LTCC Intensive Course:

From quantum algebras to total non-negativity

Stéphane Launois and Tom Lenagan

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References

K R Goodearl and R B Warfield, Jr, An introduction to noncommutative noetherian rings, LMS student texts, Vol 61

K A Brown and K R Goodearl, Lectures on algebraic quantum groups, Advanced Courses in Mathematics, CRM Barcelona, Birkhäuser Verlag

The quantum world

Recall that a ring R is *right noetherian* if each of the following three equivalent conditions hold:

- Each right ideal is finitely generated
- There is no infinite ascending chain of right ideals
- Each nonempty set of right ideals has a maximal member

All the rings in this course will be (two-sided) noetherian.

An ideal P is a *prime ideal* of R if either $A \subseteq P$ or $B \subseteq P$ for ideals A, B with $AB \subseteq P$, and is *completely prime* if $ab \in P$ implies that $a \in P$ or $b \in P$ whenever $a, b \in R$.

Example The zero ideal of $M_2(\mathbb{Z})$ is prime but not completely prime.

All prime ideals in this course will be completely prime.

Recall the *Ore condition* for the existence of localisations in (noncommutative) rings.

Let S be a set of nonzerodivisors in R.

Then there is a ring of right quotients of the form

$$RS^{-1} := \{rs^{-1} \mid r \in R, s \in S\}$$

provided that the *right ore condition* holds for S; that is, for any $a \in R$ and $c \in S$, there exist $b \in R$ and $d \in S$ with $ad = cb$

Goldie's Theorem in the case of a noetherian domain says that the right Ore condition holds for the set of nonzero elements in the ring and that the resulting ring of fractions is a division ring.

Proof Assume that $a, c \neq 0$ and that the Ore condition fails; so that $aR \cap cR = 0$.

Exercise show that the sum

$$aR + caR + c^2aR + c^3aR + \dots$$

is a direct sum.

From this one easily constructs an infinite ascending chain of right ideals, contradicting the noetherian condition.

An element u of R is a *normal element* of R provided that $uR = Ru$.

When u is a normal nonzerodivisor, the Ore conditions holds for the set $S := \{u^n\}$, and the resulting localisation is

$$R[u^{-1}] := \{ru^{-n} \mid r \in R, n \in \mathbb{N}\}.$$

If $I \lhd R[u^{-1}]$ then $I = (I \cap R)R[u^{-1}]$ and it follows that $R[u^{-1}]$ is noetherian whenever R is noetherian.

In forming polynomial rings over a noncommutative ring R, the requirement that the indeterminate x commutes with elements of R is too restrictive.

However, to have a notion of *degree*, if we agree to write polynomials with powers of x at the right side:

$$r_n x^n + r_{n-1} x^{n-1} + \cdots + r_1 x + r_0$$

then, for each $r \in R$, we must have

$$xr = sx + t$$

for some $s, t \in R$.

Write $\sigma(r) := s$ and $\delta(r) := t$.

In order to get an associative ring, the following conditions must be satisfied:

The map σ should be an *automorphism* of R and δ should be a *(left)* σ*-derivation*; that is,

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$$

In this case, one can form the ring

$$R[x; \sigma, \delta] := \{ \sum_{i=0}^{n} r_i x^i \mid i \in \mathbb{N} \}$$

where

$$xr = \sigma(r)x + \delta(r).$$

The ring $R[x;\sigma,\delta]$ is a *skew polynomial extension* of R.

Hilbert's Basis Theorem If R is noetherian then so is $R[x;\sigma,\delta]$.

There are two special cases:

Case 1 The map $\delta = 0$. In this case, we write $R[x;\sigma]$.

Case 2 The map σ is the identity map. In this case, we write $R[x;\delta]$.

Example Let $R = k[y]$ where k is a field.

Choose a nonzero element $q \in k$ and let $\sigma(y) := qy$.

Then $A_q := R[x; \sigma]$ is the *quantum plane*.

Then

$$A_q = \{\sum_{i,j} c_{ij} y^i x^j \mid c_{ij} \in k, i, j \in \mathbb{N}\}$$

and

$$xy = qyx.$$

Note that both x and y are normal elements in A_q so that one can form the algebra of *skew Laurent polynomials*

$$T_q := k[x^{\pm 1}, y^{\pm 1}] = \sum_{i,j} c_{ij} y^i x^j \mid c_{ij} \in k, i, j \in \mathbb{Z}\}$$

where

$$xy = qyx.$$

The algebra T_q is a *quantum torus*.

Theorem Suppose that q is not a root of unity. Then the quantum torus $T_q := k[x^{\pm 1}, y^{\pm 1}]$ is a simple noetherian ring.

Sketch proof Let I be a nonzero ideal in T_q. Choose an element $0 \neq f \in I$ with

$$f = f_0 + f_1 x + \cdots + f_n x^n$$

with $f_i \in k[y^{\pm 1}]$, $f_0 \neq 0$ and n minimal. Suppose $n > 0$. Then consider the element $a := q^n y f - f y \in I$. The x^n term in a is

$$q^n y f_n x^n - f_n x^n y \;=\; q^n y f_n x^n - q^n f_n y x^n \;=\; 0$$

while the constant term is

$$q^n y f_0 - f_0 y \;=\; (q^n - 1) y f_0 \;\neq\; 0.$$

this produces $0 \neq a \in I$ with a smaller n. Thus, $n = 0$ and $I \cap k[y^{\pm 1}] \neq 0$. Now play same trick with a and x to get $I \cap k \neq 0$, giving a unit in I so that $I = T_q$.

Quantum Plane k a field, $0 \neq q \in k$, not a root of unity.

$$A := k\langle x, y \mid xy = qyx \rangle$$

Problem Describe Spec(A), the set of prime ideals

Torus action: $\mathcal{H} := (k^*)^2$

$$\begin{aligned}(\alpha, \beta) \circ x &:= \alpha x \\ (\alpha, \beta) \circ y &:= \beta y\end{aligned}$$

Subproblem Find $\mathcal{H}$ − Spec(A); that is, primes P with $P^{\mathcal{H}} = P$

Note that x and y are $\mathcal{H}$-eigenvectors.

There are four obvious $\mathcal{H}$-primes:

$$0, \quad \langle x \rangle , \quad \langle y \rangle , \quad \langle x, y \rangle$$

and we claim that these are the only $\mathcal{H}$-primes.

If $P \in \mathcal{H} - \mathrm{Spec}(A)$ and either $x \in P$ or $y \in P$ then it is easy to see that P is one of

$$\langle x \rangle , \quad \langle y \rangle , \quad \langle x, y \rangle$$

Suppose that $P \in \mathcal{H} - \text{Spec}(A)$ and $x, y \notin P$.

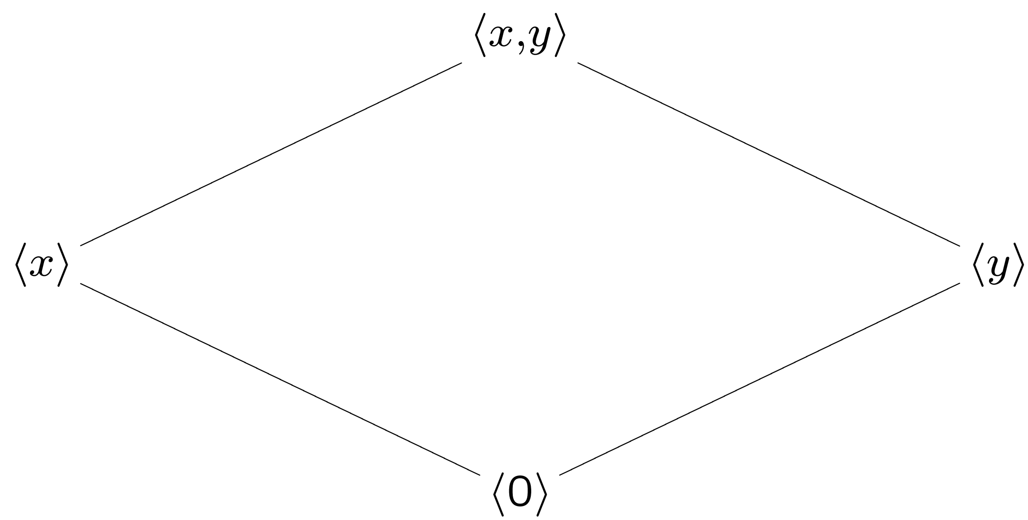
Recall that the quantum torus $T = k[x^{\pm 1}, y^{\pm 1}]$ is a simple ring.

