z}}\, v_j \otimes [i]_j$ is the homogeneous ideal with the ideals $[i]_j$ of R satisfying

$$R = [i]_{-i} \supseteq \cdots \supseteq [i]_{-1} \supseteq [i]_0, \quad [i]_0 \subseteq [i]_1 \subseteq \cdots \subseteq [i]_i = R,$$

and $[i]_j = R$, for all $|j| \geq i$. The left and right R-module $T/[i]$ is finitely generated. Clearly,

$$T \geq [1] \geq \cdots \geq [i] \geq \cdots, \qquad [i][j] \supseteq [i+j].$$

Note that any module of type $L_-({}_{\sigma_s}V, V]$ is annihilated by $[|s|]$.

Lemma 3.4 *Let M be a noetherian R-module and let $\mathcal{M}$ be a nonzero subfactor of the T-module $T(M)$. Let $\mathcal{M} = \mathcal{M}_0 > \mathcal{M}_1 > \cdots > \mathcal{M}_{n-1} > \mathcal{M}_n = 0$ be the chain of sumbodules from Proposition 2.13. Then*

1. *$[m]\mathcal{M} = 0$ for some $m \geq 1$ if and only if all subfactors $\mathcal{M}_i/\mathcal{M}_{i+1}$ belong to category 3.*
2. *$X^m\mathcal{M} = 0$ for some $m \geq 1$ if and only if all subfactors $\mathcal{M}_i/\mathcal{M}_{i+1}$ belong to categories 2, 3, 6 and the module V in case of categories 2 or 6 has finite orbit $\mathcal{O}\{V\}$.*
3. *$Y^m\mathcal{M} = 0$ for some $m \geq 1$ if and only if all subfactors $\mathcal{M}_i/\mathcal{M}_{i+1}$ belong to categories 3, 4, 7 and the module V in case of categories 4 or 7 has finite orbit $\mathcal{O}\{V\}$.*

Proof. 1. ($\Leftarrow$) Suppose that all subfactors $\mathcal{M}_i/\mathcal{M}_{i+1}$ belong to category 3. Let j be the maximum of the corresponding $s = s(i)$ as in Proposition 2.13. Then the ideal $[j]$ annihilates all subfactors $\mathcal{M}_i/\mathcal{M}_{i+1}$, hence $[j]^n$ annihilates $\mathcal{M}$. Observe that $[jn] \subseteq [j]^n$.

($\Rightarrow$) Suppose $[m]\mathcal{M} = 0$, for some $m \geq 1$. Then $[m]\mathcal{M}_i/\mathcal{M}_{i+1} = 0$ for all i. Let $\mathcal{V}$ be one of the T-modules $T(V)$, $V_\pm(V)$. The T-submodule $[m]\mathcal{V}$ of $\mathcal{V}$ is nonzero and homogeneous, hence each $\mathcal{M}_i/\mathcal{M}_{i+1}$ does not belong to classes 1, 2, 4-7.

2. ($\Leftarrow$) Let j be the maximum of the corresponding $s = s(i)$ and $|\mathcal{O}(V)|$. Then evidently X^j annihilates all subfactors, hence X^{jn} annihilates $\mathcal{M}$.

($\Rightarrow$) Suppose $X^m\mathcal{M} = 0$ for some $m \geq 1$. Then $X^m(\mathcal{M}_i/\mathcal{M}_{i+1}) = 0$ for all i. Let $\mathcal{V}$ be either $T(V)$ or $V_+(V)$. The T-submodule $TX^m\mathcal{V}$ of $\mathcal{V}$ is nonzero and homogeneous, hence each $\mathcal{M}_i/\mathcal{M}_{i+1}$ does not belong to classes 1, 4, 5, 7.

3. The proof is similar to case 2. ∎

Let S be any ring and let B be a submodule of an S-module A. Take an additive complement, $B^\bullet$, of B in A; that is, a submodule that is maximal subject to having zero intersection with $B^\bullet$. Then $B \oplus B^\bullet$ is an essential submodule of A and $B \equiv (B \oplus B^\bullet)/B^\bullet$ is an essential submodule of $A/B^\bullet$. Whenever we pass from the chain of submodules $0 \leq B \leq A$ to the chain $0 \leq B^\bullet \leq B \oplus B^\bullet \leq A$, we will say that we make a **perestroika** with the first chain.

Theorem 3.5 *Let M be a nonzero, non-simple T-clean noetherian R-module of finite Krull dimension and suppose that the ideal Ra of R have the left Artin-Rees property (for example, if R is left noetherian). Then*

$$\mathcal{K}(T \otimes_R M) = \max\{\alpha_-, \alpha_0, \alpha_+\} + 1$$

where $\alpha_0 = \max\{\mathcal{K}(T \otimes_R N) \mid N \in \mathcal{A}_0\}$, and $\mathcal{A}_0$ is the set of T-clean minor subfactors of M, while $\alpha_\pm = \max\{\mathcal{K}(\mathcal{V}_\pm(N)) \mid N \in \mathcal{A}_\pm\}$, and $\mathcal{A}_\pm$ is the set of $(T, \pm)$-clean minor subfactors of ${}_{\sigma^{\pm i}}M$, for $i \geq 0$.

Remark. It follows, from Lemma 2.6.(5) and Lemma 2.4, that $\alpha_\pm = \max\{\mathcal{K}(V_\pm(N)) \mid N \in \mathcal{A}'_\pm\}$, where $\mathcal{A}'_\pm$ is the set of $(T, \pm)$-clean minor subfactors of ${}_{\sigma^i}M$, for $i \in \mathbb{Z}$.

Proof. At least one of the sets $\mathcal{A}_0, \mathcal{A}_\pm$ is nonempty, by Lemma 2.5, since the R-module M is nonzero and non-simple. Set $\beta = \mathcal{K}(M) < \infty$, and note that $\mathcal{K}(T(M)) \leq \beta + 1 < \infty$, and that $\alpha := \max\{\alpha_-, \alpha_0, \alpha_+\} < \infty$. Clearly, $\mathcal{K}(T(M)) \geq \alpha + 1$. Suppose that $\mathcal{K}(T(M)) = \mathcal{K}(M) = \beta$. There exists either a T-clean, or a $(T, +)$-clean, or a $(T, -)$-clean minor subfactor, V say, of M with Krull dimension $\beta - 1$. Now, $\beta - 1 = \mathcal{K}(V) \leq \mathcal{K}(\mathcal{W}(V))$, where $\mathcal{W}(V) = T(V),\ \mathcal{V}_+(V),\ \mathcal{V}_-(V)$, respectively; so $\alpha \geq \beta - 1$. Thus the claim of the theorem holds. Hence, we need only concentrate on the case where $\mathcal{K}(T(M)) = \beta + 1$. Choose $N \in \mathcal{A} := \mathcal{A}_- \cup \mathcal{A}_0 \cup \mathcal{A}_+$ with $\mathcal{K}(\mathcal{N}) = \alpha$, where $\mathcal{N} = T(N), \mathcal{V}_+(N)$ or $\mathcal{V}_-(N)$, respectively, according to whether $N \in \mathcal{A}_0, \mathcal{A}_+$, or $\mathcal{A}_-$. Then $\mathcal{N}$ is isomorphic to a minor subfactor of $T(M)$, and so $\mathcal{K}(T(M)) > \alpha$.