Now, $PT \lhd T$; so either $PT = T$ or $PT = 0$.

If $PT = T$ then either $x \in P$ or $y \in P$, a contradiction.

Thus, $PT = 0$ and so $P = 0$.

$$\mathcal{H}-\mathsf{Spec} = \{0, \langle x\rangle, \langle y\rangle, \langle x, y\rangle\}$$



Example Let's determine all prime ideals in the quantum plane A_q at a nonroot of unity when k is algebraically closed.

$$\mathcal{H} - \text{Spec} = \{0, \langle x \rangle, \langle y \rangle, \langle x, y \rangle\}$$

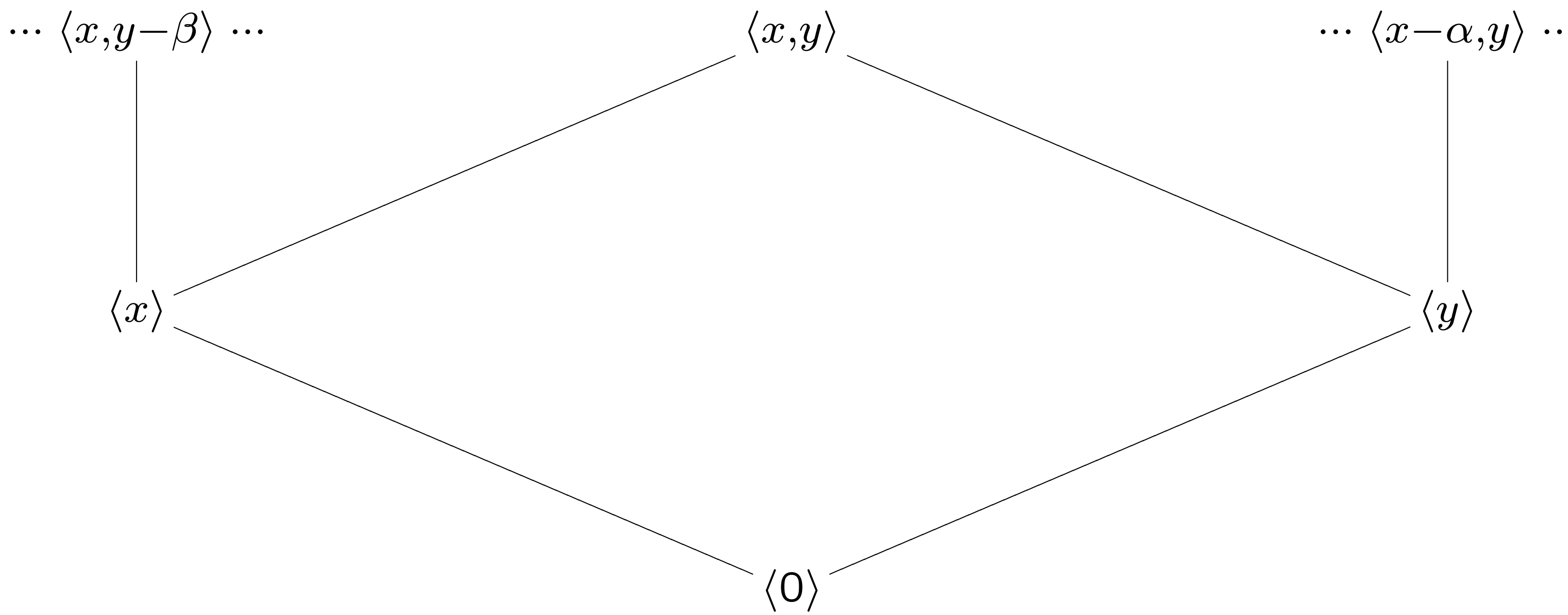
¿ Other primes? eg. $x \in P, y \notin P$

$$\left(\frac{P}{\langle x \rangle}\right)[y^{-1}] \in \text{Spec}\left(\frac{k[x,y]}{\langle x \rangle}[y^{-1}]\right) \cong k[y, y^{-1}]$$

¿ This leaves $x \notin P, y \notin P$.

As above, using the fact that the quantum torus is simple, $P = 0$.

Here is the picture of the prime spectrum of the quantum plane



with $\alpha, \beta \neq 0$.

Quantum affine n-space

$$A := k\left\langle x_1, \ldots, x_n \mid i < j,\ x_i x_j = p_{ij} x_j x_i,\ p_{ij}^m \neq 1 \right\rangle$$

$\mathcal{H} = (k^*)^n$ acts: $(\alpha_1, \ldots, \alpha_n) \circ x_i := \alpha_i x_i$

Set $P_I := \langle x_i \rangle_{i \in I}$ for each subset $I \subseteq \{1, \ldots, n\}$

$\mathcal{H} - \mathrm{Spec}(A) = \{P_I\}$

$$|\mathcal{H} - \mathrm{Spec}(A)| = 2^n < \infty$$

Exercise Calculate $(x+y)^n$ for the quantum plane

$(x+y)^2 =$

$(x+y)^3 =$

Define $[m]_q := 1 + q + q^2 + \cdots + q^{m-1}$

Note that $[m]_1 = m$ and that for $q \neq 1$,

$$[m]_q := 1 + q + q^2 + \cdots + q^{m-1} = \frac{q^m - 1}{q - 1}$$

Define $[m]_q! := [m]_q \times [m-1]_q!$, and

$$\binom{m}{r}_q := \frac{[m]_q!}{[m-r]_q![r]_q!}.$$

The quantum binomial theorem

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r}_q y^r x^{n-r}$$

Exercise The construction of Pascal's triangle is justified by the identity

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

Find the corresponding identity for q-binomial coefficients (there are two versions)

A good reference for such calculations is the book:

Victor Kac and Pokman Cheung, Quantum Calculus, Springer

Exercise Quantum Weyl algebra

Let $\sigma : k[y] \longrightarrow k[y]$ be given by $\sigma(y) := qy$

Is there a σ-derivation with $\delta(y) := 1$?

$\delta(y^2) = \sigma(y)\delta(y) + \delta(y)y =$

$\delta(y^3) =$

$\delta(y^n) =$

The **Quantum Weyl Algebra** is $k[x, y]$ with $xy - qyx = 1$.

Exercise The element $z := xy - yx$ is normal; so the quantum Weyl algebra is not simple.

Quantum matrices

$\mathcal{O}_q(\mathcal{M}_2)$, the *quantised coordinate ring of* 2×2 *matrices*

$$\mathcal{O}_q(\mathcal{M}_2) := k\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with relations

$$ab = qba \qquad ac = qca \qquad bc = cb$$

$$bd = qdb \qquad cd = qdc \qquad ad - da = (q - q^{-1})bc.$$

The *quantum determinant* is $D_q := ad - qbc$

Exercise Check that the quantum determinant is a central element and that b and c are normal elements.

Note that

$$\mathcal{O}_q(\mathcal{M}_2) = k[a][b;\tau_2][c;\tau_3][d;\tau_4,\delta_4]$$

where

$$\tau_2(a) = q^{-1}a$$

$$\tau_3(a) = q^{-1}a \qquad \tau_3(b) = b$$

and

$$\tau_4(a) = a \qquad \tau_4(b) = q^{-1}b \qquad \tau_4(c) = q^{-1}c,$$

while δ_4 is the k-linear τ_4-derivation such that

$$\delta_4(b) = \delta_4(c) = 0 \qquad \delta_4(a) = (q^{-1} - q)bc.$$

So, $\mathcal{O}_q(\mathcal{M}_2)$ is a noetherian domain and has a vector space basis consisting of monomials $a^i b^j c^l d^m$.

Overall problem Describe Spec($\mathcal{O}_q(\mathcal{M}_2)$), q generic ($q^m \neq 1$)

Set $\mathcal{H} := (k^*)^4$.

There is an action of $\mathcal{H}$ on $\mathcal{O}_q(\mathcal{M}_2)$ given by

$$(\alpha, \beta; \gamma, \delta) \circ \begin{bmatrix} a & b \\ c & d \end{bmatrix} := \begin{bmatrix} \alpha\gamma a & \alpha\delta b \\ \beta\gamma c & \beta\delta d \end{bmatrix};$$

that is, by row and column multiplications.

Subproblem Identify all of the prime ideals of $\mathcal{O}_q(\mathcal{M}_2)$ that are $\mathcal{H}$-invariant.

• **Overall problem**: describe $\text{Spec}(\mathcal{O}_q(\mathcal{M}_2))$, when q is not a root of unity.

Theorem (Goodearl-Letzter) Let $P \in \text{Spec}(\mathcal{O}_q(\mathcal{M}_2))$. Then $\mathcal{O}_q(\mathcal{M}_2)/P$ is an integral domain; that is, all primes are completely prime.

Theorem (Goodearl-Letzter)

$$|\mathcal{H} - \text{Spec}(\mathcal{O}_q(\mathcal{M}_2))| \leq 2^4 = 16 < \infty$$

• **Sub-problem**: describe $\mathcal{H} - \text{Spec}(\mathcal{O}_q(\mathcal{M}_2))$

For $P \in$ Spec($\mathcal{O}_q(\mathcal{M}_2)$) set $\mathcal{H}(P) := \cap_{h\in\mathcal{H}} P^h$. Then $\mathcal{H}(P)$ is an $\mathcal{H}$-invariant prime ideal.

For any $\mathcal{H}$-prime Q set

$$\text{Spec}_Q(\mathcal{O}_q(\mathcal{M}_2)) := \{P \text{ prime } \mid \mathcal{H}(P) = Q\}$$

The Goodearl-Letzter Stratification Theorem

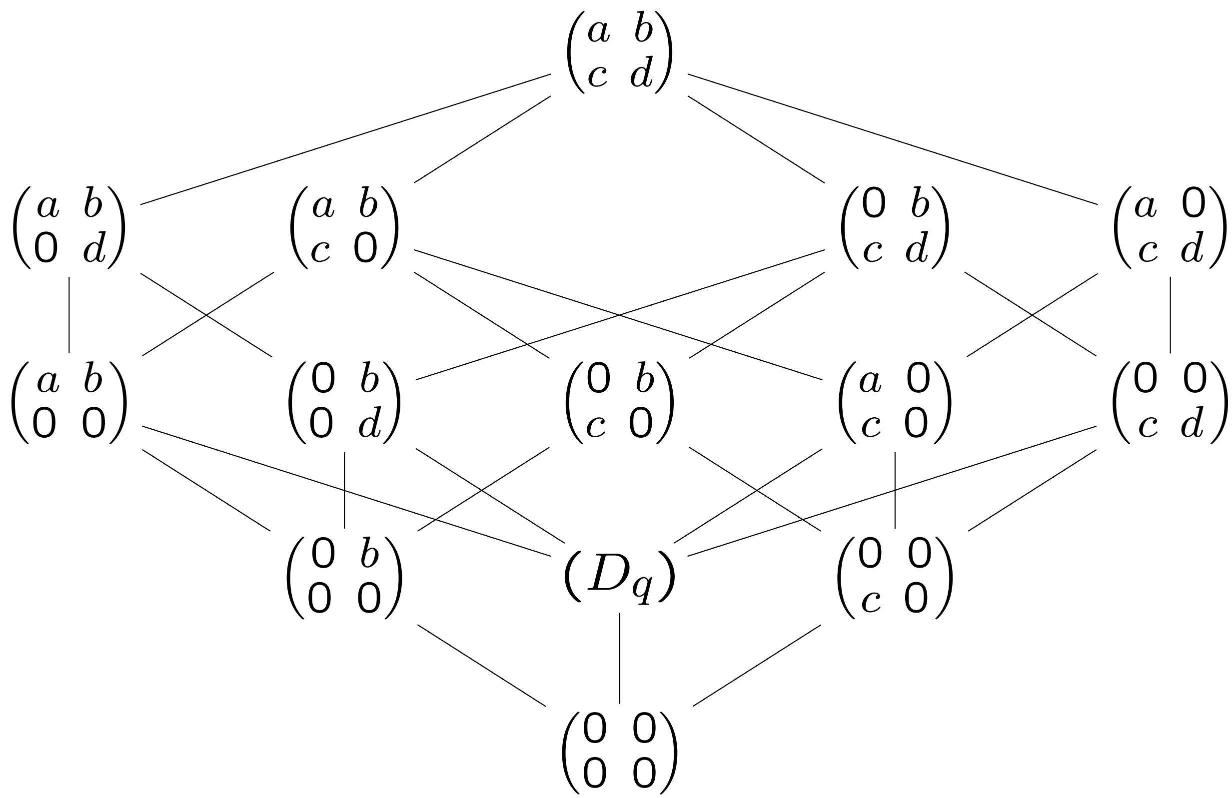
For any $Q \in \mathcal{H} - \text{Spec}$, Spec$_Q(\mathcal{O}_q(\mathcal{M}_2))$ is homeomorphic to

$$\text{Spec}(k[t_1^{\pm 1}, \ldots, t_d^{\pm 1}])$$

for some d.

Further, the primitive ideals of $\mathcal{O}_q(\mathcal{M}_2)$ are precisely the maximal elements of Spec$_Q$ for $Q \in$ Spec($\mathcal{O}_q(\mathcal{M}_2)$).

Claim The following 14 $\mathcal{H}$-invariant ideals are all prime and these are the only $\mathcal{H}$-prime ideals in $\mathcal{O}_q(\mathcal{M}_2)$.



To justify the claim, we need to show that each of the 14 ideals is a prime ideal and that there are no other $\mathcal{H}$-prime ideals.

It is easy to check that 13 of the ideals are prime.

For example, let P be the ideal generated by b and d. Then

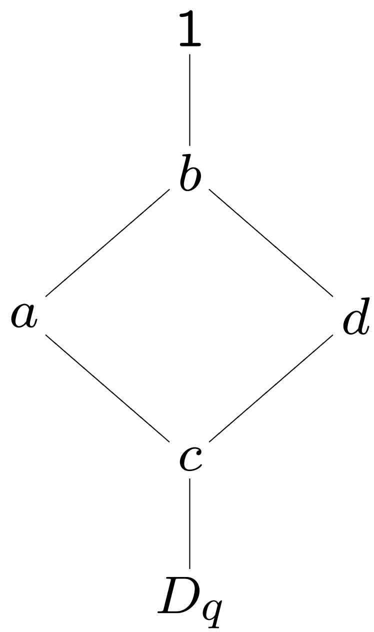
$$\frac{\mathcal{O}_q(\mathcal{M}_2)}{P} \cong k[a,c]$$

and $k[a,c]$ is a quantum plane.

The only problem is to show that the determinant generates a prime ideal.

This was originally proved by Jordan, and, independently, by Levasseur and Stafford.