If $\mathcal{K}(T(M)) > \alpha + 1$ then $\alpha = \beta - 1$, and $\mathcal{K}(T(M)/C) > \alpha$, for some nonzero T-submodule C of $T(M)$ such that $T(M)/C$ is critical. Hence, there exists a chain of T-submodules

$$T(M) \geq C_1 \geq C_2 \geq \ldots > C > 0$$

such that $\mathcal{K}(C_j/C_{j+1}) = \alpha$, for infinitely many j. After refining the chain, we may assume that each C_j/C_{j+1} is one of the modules from Proposition 2.13. By Corollary 2.3, there exists a positive integer p such that, for all $j \geq p$, any finitely generated R-module subfactor of C_j/C_{j+1} has Krull dimension less than β. Hence, for each $j \geq p$, every C_j/C_{j+1} from the categories 5-7 in Proposition 2.13 has Krull dimension less than α.

Thus, if, for some $j \geq p$, we have $\mathcal{K}(C_j/C_{j+1}) = \alpha$, then C_j/C_{j+1} belongs to one of the sets 1-4 in Proposition 2.13.

Set $\lambda = \lambda_+$ or λ_-. Inside M there is a chain of R-submodules

$$M \geq \lambda(C_1) \geq \lambda(C_2) \geq \ldots \geq \lambda(C) > 0$$

(note that $\lambda(C) \neq 0$, since M is a-torsionfree and $C \neq 0$).

By Lemma 2.10, there exists a positive integer q such that, for all $j \geq q$,
(i) each module $\lambda(C_j)/\lambda(C_{j+1})$ has no T-clean subfactors N for which $h(N : M) = 1$;
(ii) each module ${}_{\sigma^{\pm i}}(\lambda(C_j)/\lambda(C_{j+1}))$, for $i \geq 0$, has no $(T, \pm)$-clean submodule N such that $h_{\pm}(N : {}_{\sigma^{\pm i}}M) = 1$.

Choose $m \geq \max\{p, q\}$ such that $\mathcal{K}(C_m/C_{m+1}) = \alpha$. We aim to show that C_m/C_{m+1} belongs to category 3 of Proposition 2.13. It is enough to prove that C_m/C_{m+1} is not in categories 1 and 4 (since the category 2 is dual to category 4). Suppose that the module $C_m/C_{m+1} = T(V)$, or $C_m/C_{m+1} = V_+(V)$, belongs to category 1 or 4, respectively. Then $\mathcal{K}(V) < \beta$, since $m \geq p$, and hence V is a minor subfactor of some ${}_{\sigma^k}M$, since M is a critical R-module. Set $\lambda = \lambda_+$. By Proposition 1.3, there exists a nonzero subfactor N of $\lambda(C_m)/\lambda(C_{m+1})$ such that $N \leq {}_{\sigma^i}V$, for some $i \in \mathbb{Z}$, or $i \geq 0$ respectively. The R-module V is T-clean in case 1, or $(T, +)$-clean in case 4, and hence so is ${}_{\sigma^i}V$, for $i \in \mathbb{Z}$, or $i \geq 0$, respectively. In any case, the R-module N is either T-clean (in case 1) or $(T, +)$-clean (in case 4); and

$$\alpha = \mathcal{K}(T(V)) = \mathcal{K}(T({}_{\sigma^i}V)) = \mathcal{K}(T(N))$$

in case 1, while

$$\alpha = \mathcal{K}(V_+(V)) = \mathcal{K}(\mathcal{V}_+({}_{\sigma^i}V)) = \mathcal{K}(\mathcal{V}_+(N))$$

in case 4.

The R-module N is T-clean (or $(T, +)$-clean) and $m \geq q$. Thus there exists either
(a) a nonzero R-submodule L of N and a T-clean minor subfactor G of M such that L is isomorphic to a minor subfactor of G (case 1), or
(b) a nonzero R-submodule L of ${}_{\sigma^n}N$, for some $n \geq 0$, and a $(T, +)$-clean minor subfactor G of ${}_{\sigma^n}M$ such that L is isomorphic to a minor subfactor of G (case 1 or 4).

Suppose that case (a) holds. Then

$$\alpha = \mathcal{K}(T(N)) = \mathcal{K}(T(L)) < \mathcal{K}(T(G)) \leq \alpha,$$

a contradiction.

Similarly, if case 1 and case (b) hold, then

$$\alpha = \mathcal{K}(\mathcal{V}_+(N)) = \mathcal{K}(\mathcal{V}_+(L)) < \mathcal{K}(T(G)) \leq \alpha,$$

while if case 4 and case (b) hold, then

$$\alpha = \mathcal{K}(\mathcal{V}_+(N)) = \mathcal{K}(\mathcal{V}_+(L)) < \mathcal{K}(\mathcal{V}_+(G)) \leq \alpha.$$

In each case, we have a contradiction.

So, for every $m \geq r = \max\{p, q\}$, a factor C_m/C_{m+1} that has Krull dimension $\beta - 1$ must belong to category 3 of Proposition 2.13. The set $\{C_{m(i)}/C_{m(i)+1} \mid i = 1, 2, \ldots\}$ of all such factors is infinite.

Consider the chain of submodules

$$A_1 \geq B_1 \geq A_2 \geq B_2 \geq A_3 \geq \ldots > C \tag{9}$$

where $A_1 = C_r$, $B_i = C_{m(i)}$ and $A_{i+1} = C_{m(i)+1}$. Observe that $\mathcal{K}(B_i/A_{i+1}) = \beta - 1$ and $\mathcal{K}(A_i/B_i) < \beta - 1$, for all i. Making a perestroika with the chain $A_1 \geq B_1 \geq A_2$, we obtain a chain $A_1 \geq B_1' \geq A_2' \geq A_2$, where the module B_1'/A_2' is essential in A_1/A_2', and is isomorphic to B_1/A_2, and the first and third factors in the chain have Krull dimension less than $\beta - 1$. Continuing, by making a perestroika with the chain $A_2' \geq B_2 \geq A_3$, we get a chain $A_2' \geq B_2' \geq A_3' \geq A_3$ with the same properties as the new chain above. After making all perestroikas, we have a chain

$$A_1' = A_1 \geq B_1' \geq A_2' \geq B_2' \geq A_3' \geq \ldots > C \tag{10}$$

where every subfactor B_i'/A_{i+1}' is essential in A_i'/A_{i+1}', belongs to category 3 of Proposition 2.13 and has Krull dimension $\beta - 1$. Further, all other subfactors in the chain have Krull dimension less than $\beta - 1$.

The module $T(M)/C$ is critical with the same Krull dimension as A_1'/C', where $C' := \cap_{i=1}^{\infty} A_i$, hence, $C' = C$. Without loss of generality, we may assume that chains 9 and 10 coincide. Now, note that the T-module B_i/A_{i+1} is isomorphic to $L_-({}_{\sigma^{s(i)}}V_i, V_i]$, by Proposition 2.13.(3) and $T(M) = \oplus_{j \in \mathbb{Z}} (v_j \otimes M)$, where ${}_R(v_j \otimes M) \simeq {}_{\sigma^j}M$. Thus, we conclude that the critical R-module V_i is isomorphic to a subfactor of some ${}_{\sigma^j}M$, for $j = j(i)$.

The R-module M is a-torsionfree, thus

$$\mathcal{K}({}_{\sigma^j}M/a\,{}_{\sigma^j}M) < \mathcal{K}({}_{\sigma^j}M) = \beta.$$

By Lemma 2.6.(2), $\mathcal{K}(V_i) = \beta - 1$ and, by Proposition 3.1, V_i is isomorphic to a subfactor of ${}_{\sigma^j}M/a_{\sigma^j}M$. It follows that

$$\beta - 1 = \mathcal{K}(V_i) = \mathcal{K}({}_{\sigma^j}M/a\,{}_{\sigma^j}M) = \mathcal{K}(M/aM),$$

that V_i is a $(\beta-1)$-critical subfactor of ${}_{\sigma^j}M/a_{\sigma^j}M$, and that the orbit $\mathcal{O}\{V_i\}$ belongs to the finite set of orbits, Λ, of the $(\beta-1)$-critical subfactors of the R-module M/aM. Note that the orbit $\mathcal{O}\{V_i\}$ is infinite, and $St(V_i)$ is finite, since $\alpha=\beta-1$, using Lemma 2.6.(3). Denote by Ω the subset of Λ which consists of the infinite orbits $\mathcal{O}_\gamma=\mathcal{O}_\gamma\{U_\gamma\}$, where $\gamma=1,\dots,s$, which have finitely many, but at least two, stars; that is, subisomorphism classes annihilated by a. Note that $\{W\}$ is a star of $\mathcal{O}_\gamma$ if and only if $\{W\}=\{{}_{\sigma^j}U_\gamma\}$, for $j\in St(U_\gamma)$. Observe that $\mathcal{O}\{V_i\}\in\Omega$ and V_i is a star of $\mathcal{O}\{V_i\}$.

Denote by m the maximum value of $m_\gamma:=S_\gamma^{\max}-S_\gamma^{\min}$, for $\gamma=1,\dots,s$, where $S_\gamma^{\max}$ and $S_\gamma^{\min}$ are the maximal and minimal elements of $St(U_\gamma)$. Note that m_γ does not depend on the choice of U_γ but only on the orbit $\mathcal{O}_\gamma$.

Let

$$\mathrm{Int}:=\{\{{}_{\sigma^j}U_\gamma\}\mid\gamma=1,\dots,s;\ S_\gamma^{\min}<j\le S_\gamma^{\max}\}.$$

Then every T-module B_i/A_{i+1} is annihilated by the homogeneous ideal $[m]$ of T that is generated by X^m and Y^m.

The T-submodule B_i/A_{i+1} of A_i/A_{i+1} is essential and is an epimorphic image of $T(V_i)$. Every other subfactor C_j/C_{j+1} of A_i/A_{i+1} belongs to one of the categories 3, 5-7 in Proposition 2.13 and is an epimorphic image of $T(V_j)$, where $\mathcal{K}(V_j)\le\beta-1=\mathcal{K}(V_i)$. Every orbit $\mathcal{O}\{V_j\}$ is distinct from $\mathcal{O}\{V_i\}$, since in the last three categories, $\mathcal{O}\{V_j\}$ is finite, by Lemma 3.2, and $\mathcal{K}(V_j)<\beta-1=\mathcal{K}(V_i)$, in the first case. Thus, applying Lemma 3.3, step by step, to the chain of submodules

$$0\le B_i/A_{i+1}=C_{m(i)}/C_{m(i)+1}\le C_{m(i)-1}/C_{m(i)+1}\le\dots\le C_{m(i-1)+1}/C_{m(i)+1}=A_i/A_{i+1},$$

we conclude that $[m](A_i/A_{i+1})=0$. Then, by Lemma 3.4, all subfactors $C_j/C_{j+1}\simeq L_-({}_{\sigma^{s(j)}}V_j,V_j]$ are from category 3 in Proposition 2.13.

Therefore, every A_1/A_i has a finite chain of R-submodules

$$A_1=D_1>D_2>\dots>D_t=A_i,\qquad t=t(i)$$

with factors isomorphic to one of the direct summands ${}_{\sigma^k}V_j$ of $L_-({}_{\sigma^{s(j)}}V_j,V_j]=\oplus_{s(j)<k\le 0}\,{}_{\sigma^k}V_j$. We use induction on i to show that $[m]$ annihilates A_1/A_i. This is certainly true for $i=2$; so assume, by induction, that $[m](A_1/A_{i-1})=0$, or, equivalently, that

$$[m]A_1=TX^mA_1+TY^mA_1\le A_{i-1}.$$

Hence, $E=[m](A_1/A_i)$ is a T-submodule of A_{i-1}/A_i. If $E\neq0$, then $E\cap(B_{i-1}/A_i)\neq0$ and so the Krull dimension of the T-module E is $\beta-1$. On the other hand, inside A_{i-1}/A_i there is a chain of R-modules $X^mA_1+A_i=X^mD_1+A_i\ge X^mD_2+A_i\ge\dots\ge X^mD_t+A_i=A_i$, and each factor $(X^mD_j+A_i)/(X^mD_{j+1}+A_i)$ is an epimorphic image of the R-module ${}_{\sigma^m}(D_j/D_{j+1})$. Clearly, $\mathcal{K}_R(A_{i-1}/A_i) = \beta - 1$ and the subisomorphism class of every $(\beta-1)$-critical subfactor of the R-module A_{i-1}/A_i belongs to Int.

If $\mathcal{K}(D_j/D_{j+1}) = \beta - 1$, then $\{{}_{\sigma^m}(D_j/D_{j+1})\} \notin$ Int, by the choice of m, hence the factor $(X^m D_j + A_i)/(X^m D_{j+1} + A_i)$ is a proper epimorphic image of ${}_{\sigma^m}(D_j/D_{j+1})$ and so has Krull dimension less than $\beta - 1$. Finally, $\mathcal{K}_R(X^m(A_1/A_i)) < \beta - 1$, and similarly, $\mathcal{K}_R(Y^m(A_1/A_i)) < \beta - 1$.
Now,

$$\beta - 1 = \mathcal{K}_T([m](A_1/A_i)) \leq \mathcal{K}_R([m](A_1/A_i)) = \mathcal{K}_R(T \cdot U),$$

where $U = X^m(A_1/A_i) + Y^m(A_1/A_i)$ is an R-submodule of A_{i-1}/A_i of Krull dimension $< \beta - 1$. Consider the R-module epimorphism

$$T \otimes_R U = \oplus_{j \in \mathbb{Z}} (v_j \otimes U) \twoheadrightarrow T \cdot U, \qquad t \otimes u \mapsto tu.$$

The R-module A_{i-1}/A_i is noetherian, so there is a finite sum $F = \oplus (v_j \otimes U)$ which maps onto $T \cdot U$. However, $\mathcal{K}_R(F) = \mathcal{K}_R(\oplus_{\sigma^j} U) < \beta - 1$. Hence $\mathcal{K}_R(T \cdot U) < \beta - 1$, a contradiction, so that $[m]A_1 \subseteq A_i$. This now holds for each i, and so $[m]A_1 \subseteq \cap_{i=1}^{\infty} A_i = C$. Hence, $[m]$ annihilates the module A_1/C.