However, we will prove this in a different way and also show that there are no other $\mathcal{H}$-invariant prime ideals.



Consider the poset on the left.

Note that elements are in the poset are normal modulo lower elements.

We can use the **commutation rules** to bring a and d together in any monomial, and then use the **straightening law**

$$ad \rightsquigarrow qcb + D_q$$

to get a spanning set of the form

$$\{D_q^i c^j a^l b^m, D_q^i c^j d^l b^m\}.$$

In fact, this is a basis of $\mathcal{O}_q(\mathcal{M}_2)$, the **preferred basis**

Cauchon's theory of deleting derivations

Recall

$$\mathcal{O}_q(\mathcal{M}_2) := k\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with relations

$$ab = qba \qquad ac = qca \qquad bc = cb$$

$$bd = qdb \qquad cd = qdc \qquad ad - da = (q - q^{-1})bc.$$

Set $a' := a - bd^{-1}c = (ad - qbc)d^{-1} = D_q d^{-1}$

Calculate

$$a'b = qba' \qquad a'c = qca' \qquad bc = cb$$

$$bd = qdb \qquad cd = qdc \qquad a'd = da'$$

All calculations take place in the division ring of fractions of $\mathcal{O}_q(\mathcal{M}_2)$.

Note that $\widehat{A} := k[a', b, c] \cong k[a, b, c] =: A$ and that

$$\widehat{R} := k[a', b, c, d] \cong \widehat{A}[d; \sigma]$$

whereas

$$R := \mathcal{O}_q(\mathcal{M}_2) \cong A[d; \sigma, \delta]$$

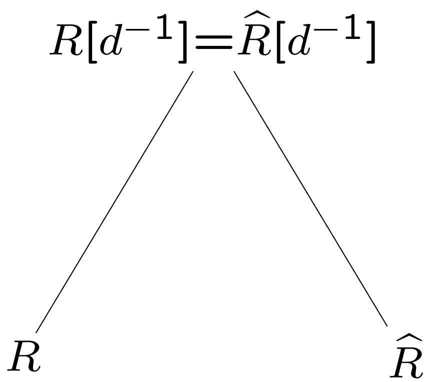
There is an induced action of $\mathcal{H}$ on $\widehat{R}$ and, as $\widehat{R}$ is a quantum affine 4-space, we know that $\widehat{R}$ has 16 $\mathcal{H}$-primes, corresponding to the subsets of $\{a', b, c, d\}$.

We will relate the $\mathcal{H}$-prime ideals of $\mathcal{O}_q(\mathcal{M}_2)$ with a subset of the $\mathcal{H}$-prime ideals of $\widehat{R}$.

Exercise Show that the set $S := \{d^n\}$ is a right (and left) ore set in $\mathcal{O}_q(\mathcal{M}_2)$; so that one can form the localisation $R[d^{-1}]$.

As d is a normal element of $\widehat{R}$, we can form $\widehat{R}[d^{-1}]$.

Check that $\widehat{R}[d^{-1}] = R[d^{-1}]$.



Note that $\langle a' \rangle$ is a prime ideal in $\widehat{R}$.

Claim: $\langle a' \rangle \, \widehat{R}[d^{-1}] \cap R = \langle D_q \rangle$.

This will show that $\langle D_q \rangle$ is a prime ideal of $\mathcal{O}_q(\mathcal{M}_2)$.

Claim: $\langle a' \rangle \widehat{R}[d^{-1}] \cap R = \langle D_q \rangle$

Sketch proof Note that

$$\left\langle a' \right\rangle \widehat{R}[d^{-1}] \cap R = a'\widehat{R}[d^{-1}] \cap R = a'R[d^{-1}] \cap R$$

Let $r \in \langle a' \rangle \widehat{R}[d^{-1}] \cap R$. Then, there exists $s \in R$ such that $r = a'sd^{-n}$ for some n.

Now, there exists $t \in R$ with $d^{-1}s = td^{-m}$ for some m; so

$$r = a'sd^{-n} = a'dd^{-1}sd^{-n} = D_q td^{-(n+m)}$$

and $rd^{(n+m)} = D_q t$.

Writing r and t in terms of the preferred basis

$$\{D_q^i c^j a^l b^m, D_q^i c^j d^l b^m\}$$

leads to $r \in \langle D_q \rangle$

Given an $\mathcal{H}$-prime P in $\mathcal{O}_q(\mathcal{M}_2)$, we associate an $\mathcal{H}$-prime in $\widehat{R}$ in the following way.

Case 1 Suppose that $d \notin P$. Then

$$P \quad \mapsto \quad PR[d^{-1}] = P\widehat{R}[d^{-1}] \quad \mapsto \quad P\widehat{R}[d^{-1}] \cap \widehat{R}.$$

For example,

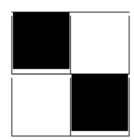
$$\langle D_q \rangle \quad \mapsto \quad \langle D_q \rangle [d^{-1}] = \langle D_q d^{-1} \rangle [d^{-1}] = \langle a' \rangle [d^{-1}] \quad \mapsto \quad \langle a' \rangle.$$

Any $\mathcal{H}$-prime in $\widehat{R}$ is specified by a subset of the four elements

a'	b
c	d

.

We will record a subset by putting taking a two-by-two array and filling in a square with black if the corresponding element is in the subset. For example, the $\mathcal{H}$-prime generated by a' and d is denoted



There are 16 possible fillings, corresponding to the 16 $\mathcal{H}$-prime ideals in $\widehat{R}$

Given an $\mathcal{H}$-prime P in $\mathcal{O}_q(\mathcal{M}_2)$, we associate such a diagram to it in the following way.

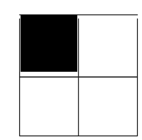
Case 1 Suppose that $d \notin P$. Then

$$P \quad \mapsto \quad PR[d^{-1}] = P\widehat{R}[d^{-1}] \quad \mapsto \quad P\widehat{R}[d^{-1}] \cap \widehat{R}.$$

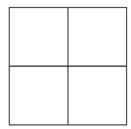
Now, $P\widehat{R}[d^{-1}] \cap \widehat{R}$ is an $\mathcal{H}$-prime in $\widehat{R}$ and so corresponds to a diagram.

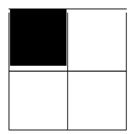
For example,

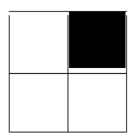
$$\langle D_q \rangle \;\mapsto\; \langle D_q \rangle[d^{-1}] = \langle D_q d^{-1} \rangle[d^{-1}] = \langle a' \rangle[d^{-1}] \;\mapsto\; \langle a' \rangle \;\mapsto$$

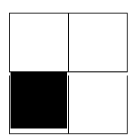


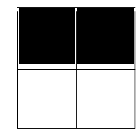
We know 8 $\mathcal{H}$-prime ideals in $\mathcal{O}_q(\mathcal{M}_2)$ that do not contain d, and so make the following associations:

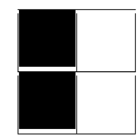
$\langle 0 \rangle \quad \longmapsto$ 

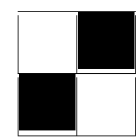
$\langle D_q \rangle \quad \longmapsto$ 

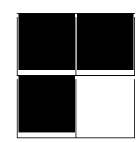
$\langle b \rangle \quad \longmapsto$ 

$\langle c \rangle \quad \longmapsto$ 

$\langle a, b \rangle \quad \longmapsto$ 

$\langle a, c \rangle \quad \longmapsto$ 

$\langle b, c \rangle \quad \longmapsto$ 

$\langle a, b, c \rangle \quad \longmapsto$ 

Case 2 Suppose that $d \in P$.

We find an $\mathcal{H}$-prime Q in $\widehat{R}$ such that $\widehat{R}/Q \cong R/P$ and then associate to P the diagram of Q.