We conclude that

$$\beta = \mathcal{K}_T(A_1/C) = \mathcal{K}_T(L_-({}_{\sigma^k}W, W]) = \mathcal{K}_R(W),$$

by Lemma 3.4 and Lemma 2.6.(2), where $L_-({}_{\sigma^k}W, W]$ is isomorphic to a subfactor of A_1/C. We have shown above that W is isomorphic to a subfactor of some ${}_{\sigma^r}M$. Since both of the R-modules W and ${}_{\sigma^r}M$ are critical and have the same Krull dimension, β, this forces W to be a submodule of ${}_{\sigma^r}M$, leading to $\mathcal{O}\{W\} = \mathcal{O}\{M\}$, which is contradiction, since M is a-torsionfree, whereas $aW = 0$. ∎

4 Krull dimension formulae

The formulae given in Proposition 2.8 and Theorem 3.5 will be used as an inductive step by which the Krull dimension of an induced module $T \otimes_R M$ may be computed in terms of the Krull dimension of the modules $T \otimes_R A$, $\mathcal{V}_\pm(A)$, where A runs through the simple subfactors of M.

Definition. Let A be a simple R-module and let M be an arbitrary R-module.

1. If A is a-torsionfree, then $h(A : M)$ is defined to be the supremum of those non-negative integers n for which there exists a sequence $A = A_0, A_1, \dots, A_n$ of T-clean R-modules

such that A_i is isomorphic to a minor subfactor of A_{i+1}, for $i = 0, \dots, n-1$, while A_n is isomorphic to a subfactor (not necessarily minor) of M.

2. If A is a-torsion, then $h_+(A : M)$, $(h_-(A : M))$, is defined to be the supremum of those non-negative integers n for which there exists a sequence $A = A_0, A_1, \dots, A_n$ of R-modules such that each of the R-modules $A = A_0, A_1, \dots, A_i$, for some $i \geq 0$, is $(T, +)$-clean, $((T, -)$-clean), while each of the R-modules $A_{i+1}, \dots, A_n$ is T-clean; and A_j is isomorphic to a minor subfactor of A_{j+1}, for $j = 0, \dots n-1$, while A_n is isomorphic to a subfactor (not necessarily minor) of M.

Remark. Any simple a-torsionfree R-module is T-clean. However, a simple a-torsion R-module may fail to be either $(T, +)$-clean or $(T, -)$-clean. So, $h_\pm(A, M)$ is only defined for the simple a-torsion modules which are $(T, \pm)$-clean.

The sequence of R-modules $A = A_0, A_1, \dots, A_n$ is called a $(T, \bullet)$**-clean sequence associated with** A in M, where $\bullet$ stands for $\emptyset, +$ or $-$. The corresponding $h_\bullet(A : M)$ is called the $\bullet$**-height** of A in M. If M has Krull dimension, then $h_\bullet(A : M) \leq \mathcal{K}(M)$, since $\mathcal{K}(A_n) > \mathcal{K}(A_{n-1}) > \dots > \mathcal{K}(A_0) = 0$, so $\mathcal{K}(M) \geq n$.

Observe that if $A = A_0, A_1, \dots, A_n$ is a T-clean (or $(T, \pm)$-clean) sequence associated with A in M, then ${}_{\sigma^i}A = {}_{\sigma^i}A_0, {}_{\sigma^i}A_1, \dots, {}_{\sigma^i}A_n$ is a T-clean (or $(T, \pm)$-clean) sequence associated with ${}_{\sigma^i}A$ in ${}_{\sigma^i}M$, for $i \in \mathbb{Z}$ (or $\pm i \geq 0$). Thus, $h(A : M) = h({}_{\sigma^i}A : {}_{\sigma^i}M)$, for $i \in \mathbb{Z}$ (or $h_\pm(A, M) \leq h_\pm({}_{\sigma^i}A, {}_{\sigma^i}M)$, for $\pm i \geq 0$).

Theorem 4.1 *Let M be a nonzero noetherian R-module with finite Krull dimension. Suppose that the ideal Ra of R has the left AR property.*

Then
1.

$$\mathcal{K}(T \otimes_R M) = \mu(M) := \max\{\mu_-, \mu_0, \mu_+\}$$

where

$$\mu_0 = \max\{\mathcal{K}(T \otimes_R A) + h(A : M) \mid A \in \mathcal{M}\}$$

and $\mathcal{M}$ is the set of a-torsionfree simple subfactors of M;

$$\mu_\pm = \max\{\mathcal{K}(\mathcal{V}_\pm(A_i)) + h_\pm(A_i : {}_{\sigma^{\pm i}}M) \mid A_i \in \mathcal{M}_{\pm,i},\ i \geq 0\}$$

and $\mathcal{M}_{\pm,i}$ is the set of $(T, \pm)$-clean a-torsion simple subfactors of ${}_{\sigma^{\pm i}}M$.

2. Let $A = R/I$ be a simple R-module, where I is a maximal left ideal of R.

(i) If A is a-torsionfree, then $\mathcal{K}(T \otimes_R A) = 1$ if and only if the orbit, $\mathcal{O}\{A\}$, is finite; that is, ${}^{\sigma^n}A \simeq A$, for some $n \geq 1$, or, equivalently, $\sigma^n(I)x \subseteq I$, for some $x \in R \backslash I$, and some $n \geq 1$.

(ii) If A is a-torsion, then $\mathcal{K}(\mathcal{V}_\pm(A)) = 1$ if and only if the set

$$St_\pm(A) := \{0 < \pm i \in \mathbb{Z} \mid a\,{}_{\sigma^i}A = \sigma^{-i}(a)A = 0\}$$

is infinite.

Remark. If A is a simple a-torsion R-module, then ${}_{\sigma^{\pm i}}A$ is $(T,\pm)$-clean for $i \gg 0$, by Lemma 2.5.(2,3).

Proof. Set $n = \mathcal{K}(M)$. If $n = 0$, then every $(T,\bullet)$-clean subfactor of ${}_{\sigma^j}M$, for $j \in \mathbb{Z}$, is a simple R-module of zero height, so the result follows from Lemma 2.7.(2).

Now, let $n \geq 1$, and assume that the result holds for noetherian R-modules of Krull dimension less than n. If $A \in \mathcal{M}$ and $h(A : M) = m$, then there exists a T-clean sequence $A = A_0, A_1, \ldots, A_m$ associated with A in M. Then each $T \otimes_R A_i$ is isomorphic to a minor subfactor of the critical T-module $T \otimes_R A_{i+1}$, hence

$$\mathcal{K}(T \otimes_R M) \geq \mathcal{K}(T \otimes_R A_m) > \mathcal{K}(T \otimes_R A_{m-1}) > \ldots > \mathcal{K}(T \otimes_R A_0).$$

Thus,

$$\mathcal{K}(T \otimes_R M) \geq \mathcal{K}(T \otimes_R A_0) + h(A : M)$$

and so $\mathcal{K}(T \otimes_R M) \geq \mu_0$.

If $A \in \mathcal{M}_{\pm,i}$, for some $i \geq 0$, and $h_\pm(A : {}_{\sigma^{\pm i}}M) = m$, then there exists a $(T,\pm)$-clean sequence $A = A_0, A_1, \ldots, A_m$ associated with A in ${}_{\sigma^{\pm i}}M$, with each of the R-submodules $A_0, \ldots, A_j$, for some $0 < j \leq m$, being $(T,\pm)$-clean, while each of the R-modules $A_{j+1}, \ldots, A_m$ is T-clean (if $j < m$). Then each $\mathcal{V}_\pm(A_k)$, for $0 \leq k < j$, is isomorphic to a minor subfactor of the critical T-module $\mathcal{V}_\pm(A_{k+1})$; while $\mathcal{V}_\pm(A_j)$ is isomorphic to a minor subfactor of the T-critical module $T \otimes_R A_{j+1}$, and each $T \otimes_R A_l$, for $j+1 \leq l < m$, is isomorphic to a minor subfactor of the T-critical module $T \otimes_R A_{l+1}$. Hence,

$$\begin{aligned} \mathcal{K}(T \otimes_R M) &\geq \mathcal{K}(T \otimes_R A_m) > \ldots > \mathcal{K}(T \otimes_R A_{j+1}) \\ &> \mathcal{K}(\mathcal{V}_\pm(A_j)) > \mathcal{K}(\mathcal{V}_\pm(A_{j-1})) > \ldots > \mathcal{K}(\mathcal{V}_\pm(A_0)). \end{aligned}$$

Thus, we obtain

$$\mathcal{K}(T \otimes_R M) \geq \mathcal{K}(\mathcal{V}_\bullet(A_m)) \geq \mathcal{K}(\mathcal{V}_\pm(A_0)) + m = \mathcal{K}(\mathcal{V}_\pm(A)) + h_\pm(A : {}_{\sigma^{\pm i}}M),$$

where $\mathcal{V}_\bullet(A_m) = T \otimes_R A_m$, if $j < m$, while $\mathcal{V}_\bullet(A_m) = \mathcal{V}_\pm(A_m)$, if $j = m$. Consequently, $\mathcal{K}(T \otimes_R M) \geq \mu_\pm$, and, finally, $\mathcal{K}(T \otimes_R M) \geq \mu$.

Without loss of generality, we may suppose that M is $(T,\bullet)$-clean, by Lemma 2.7.(2).

If M is $(T, \pm)$-clean then, by Proposition 2.8, there exists a $(T, \pm)$-clean minor subfactor, N say, of ${}_{\sigma^{\pm i}}M$, for some $i \geq 0$, such that

$$\mathcal{K}(\mathcal{V}_{\pm}(M)) = \mathcal{K}(\mathcal{V}_{\pm}(N)) + 1.$$

Note that here we are using the fact that if N is $(T, \pm)$-clean then so is ${}_{\sigma^{\pm i}}N$, for all $i \geq 0$.

Since M is critical, so is ${}_{\sigma^{\pm i}}M$, and so $\mathcal{K}(N) < n$. Hence, using the inductive hypothesis, there exists a simple subfactor A of ${}_{\sigma^{\pm j}}N$, for some $j \geq 0$, such that

$$\mathcal{K}(\mathcal{V}_{\pm}(N)) = \mathcal{K}(\mathcal{V}_{\pm}(A)) + h_{\pm}(A : {}_{\sigma^{\pm j}}N).$$

Consequently,

$$\mathcal{K}(\mathcal{V}_{\pm}(M)) = \mathcal{K}(\mathcal{V}_{\pm}(A)) + h_{\pm}(A : {}_{\sigma^{\pm j}}N) + 1.$$

In addition, $A \in \mathcal{M}_{\pm,(i+j)}$, and $h_{\pm}(A : {}_{\sigma^{\pm(i+j)}}(M)) \geq h_{\pm}(A : {}_{\sigma^{\pm j}}N) + 1$, since N is a minor subfactor of the $(T, \pm)$-clean module ${}_{\sigma^{\pm i}}M$. Hence,

$$\mathcal{K}(\mathcal{V}_{\pm}(M)) \leq \mathcal{K}(\mathcal{V}_{\pm}(A)) + h_{\pm}(A : {}_{\sigma^{\pm(i+j)}}(M)) \leq \mu(M).$$

If M is T-clean, then by Theorem 3.5, there exists either a T-clean minor subfactor N of M such that $\mathcal{K}(T \otimes_R M) = \mathcal{K}(T \otimes_R N) + 1$, or a $(T, \pm)$-clean minor subfactor $N_{\pm}$ of ${}_{\sigma^{\pm i}}M$, for some $i \geq 0$, such that $\mathcal{K}(T \otimes_R M) = \mathcal{K}(\mathcal{V}_{\pm}(N)) + 1$. Note that $\mathcal{K}(N) < n$, since M is critical.

In the latter two cases, we use the same arguments as above to show that $\mathcal{K}(T \otimes_R M) \leq \mu_{\pm}$. In the first case, $\mathcal{K}(T \otimes_R N) = \mu(N)$, by the inductive hypothesis. Clearly, $\mu(M) \geq \mu(N) + 1$, hence $\mathcal{K}(T \otimes_R M) = \mu(N) + 1 \leq \mu(M)$.

2.(i) As an R-module, $T \otimes_R A$ is the direct sum of simple modules $\oplus_{i \in \mathbb{Z}}\, {}_{\sigma^i}A$. Thus, if the orbit, $\mathcal{O}\{A\}$, of A is infinite (that is, ${}_{\sigma^i}A \not\simeq {}_{\sigma^j}A$, for all $i \neq j$), then $T \otimes_R A$ is a simple T-module and so $\mathcal{K}(T \otimes_R A) = 0$. If the orbit $\mathcal{O}(A)$ is finite, and contains n elements, then ${}^{\sigma^n}A \simeq A$, and there exists $x \in R \backslash I$ with $\sigma^n(I)x \subseteq I$. The nonzero element $u := 1 \otimes (1 + I) + (Y^n \otimes x)$ is annihilated by I; moreover, the submodule Tu of $T \otimes_R A$ is proper and isomorphic to $T \otimes_R A$, by using the $\mathbb{Z}$-grading of $T \otimes_R A$ and the fact that A is a-torsionfree. Therefore, $1 \leq \mathcal{K}(T \otimes_R A) \leq \mathcal{K}(A) + 1$, and so $\mathcal{K}(T \otimes_R A) = 1$.

(ii)If $St_{\pm}(A)$ is infinite, then $\mathcal{K}(\mathcal{V}_{\pm}(A)) = 1$, by (4) or (5). If $St_{\pm}(A)$ is finite, then the orbit $\mathcal{O}\{A\}$ is infinite. Let h be the maximal hole in $Ho_{+}(A)$, and let s be the minimal star in $St_{-}(A)$. Then $\mathcal{K}(\mathcal{V}_{+}(A)) = \mathcal{K}(V_{+}({}_{\sigma^h}A))$ and $\mathcal{K}(\mathcal{V}_{-}(A)) = \mathcal{K}(V_{-}({}_{\sigma^s}A))$.