Consider the two maps

$$\rho : \widehat{R} = k[a', b, c][d; \sigma] \quad \twoheadrightarrow \quad k[a', b, c] \quad \cong \quad k[a, b, c]$$

and

$$\eta_P : k[a, b, c] \quad \twoheadrightarrow \quad k[a, b, c, d]/P \quad = \quad \mathcal{O}_q(\mathcal{M}_2)/P$$

Then

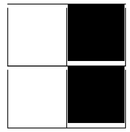
$$P \quad \rightsquigarrow \quad \ker(\eta_P \circ \rho)$$

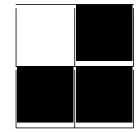
Note that $(q - q^{-1})bc = ad - da \in P$ so that either $b \in P$ or $c \in P$ (or both).

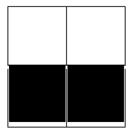
So, it is impossible to associate the two diagrams

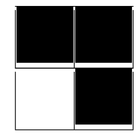
to any $\mathcal{H}$-prime ideal of $\mathcal{O}_q(\mathcal{M}_2)$.

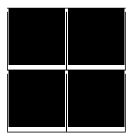
We know 6 $\mathcal{H}$-prime ideals in $\mathcal{O}_q(\mathcal{M}_2)$ that do contain d, and so make the following associations:

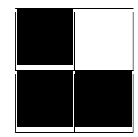
$\langle b, d \rangle \quad \mapsto$ 

$\langle b, c, d \rangle \quad \mapsto$ 

$\langle c, d \rangle \quad \mapsto$ 

$\langle a, b, d \rangle \quad \mapsto$ 

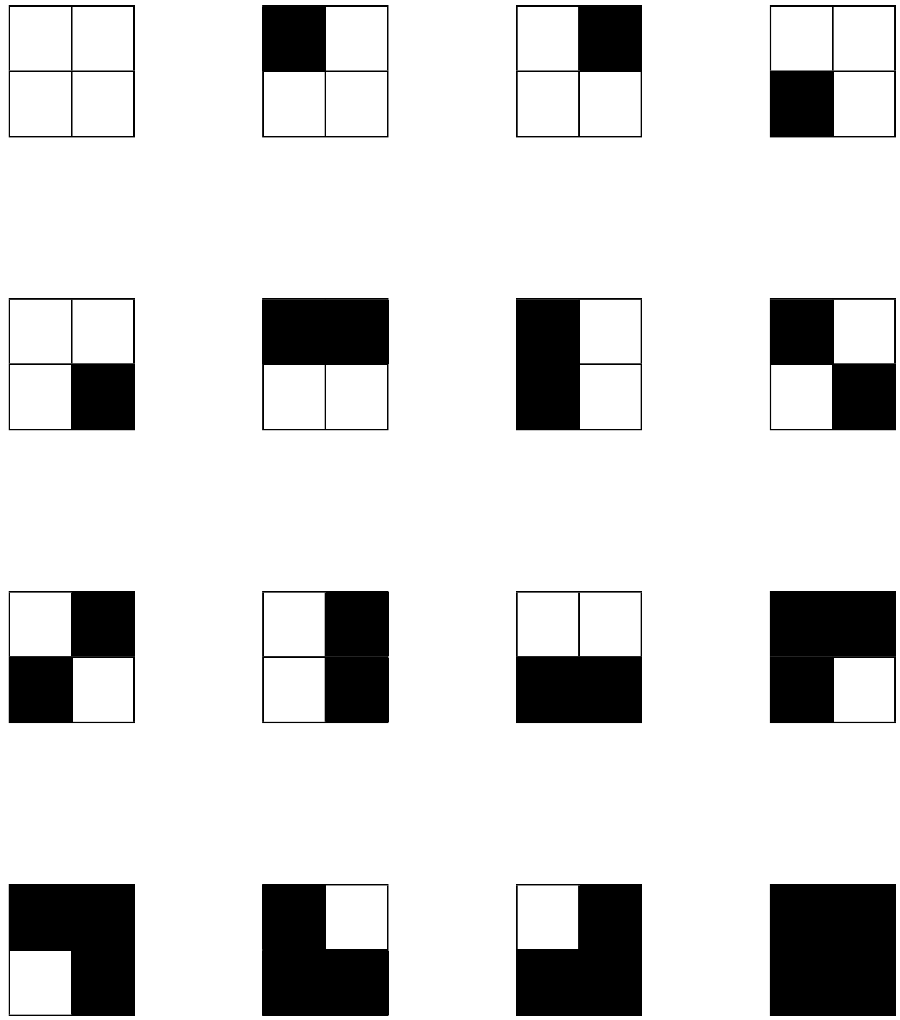
$\langle a, b, c, d \rangle \quad \mapsto$ 

$\langle a, c, d \rangle \quad \mapsto$ 

The diagrams that can be associated to $\mathcal{H}$-primes in $\mathcal{O}_q(\mathcal{M}_2)$ are known as **Cauchon Diagrams**.

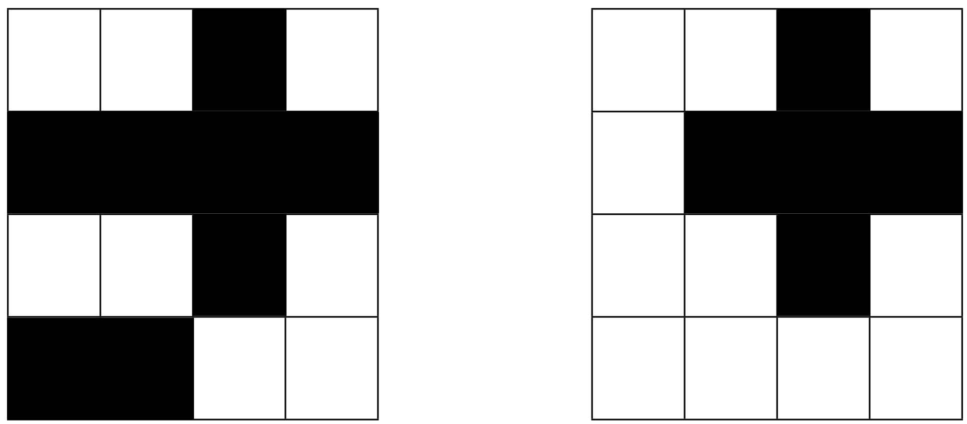
We've seen that 14 of the possible 16 black-white fillings of the 2×2 array are Cauchon diagrams; so there are 14 $\mathcal{H}$-prime ideals in $\mathcal{O}_q(\mathcal{M}_2)$.

2×2 Cauchon Diagrams



All of this works for quantum $m \times p$ matrices where there is an action of $\mathcal{H} = (k^*)^{m+p}$ and Cauchon shows that the $\mathcal{H}$-prime ideals are in bijection with $m \times p$ Cauchon diagrams:

Cauchon Diagrams



The rule for a Cauchon diagram is that if a square is black then either each square to the left of it is black, or each square above it is black.

The Poisson world

Lie algebra: definition

A Lie algebra is a $\mathbb{C}$-vector space V with a "Lie bracket" $[-,-] : V \times V \to V$ such that

1. *skew-symmetry*: $[v,w] = -[w,v]$ for all $v, w \in V$;

2. *Jacobi identity*:

$$[[u,v],w] + [[v,w],u] + [[w,u],v] = 0$$

for all $u, v, w \in V$.

Example. Let A be a $\mathbb{C}$-algebra. Set $[a,b] := ab - ba$. Then $(A, [-,-])$ is a Lie algebra.