As R-modules $V_{+}({}_{\sigma^h}A) = \oplus_{j \geq h}\, {}_{\sigma^j}A$ and $V_{-}({}_{\sigma^s}A) = \oplus_{j \leq s}\, {}_{\sigma^j}A$ are the direct sums of non-isomorphic R-modules, hence, as T-modules, they are simple, and so $\mathcal{K}(\mathcal{V}_{\pm}(A)) = 0$. ∎

Corollary 4.2 *Let R be a left noetherian ring with finite Krull dimension. Then*

$$\mathcal{K}(T) = \max\{\mathcal{K}(\mathcal{V}_\bullet(A)) + h_\bullet(A : R) \mid A \in \mathcal{M},\ \bullet \in \{\emptyset, \pm\}\}$$

where $\mathcal{M}$ is the family of simple $(T, \bullet)$-clean R-modules, and $\mathcal{V}_\bullet(A)$ is equal to $T \otimes_R A$ when A is a-torsionfree, and is equal to $\mathcal{V}_\pm(A)$ when A is a-torsion.

Proof. Observe that $T \simeq T \otimes_R R$, and that ${}_{\sigma^i}R \simeq R$, for all $i \in \mathbb{Z}$. ∎

Corollary 4.3 *Let R be a left noetherian ring with finite Krull dimension. Then*

$$\mathcal{K}(T) = \mathcal{K}(R)$$

unless there exists a simple $(T, \bullet)$-clean R-module A such that $\mathcal{K}(\mathcal{V}_\bullet(A)) = 1$ and $h_\bullet(A : R) = \mathcal{K}(R)$, in which case

$$\mathcal{K}(T) = \mathcal{K}(R) + 1. \quad ∎$$

Corollary 4.4 *Let R be a left noetherian ring with finite Krull dimension. Then*

$$\mathcal{K}(T) = \mathcal{K}(R)$$

unless there exists either
(i) a simple a-torsionfree R-module A such that $h(A : R) = \mathcal{K}(R)$ and $\mathcal{O}\{A\}$ is finite; or
(ii) a simple $(T, +)$-clean R-module A such that $h_+(A : R) = \mathcal{K}(R)$ and the set $St_+(A)$ is infinite; or
(iii) a simple $(T, -)$-clean R-module A such that $h_-(A : R) = \mathcal{K}(R)$ and the set $St_-(A)$ is infinite. ∎

5 Fully Bounded Noetherian rings

In this section, we specialize our results to the case of Generalized Weyl Algebras, $T = R(\sigma, a)$, where R is a fully bounded noetherian ring. In this case we recover the same formula as that obtained in the case of a commutative noetherian base ring in [7].

Lemma 5.1 *Let R be a left noetherian ring, and let A be a simple R-module. Set $\mathbf{p} = \mathrm{Ann}_R(A)$, and assume that $R/\mathbf{p}$ is a simple artinian ring. Then*

1. *If A is a-torsionfree, then $\mathcal{K}(T \otimes_R A) = 1$ if and only if $\mathbf{p}$ is invariant under some nonzero power of σ.*

2. *If A is a-torsion, then $\mathcal{K}(\mathcal{V}_\pm(A)) = 1$ if and only if the set $\mathbf{Z}_\pm(\mathbf{p}) := \{i \geq 0 \mid a \in \sigma^{\pm i}(\mathbf{p})\}$ is finite.*

Proof. 1. By Theorem 4.1.2(i), $\mathcal{K}(T \otimes_R A) = 1$ if and only if the orbit $\mathcal{O}\{A\}$ is finite, or, equivalently, $\sigma^n(\mathbf{p}) = \mathbf{p}$, for some $n \geq 1$, since $R/\mathbf{p}$ is a simple artinian ring.

2. By Theorem 4.1.2(ii), $\mathcal{K}(\mathcal{V}_\pm(A)) = 1$ if and only if the set $St_\pm$ is infinite, or, equivalently, if and only if the set $\mathbf{Z}_\pm(\mathbf{p})$ is infinite, since $R/\mathbf{p}$ is a simple artinian ring. ▪

Proposition 5.2 *Let R be a fully bounded noetherian ring, and let A be a simple R-module. Set $\mathbf{p} = \mathrm{Ann}_R(A)$.*

If A is a-torsionfree, then $h(A : R) = \mathrm{height}(\mathbf{p})$.

If A is a-torsion, then

$$\mathrm{height}(\mathbf{p}) = \max\{h_+({}_{\sigma^j}A : R) \mid j \gg 0\} = \max\{h_-({}_{\sigma^j}A : R) \mid j \ll 0\}.$$

Proof. If A is a-torsion, then ${}_{\sigma^{\pm j}}A$ is $(T, \pm)$-clean, for $j \gg 0$; and $h_\pm({}_{\sigma^{\pm j}}(A : R)) \leq h_\pm({}_{\sigma^{\pm k}}(A : R))$, for $k \geq j \gg 0$. Hence, without loss of generality, in the case that A is a-torsion, we may assume that A is $(T, \pm)$-clean. Suppose that there is a $(T, \bullet)$-clean sequence $A = A_0, A_1, \ldots, A_n$ associated with A in R. Each of the annihilators $\mathbf{p}_i := \mathrm{Ann}_R(A_i)$ is a prime ideal of R and A_i is a nonsingular $R/\mathbf{p}_i$-module, so that $\mathcal{K}(A_i) = \mathcal{K}(R/\mathbf{p}_i)$, see, for example [10, Theorem 2.5, Proposition 1.4]. Now, $\mathbf{p}_i > \mathbf{p}_{i+1}$, for each i, since $\mathbf{p}_i \geq \mathbf{p}_{i+1}$ and $\mathcal{K}(\mathbf{p}_i) > \mathcal{K}(\mathbf{p}_{i+1})$. Thus, $\mathrm{height}(\mathbf{p}) \geq n$, and so $\mathrm{height}(\mathbf{p}) \geq h_\bullet(A : R)$, By combining this with the facts that $\mathrm{Ann}_R({}_{\sigma^j}A) = \sigma^j(\mathbf{p})$ and $\mathrm{height}(\sigma^j(\mathbf{p})) = \mathrm{height}(\mathbf{p})$, for all $j \in \mathbb{Z}$, we obtain, in the case that A is a-torsion,

$$\mathrm{height}(\mathbf{p}) \geq h_\pm := \max\{h_\pm({}_{\sigma^{\pm j}}A : R) \mid j \gg 0\}.$$

Next, consider a strictly descending chain of prime ideals $\mathbf{p} = \mathbf{p}_0 > \mathbf{p}_1 > \ldots > \mathbf{p}_n$. Set $A = A_0$. By repeating the argument in [9, Proposition 6.2], we can construct cyclic critical $R/\mathbf{p}_i$-modules A_i, for each $i < n$, such that A_i is isomorphic to a minor subfactor of A_{i+1}. Since A_n is cyclic, it is isomorphic to a subfactor of R. Each A_i is compressible, by [10, Theorem 2.5]. If A_i is a-torsionfree, then A_i is T-clean, by Lemma 2.5, and so ${}_{\sigma^j}A_i$ is T-clean, for each $j \in \mathbb{Z}$. If A_i is a-torsion, then, by using Lemma 2.5.(2), or Lemma 2.5.(3), we conclude either that ${}_{\sigma^j}A_i$ is $(T, +)$-clean, for almost all positive integers j, or that ${}_{\sigma^j}A_i$ is $(T, -)$-clean, for almost all negative integers j.