Poisson algebra: definition

A Poisson algebra is a commutative finitely generated $\mathbb{C}$-algebra A with a "Poisson bracket" $\{-,-\} : A \times A \to A$ such that

1. $(A, \{-,-\})$ is a Lie algebra;

2. for all $a \in A$, the linear map $\{a,-\} : A \to A$ is a derivation, that is:

$$\{a, bc\} = b\{a, c\} + \{a, b\}c \qquad \forall a, b, c \in A.$$

Example. $\mathbb{C}[X, Y]$ is a Poisson algebra with Poisson bracket given by:

$$\{P, Q\} := \frac{\partial P}{\partial X} \cdot \frac{\partial Q}{\partial Y} - \frac{\partial P}{\partial Y} \cdot \frac{\partial Q}{\partial X}.$$

Poisson algebra: brief history

1807: the classical Poisson bracket (Poisson).

1875: general Poisson brackets on the ring of smooth functions on a manifold (Lie).

1960s: Poisson brackets on symmetric algebra of a Lie algebra and its quotient field; informal use of the term “Poisson algebra (Dixmier et al). First steps in quantization (Berezin).

1977: first (?) formal definition of Poisson algebra (Braconnier).

current: much used in quantum algebra and integrable systems.

Poisson algebra: example

$S = \mathbb{C}[X_1, X_2, ..., X_n; Y_1, Y_2, ..., Y_n]$, the symmetric algebra on $2n$-dimensional symplectic space, is a Poisson algebra with

$$\{X_i, X_j\} = 0 = \{Y_i, Y_j\} \text{ and } \{X_i, Y_j\} = \delta_{ij}.$$

Note that

$$\{X_i, -\} = \frac{\partial}{\partial Y_i} \text{ and } \{-, Y_i\} = \frac{\partial}{\partial X_i}$$

and

$$\{P, Q\} = \sum_i \left(\frac{\partial P}{\partial X_i} \cdot \frac{\partial Q}{\partial Y_i} - \frac{\partial P}{\partial Y_i} \cdot \frac{\partial Q}{\partial X_i} \right).$$

Here $\{-, -\}$ extends the antisymmetric bilinear form.

Poisson algebra: generators

Let A be a Poisson algebra. Assume that A is generated (as a $\mathbb{C}$-algebra) by $g_1, g_2, ..., g_n$.

Then one can retrieve $\{-,-\}$ from $\{g_i, g_j\}$ by using the skew-symmetry of $\{-,-\}$ together with the Leibniz rule.

That is why we often define the Poisson bracket on a commutative algebra A just by giving its values on the generators.

Exercise. Show that for all $P \in \mathbb{C}[X,Y]$, the rule $\{X,Y\} = P$ defines a Poisson bracket on $\mathbb{C}[X,Y]$.

Poisson algebra: generators 2

Be careful however, defining all brackets $\{g_i, g_j\}$ does not ensure that you will get a Poisson bracket. For instance, one is able to define a Poisson bracket on $A = \mathbb{C}[X, Y, Z]$ via

$$\{X, Y\} = R, \quad \{Y, Z\} = P \quad \text{and} \quad \{Z, X\} = Q$$

if and only if

$$(P, Q, R) \cdot \mathrm{curl}(P, Q, R) = 0,$$

where

$$\mathrm{curl}(P, Q, R) = \left(\frac{\partial R}{\partial Y} - \frac{\partial Q}{\partial Z}\ ,\ \frac{\partial P}{\partial Z} - \frac{\partial R}{\partial X}\ ,\ \frac{\partial Q}{\partial X} - \frac{\partial P}{\partial Y}\right).$$

Semiclassical limit

Let A_λ be a finitely generated $\mathbb{C}[\lambda]$-algebra, and assume that A_λ is a noetherian domain. Assume also that $A := A_\lambda/\lambda A_\lambda$ is commutative.

We define a Poisson bracket on A as follows. Let $a, b \in A$, and choose $u, v \in A_\lambda$ so that $u + \lambda A_\lambda = a$ and $v + \lambda A_\lambda = b$. As A is abelian, one has $[u, v] \in \lambda A_\lambda$.

Hence there exists a unique $w \in A_\lambda$ such that $[u, v] = \lambda w$. We set

$$\{a, b\} := w + \lambda A_\lambda.$$

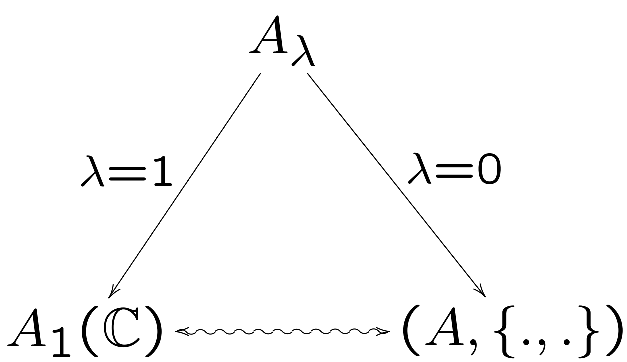
Informally, we write $\{a, b\} = \left. \frac{[a, b]}{\lambda} \right|_{\lambda=0}$.

Exercise. Check that $(A, \{-, -\})$ is a Poisson algebra.

Example 1: the first Weyl algebra

- Heisenberg algebra $A_\lambda = \mathbb{C}[\lambda, x, y]$ with $xy - yx = \lambda$, that is, $A_\lambda = \mathbb{C}[\lambda][x][y; id, \lambda\frac{\partial}{\partial x}]$.

- Weyl algebra $A_1(\mathbb{C}) = \mathbb{C}[x, y]$ with $xy - yx = 1$.



where $A = \mathbb{C}[X, Y]$

and $\{P, Q\} = \frac{[P,Q]}{\lambda}\Big|_{\lambda=0}$

Exercise. Compute $\{X, Y\}$ and $\{XY, X + Y\}$.

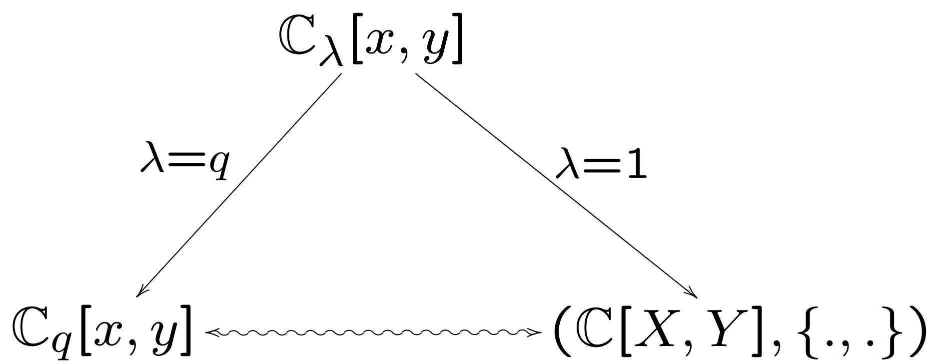
$A_1(\mathbb{C})$ is a (noncommutative) deformation of the Poisson algebra $(A, \{.,.\})$.

Example 2: quantum plane

We need to adapt the construction and work over $\mathbb{C}[\lambda^{\pm 1}]$ rather than over $\mathbb{C}[\lambda]$.

Recall that $\mathbb{C}_\lambda[x,y] := \mathbb{C}[\lambda, x, y]$ with $xy = \lambda yx$.

Let $q \in \mathbb{C}^*$, not a root of unity.



Exercise. Show that $\{P, Q\} := XY\left(\frac{\partial P}{\partial X}\frac{\partial Q}{\partial Y} - \frac{\partial P}{\partial Y}\frac{\partial Q}{\partial X}\right)$.

Semiclassical limit of $\mathcal{O}_q(\mathcal{M}_2(\mathbb{C}))$

Recall that $\mathcal{O}_q(\mathcal{M}_2(\mathbb{C})) := \mathbb{C}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is generated by four indeterminates a, b, c, d subject to the following rules:

$$ab = qba, \qquad cd = qdc$$

$$ac = qca, \qquad bd = qdb$$

$$bc = cb, \qquad ad - da = (q - q^{-1})cb.$$

The **quantum determinant** $ad - qbc$ is a central element.