Hence, if A is a-torsionfree, then $h(A : R) \geq n$, and so $h(A : R) \geq \mathrm{height}(\mathbf{p})$; while if A is a-torsion then there exist a positive integer, j say, such that each ${}_{\sigma^j}A_i$ is either T-clean or $(T, +)$-clean, so that $h_+({}_{\sigma^j}A : R) \geq n$, and $h_+ \geq \mathrm{height}(\mathbf{p})$; similarly, $h_- \geq \mathrm{height}(\mathbf{p})$. ▪

Theorem 5.3 *Let R be a fully bounded noetherian ring with finite Krull dimension. Then $\mathcal{K}(T) = \mathcal{K}(R)$ unless there exists a maximal ideal $\mathbf{p}$ of R such that* $\text{height}(\mathbf{p}) = \mathcal{K}(R)$ *and either $\mathbf{p}$ is invariant under some nonzero power of σ, or there are infinitely many $i \in \mathbb{Z}$ with $\sigma^i(a) \in \mathbf{p}$.*

Proof. This follows from Corollary 4.4 and the above Proposition. ∎

Corollary 5.4 *Let R be a fully bounded noetherian ring with finite Krull dimension. Then the left and right Krull dimensions of T coincide.*

Proof. This follows immediately form the above result, by using the $\pm$-symmetry. ∎

6 Examples

In this final section, we present a class of examples that demonstrates that the methods we have introduced and the theorems we have proved do, indeed, enable one to calculate Krull dimensions.

Let R be a ring. Suppose that we are given the following data: $\sigma \in \text{Aut}(R)$ and b, $\rho \in Z(R)$, the centre of R, with ρ being a σ-stable unit; that is, $\sigma(\rho) = \rho$. We form an overing, $E = R\langle\sigma; b, \rho\rangle$, by adjoining symbols X and Y to R subjected to the relations:

$$Xr = \sigma(r)X, \ Yr = \sigma^{-1}(r)Y, \text{ for all } r \in R; \quad XY - \rho YX = b.$$

As an example, observe that in the case that $R = K[H]$, $\rho = 1$ and σ is defined by $\sigma(H) = H - 1$, while $b = 2H$, then E is the enveloping algebra $U(sl(2))$.

We may view E as the iterated skew polynomial ring $E = R[Y; \sigma^{-1}][X; \sigma, \partial]$, where ∂ is a σ-derivation of $R[Y; \sigma^{-1}]$ and $\partial R = 0$, while $\partial Y = b$; moreover σ is extended from R to $R[Y; \sigma^{-1}]$ by setting $\sigma(Y) = \rho Y$.

When R is left Noetherian it is known, [15, 6.5.4], that

$$\mathcal{K}(R\langle\sigma; b, \rho\rangle) = \mathcal{K}(R) + 1 \text{ or } \mathcal{K}(R) + 2.$$

It is not trivial to decide which of the two cases actually happens.

The rings of type E fall within in the scope of this paper because, as the lemma below shows, the rings $E = D\langle\sigma; b, \rho\rangle$ are generalized Weyl algebras, see [4].

Lemma 6.1 *Let R be a ring; then $R\langle\sigma; b, \rho\rangle \simeq R[H](\sigma, H)$ and σ is extended from R to $R[H]$ by $\sigma(H) = \rho H + b$.* ∎

If $b = 0$, then $\partial = 0$ and $E = R[Y; \sigma^{-1}][X; \sigma]$, thus $\mathcal{K}(E) = \mathcal{K}(R) + 2$, by [15, 6.5.4.(i)]. From here on, we will assume that $b \neq 0$.

Let K be an algebraically closed field of characteristic zero, and let $A_1 = A_1(K)$ be the first Weyl Algebra; so that $A_1 = K[p, q]$ with $pq - qp = 1$. We aim to compute the Krull dimension of the algebra

$$E = A_1\langle\sigma; b(\neq 0), \rho\rangle, \qquad \sigma \in \mathrm{Aut}_K(A_1).$$

We may view E in the form $E = A_1 \otimes K[H](\sigma, H)$, where $\sigma(H) = \rho H + b$, by the previous lemma. The algebra A_1 is central simple, since the characteristic is zero; so both ρ and b are nonzero scalars. If $\rho \neq 1$, then $\sigma(H - \gamma) = \rho(H - \gamma)$, where $\gamma = b(1 - \rho)^{-1}$, while if $\rho = 1$, then $\sigma(b^{-1}H) = b^{-1}H + 1$. Thus, by making the obvious change of variables, we may assume that either (i) $\sigma(H) = \rho H$, with $\rho \neq 1$ and $a = H + \gamma$, or (ii) $\sigma(H) = H + 1$ and $a = bH$.

Lemma 6.2 *Let K be an algebraically closed field of characteristic zero. Then*

$$\mathcal{K}(A_1\langle\sigma; b(\neq 0), \rho\rangle) = 3$$

if and only if at least one of the following two cases holds:
1. $\rho \neq 1$ and there is a simple A_1-module M such that ${}^{\sigma^n}M \simeq M$, for some $n \geq 1$, or
2. $\rho \neq 1$, but $\rho^n = 1$, for some $n \geq 2$.

In all other cases, $\mathcal{K}(A_1\langle\sigma; b(\neq 0), \rho\rangle) = 2$.

Proof. ($\Leftarrow$) The algebra $A_1\langle\sigma; b, \rho\rangle$ is isomorphic to $E = A_1 \otimes K[H](\sigma, H)$, where either (i) $\sigma(H) = \rho H$, with $\rho \neq 1$ and $a = H + \gamma$, or (ii) $\sigma(H) = H + 1$ and $a = bH$. Suppose first that case 1 above holds; so that $\rho \neq 1$, and $\sigma(H) = \rho H$. Let M be a module as in case 1 above. The simple $K[H]$-module $K_0 := K[H]/(H)$ is isomorphic to ${}^{\sigma}K_0$. The simple $A_1 \otimes K[H]$-module $M_0 := M \otimes K_0$ is a-torsionfree, while ${}^{\sigma^n}M_0 \simeq M_0$, as $A_1 \otimes K[H]$-modules. Also, $h(M_0 : A_1 \otimes K[H]) = 2$, because the sequence $\{M_0,\ M \otimes K[H],\ A_1 \otimes K[H]\}$ is a T-clean sequence associated with M_0 in $A_1 \otimes K[H]$, since all the modules involved are compressible. It follows, from Theorem 4.1, that

$$\mathcal{K}(E) = \mathcal{K}(A_1 \otimes K[H]) + 1 = \mathcal{K}(A_1) + 2 = 3.$$

Suppose next that case 2 holds. Let N be a simple A_1-module, and let $K_{-\gamma} = K[H]/(a) = K[H]/(H + \gamma)$. Then the simple $A_1 \otimes K[H]$-module $N_{-\gamma} := N \otimes K_{-\gamma}$ is a-torsion, and both of the sets $St_{\pm}(N_{-\gamma})$ are infinite, since $\sigma^n(H) = \rho^n H = H$. By Lemma 2.5, there is a positive integer i such that ${}_{\sigma^{\pm i}}N_{-\gamma}$ is $(T,\pm)$-clean. Also, $h_{\pm}({}_{\sigma^{\pm i}}N_{-\gamma} : A_1 \otimes K[H]) = 2$, since the sequence $\{{}_{\sigma^{\pm i}}N_{-\gamma},\ {}_{\sigma^{\pm i}}N_{-\gamma} \otimes K[H],\ A_1 \otimes K[H]\}$ is a $(T,\pm)$-clean sequence associated with ${}_{\sigma^{\pm i}}N_{-\gamma}$ in $A_1 \otimes K[H]$, by noting that the second and third modules in the sequence are compressible. The result follows as in the first case.