Exercise. What is the semiclassical limit of $\mathcal{O}_q(\mathcal{M}_2(\mathbb{C}))$?

Symplectic leaves

Let A be the algebra of complex-valued C^∞ functions on a smooth affine variety V.

• *Hamiltonian derivations*: $H_a := \{a, -\}$ with $a \in A$.

• A *Hamiltonian path in* V is a smooth path $c : [0, 1] \to V$ such that there exists $H \in C^\infty(V)$ with

$$\frac{d}{dt}(f \circ c)(t) = \{H, f\} \circ c(t)$$

for all $0 < t < 1$.

• It is easy to check that the relation "connected by a piecewise Hamiltonian path" is an equivalence relation.

• The equivalence classes of this relation are called the *symplectic leaves* of V; they form a partition of V.

Symplectic leaves in $\mathbb{C}^2$

We consider $\mathbb{C}[X, Y]$ with the Poisson bracket defined by $\{X, Y\} = XY$; this Poisson bracket on $\mathbb{C}[X, Y] = \mathcal{O}(\mathbb{C}^2)$ extends uniquely to a Poisson bracket on $\mathcal{C}^\infty(\mathbb{C}^2)$, so that $\mathbb{C}^2$ can be viewed as a Poisson manifold. Hence $\mathbb{C}^2$ can be decomposed as the disjoint union of its symplectic leaves.

Let $a, b \in \mathbb{C}$. Then

- $c(t) = (a, be^{at})$ is a flow of H_X.

- $c(t) = (be^{at}, a)$ is a flow of H_{-Y}.

Symplectic leaves in $\mathbb{C}^2$

$\mathcal{H}$-orbits of symplectic leaves in $\mathbb{C}^2$

At the geometric level, the action of $\mathcal{H} = (\mathbb{C}^*)^2$ on $\mathbb{C}^2$ (by Poisson automorphisms) is given by:

$$(\alpha, \beta).\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ \beta y \end{pmatrix}.$$

This action of $\mathcal{H}$ on $\mathbb{C}^2$ induces an action of $\mathcal{H}$ on the set Sympl($\mathbb{C}^2$) of symplectic leaves in $\mathbb{C}^2$.

We view the $\mathcal{H}$-orbit of a symplectic leaf $\mathcal{L}$ as the set-theoretic union

$$\bigcup_{h \in \mathcal{H}} h.\mathcal{L} \subseteq \mathbb{C}^2,$$

rather than as the family $\{h.\mathcal{L} \mid h \in \mathcal{H}\}$.

$\mathcal{H}$-orbits of symplectic leaves in $\mathbb{C}^2$

Poisson prime ideals

A *Poisson ideal* of a Poisson algebra A is an ideal of A both in the associative and in the Lie sense. That is, I is an additive subgroup of A such that:

$$a.x \in I \quad \forall\ a \in A,\ x \in I$$

and

$$\{a, x\} \in I \quad \forall\ a \in A,\ x \in I.$$

An ideal I which is both Poisson and prime is called a *Poisson prime ideal*.

Exercise. Compute the Poisson prime ideals of $\mathbb{C}[X, Y]$ with Poisson bracket defined by $\{X, Y\} = XY$.

Poisson $\mathcal{H}$-primes in $A = \mathbb{C}[X, Y]$

The torus $\mathcal{H} = (\mathbb{C}^*)^2$ acts by Poisson automorphisms on A via:

$$(\alpha, \beta). \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \alpha X \\ \beta Y \end{pmatrix}.$$

Exercise. Describe the Poisson $\mathcal{H}$-primes.

Poisson $\mathcal{H}$-primes in A

Let A be a Poisson algebra, and assume that the torus $\mathcal{H} := (\mathbb{C}^*)^l$ acts rationally by Poisson automorphisms on A.

Theorem: (Goodearl)
Assume there are only finitely many $\mathcal{H}$-orbits of symplectic leaves in V, and that these are locally closed subvarieties of V. Then there is a $1:1$ correspondence between the set of $\mathcal{H}$-orbits of symplectic leaves in V and the set of prime Poisson $\mathcal{H}$-ideals in $\mathcal{O}(V)$.

Matrix Poisson varieties: 2×2

The coordinate ring of 2×2 matrices

$$\mathcal{O}(\mathcal{M}_2(\mathbb{C})) := \mathbb{C}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \mathbb{C}\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$$ is a Poisson algebra:

$$\begin{aligned} \{a, b\} &= ab, & \{c, d\} &= cd \\ \{a, c\} &= ac, & \{b, d\} &= bd \\ \{b, c\} &= 0, & \{a, d\} &= 2bc. \end{aligned}$$

Exercise. Show that the Poisson algebra $\mathcal{O}(\mathcal{M}_2(\mathbb{C}))$ is the semi-classical limit of the algebra of 2×2 quantum matrices.

Torus action

$\mathcal{H} := (\mathbb{C}^*)^4$ acts on $\mathcal{O}(\mathcal{M}_2(\mathbb{C}))$ by Poisson automorphisms via:

$$(a_1, a_2, b_1, b_2).Y_{i,\alpha} = a_i b_\alpha Y_{i,\alpha}.$$

At the geometric level, this action of the algebraic torus $\mathcal{H}$ comes from the left action of $\mathcal{H}$ on $\mathcal{M}_2(\mathbb{C})$ by Poisson isomorphisms via:

$$(a_1, a_2, b_1, b_2).M := \text{diag}(a_1, a_2) \cdot M \cdot \text{diag}(b_1, b_2).$$

We denote the set of $\mathcal{H}$-orbits by $\mathcal{H}$-Sympl$(\mathcal{M}_2(\mathbb{C}))$.

Exercise (hard). Describe $\mathcal{H}$-Sympl$(\mathcal{M}_2(\mathbb{C}))$.

Torus orbits

Proposition.

1. There is a 1 : 1 correspondence between

$$\mathcal{S} = \{w \in S_4 \mid -2 \leq w(i) - i \leq 2 \text{ for all } i = 1, 2, 3, 4\}.$$

and $\mathcal{H}$-Sympl($\mathcal{M}_2(\mathbb{C})$).

2. Each $\mathcal{H}$-orbit is defined by some rank conditions.

Exercise. Compute $|\mathcal{S}|$.

Matrix Poisson varieties: general case

$$\mathcal{O}\left(\mathcal{M}_{m,p}(\mathbb{C})\right) = \mathbb{C}\begin{bmatrix} Y_{1,1} & \cdots & Y_{1,p} \\ \vdots & \cdots & \vdots \\ Y_{m,1} & \cdots & Y_{m,p} \end{bmatrix}$$ is a Poisson algebra via

$\{Y_{i,\alpha}, Y_{i,\beta}\} = Y_{i,\alpha}Y_{i,\beta}$	$\alpha < \beta$
$\{Y_{i,\alpha}, Y_{j,\alpha}\} = Y_{i,\alpha}Y_{j,\alpha}$	$i < j$
$\{Y_{i,\alpha}, Y_{j,\beta}\} = 0$	$i < j$ and $\alpha > \beta$
$\{Y_{i,\alpha}, Y_{j,\beta}\} = 2Y_{i,\beta}Y_{j,\alpha}$	$i < j$ and $\alpha < \beta$

Exercise. Show that the Poisson algebra $\mathcal{O}\left(\mathcal{M}_{m,p}(\mathbb{C})\right)$ is the semiclassical limit of the algebra of $m \times p$ quantum matrices.