($\Rightarrow$) We now prove the converse. The field K is algebraically closed and the first Weyl algebra is almost commutative, hence, by Quillen's Lemma, [15, 9.7.3(a)], the simple $A_1 \otimes K[H]$-modules are exactly the modules which are isomorphic to modules of the form $M_\lambda := M \otimes K_\lambda$, where M is a simple A_1-module and $K_\lambda := K[H]/(H-\lambda)$, for some $\lambda \in K$.

Now assume that neither case 1 nor 2 holds. There are two possibilities: either $\rho = 1$, or the orbit $\mathcal{O}\{M\}$ is infinite for every simple A_1-module M and ρ is not a root of unity. In either case, it can easily be checked that the orbit $\mathcal{O}\{M_\lambda\}$ is infinite and the set $St(M_\lambda)$ is finite, for every simple $A_1 \otimes K[H]$-module M_λ. Thus, $\mathcal{K}(E) = \mathcal{K}(A_1 \otimes K[H]) = 2$, by Theorem 4.1. ∎

It is easy to give examples of the two possibilities. If $\rho \neq 1$ is a root of unity then $\mathcal{K}(E) = 3$, while if we choose σ to be the identity automorphism and $b = \rho = 1$, the algebra E is isomorphic to the second Weyl algebra $A_2 = A_1 \otimes A_1$. In this case, we have $\mathcal{K}(A_2) = 2$, since the characteristic is zero.

In fact, the analysis of this example can also be used to establish the following corollary. First, recall that a K-algebra A is said to have the **endomorphism property** over K if, for each simple A-module M, the endomorphism ring $\mathrm{End}_K(M)$ is algebraic over K, see, for example [15, 9.1.4]. Every countably generated algebra over an uncountable field has the endomorphism property, [15, 9.1.7].

Corollary 6.3 *Let K be an algebraically closed field of characteristic zero, and let A be a left noetherian K-algebra which is a domain, has Krull dimension equal to one, and has the endomorphism property over K. Let $E = A\langle\sigma; b, \rho\rangle$, with $\sigma \in \mathrm{Aut}_K(A)$ and $b, \rho \in K^* = K\backslash\{0\}$. Then $\mathcal{K}(E) = 3$ if and only if at least one of the following cases holds.*
1. $\rho \neq 1$ and there is a simple A_1-module M such that ${}^{\sigma^n}M \simeq M$, for some $n \geq 1$, or
2. $\rho \neq 1$, but $\rho^n = 1$, for some $n \geq 2$.

In all other cases, $\mathcal{K}(A\langle\sigma; b, \rho\rangle) = 2$.

Proof. Quillen's Lemma is used in the proof of the previous lemma to establish that the first Weyl algebra has the endomorphism property. Here we given the endomorphism property and the proof is then the same. ∎

A particular class of algebras to which we could apply this corollary would be the McConnell-Pettit algebras, [14]. Let K be an uncountable algebraically closed field of characteristic zero, and let $\Lambda = (\lambda_{ij})$ be a multiplicatively antisymmetric matrix of nonzero elements of K. The McConnell-Pettit algebra, $P(\Lambda)$, associated with this data, is the K-algebra generated by $X_1^{\pm 1}, \ldots, X_n^{\pm 1}$, subject to the commutation relations $X_i X_j = \lambda_{ij} X_j X_i$. In the case that the multiplicative subgroup of K^* generated by the λ_{ij} has maximal rank, $n(n-1)/2$, the algebra $P(\Lambda)$ is known to have Krull dimension one, [14, Corollary 3.10]; and so the Corollary above applies.

References

[1] V V Bavula, *Finite-dimensionality of* Ext^n *and* Tor_n *of simple modules over a class of algebras*, (Russian) Funktsional. Anal. i Prilozhen. 25 (1991), no. 3, 80-82; translation in Functional Anal. Appl. 25 (1991), no. 3, 229-230 (1992)

[2] V V Bavula, *Generalized Weyl algebras, kernel and tensor-simple algebras, their simple modules*, Representations of algebras (Ottawa, ON, 1992), 83-107, CMS Conf. Proc., **14**, Amer. Math. Soc., Providence, RI, 1993.

[3] V V Bavula, *Generalized Weyl algebras and their representations*, Algebra i Analiz **4** (1992), 75-97; translation in St. Petersburgh Math. J., **4** (1993), no. 1, 71-92.

[4] V V Bavula, *Global dimension of Generalized Weyl Algebras*, Representation theory of algebras (Cocoyoc, 1994), 81-107, CMS Conf. Proc., **18**, Amer. Math. Soc., Providence, RI, 1996.

[5] V V Bavula and T H Lenagan, *Generalised Weyl Algebras are Krull Tensor Minimal*, preprint, Edinburgh 1998.

[6] V V Bavula and F van Oystaeyen, *The simple modules of certain generalized crossed products*. J. Algebra **194** (1997), 521-566.

[7] V V Bavula and F van Oystaeyen, *Krull dimension of generalized Weyl algebras and iterated skew polynomial rings. Commutative coefficients*, J. Algebra to appear.

[8] K R Goodearl and T H Lenagan, *Krull dimension of skew-Laurent extensions*, Pacific J. of Maths., **114** (1984), 109-147.

[9] K R Goodearl and T H Lenagan, *Krull dimension of differential operator rings III: noncommutative coefficients*, Trans. Amer. Math. Soc., **275** (1983). 833-859.

[10] A V Jategaonkar, *Jacobson's conjecture and modules over fully bounded Noetherian rings*, J. Algebra 30 (1974), 103-121.

[11] D A Jordan, *Krull and global dimension of certain iterated skew polynomial rings*, Cont. Maths, **130** (1992), 201-213.

[12] G R Krause, *Descending chains of submodules and the Krull-dimension of Noetherian modules*, J. Pure Appl. Algebra **3** (1973), 385-397.

[13] T Levasseur, *La dimension de Krull de* $U(\mathrm{sl}(3))$, J. Algebra **102** (1986), 39–59.

[14] J C McConnell and J J Pettit, *Crossed products and multiplicative analogues of Weyl algebras*, J. London Math. Soc., **38** (1988), 47-55.

[15] J C McConnell and J C Robson, *Noncommutative noetherian rings*, Wiley, Chichester, 1987.

[16] S P Smith, *Krull dimension of the enveloping algebra of* $sl(2)$, J. Algebra, **71** (1981). 89-94.