Torus action

$\mathcal{H} := (\mathbb{C}^*)^{m+p}$ acts on $\mathcal{O}(\mathcal{M}_{m,p}(\mathbb{C}))$ by Poisson automorphisms via:

$$(a_1, \ldots, a_m, b_1, \ldots, b_p).Y_{i,\alpha} = a_i b_\alpha Y_{i,\alpha}$$

At the geometric level, this action of the algebraic torus $\mathcal{H}$ comes from the left action of $\mathcal{H}$ on $\mathcal{M}_{m,p}(\mathbb{C})$ by Poisson isomorphisms via:

$$(a_1, \ldots, a_m, b_1, \ldots, b_p).M := \text{diag}(a_1, \ldots, a_m) \cdot M \cdot \text{diag}(b_1, \ldots, b_p).$$

We denote the set of $\mathcal{H}$-orbits by $\mathcal{H}$-Sympl$(\mathcal{M}_{m,p}(\mathbb{C}))$.

Exercise (very hard). Describe $\mathcal{H}$-Sympl$(\mathcal{M}_{m,p}(\mathbb{C}))$.

Torus orbits

The orbits $\mathcal{H}$-Sympl($\mathcal{M}_{m,p}(\mathbb{C})$) have been described by Brown, Goodearl and Yakimov.

We set

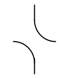
$$\mathcal{S} = \{w \in S_{m+p} \mid -p \leq w(i) - i \leq m \text{ for all } i = 1, 2, \ldots, m+p\}.$$

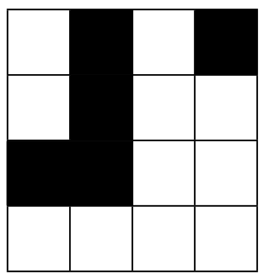
Theorem.

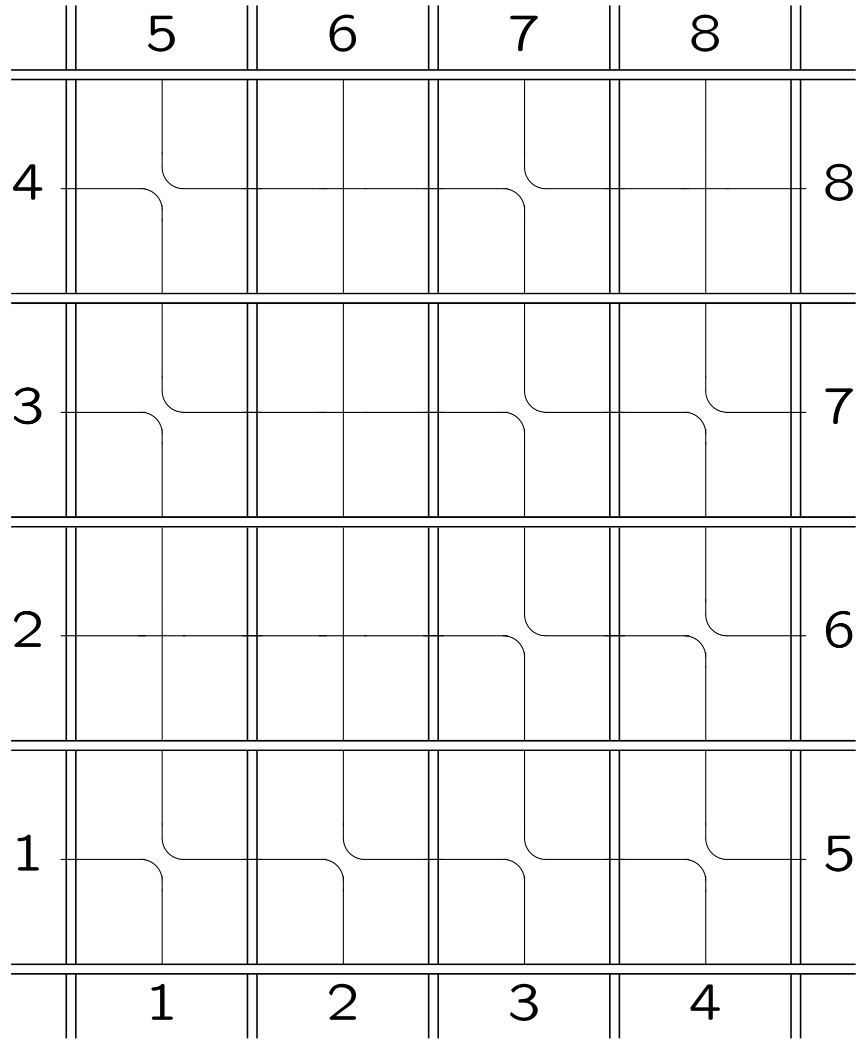
1. There is an explicit $1:1$ correspondence between $\mathcal{S}$ and $\mathcal{H}$-Sympl($\mathcal{M}_{m,p}(\mathbb{C})$).

2. Each $\mathcal{H}$-orbit is defined by some rank conditions.

Restricted permutations versus Cauchon diagrams

Replace ■ by + and □ by .

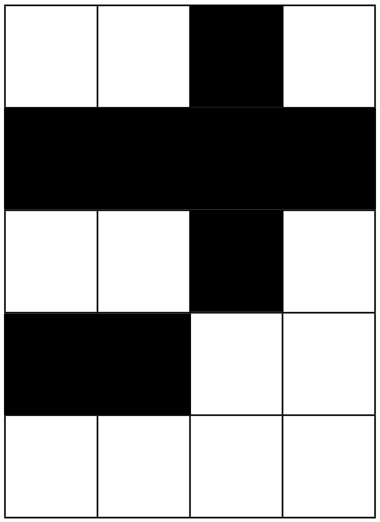
$C =$ 



Exercise

What are the restricted permutations associated to the 2×2 Cauchon diagrams?

What is the restricted permutation associated to



Additional exercises

1. Let $A = C^{\infty}(V)$ be a Poisson algebra and z be a Casimir element of A, that is, $\{z, f\} = 0$ for all $f \in A$. Show that z is constant on a symplectic leaf.

2. Let $\alpha \in \mathbb{C} \setminus \mathbb{Q}$. Check that one defines a Poisson structure on $\mathcal{O}(\mathbb{C}^3) = \mathbb{C}[X, Y, Z]$ via

$$\{X, Y\} = 0, \quad \{X, Z\} = \alpha X \text{ and } \{Y, Z\} = -Y.$$

Prove that $\{(a, b, c) \in \mathbb{C}^3 \mid ab^{\alpha} = 1\}$ is a symplectic leaf of $\mathbb{C}^3$.

3. One defines a Poisson bracket on $\mathcal{O}(\mathbb{R}^3) = \mathbb{R}[X, Y, Z]$ via

$$\{X, Y\} = Z, \quad \{X, Z\} = -Z \text{ and } \{Y, Z\} = X.$$

Compute the symplectic leaves.

4. Describe the semiclassical limit of the quantum special linear group $\mathcal{O}_q(SL_2(\mathbb{C})) := \mathcal{O}_q(M_2(\mathbb{C}))/\langle \det_q -1 \rangle$. Compute the symplectic leaves of $SL_2(\mathbb{C})$.

The non-negative world

- A matrix is **totally positive** if each of its minors is positive.

- A matrix is **totally non-negative** if each of its minors is non-negative.

History

- Fekete (1910s)
- Gantmacher and Krein, Schoenberg (1930s): small oscillations, eigenvalues
- Karlin and McGregor (1950s): statistics, birth and death processes
- Lindström (1970s): planar networks
- Gessel and Viennot (1985): binomial determinants, Young tableaux
- Gasca and Peña (1992): optimal checking
- Lusztig (1990s): reductive groups, canonical bases
- Fomin and Zelevinsky (1999/2000): survey articles (eg Math Intelligencer)
- Postnikov (2007): the totally non-negative grassmannian

Examples

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 \end{pmatrix} \qquad \begin{pmatrix} 5 & 6 & 3 & 0 \\ 4 & 7 & 4 & 0 \\ 1 & 4 & 4 & 2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

¿ How much work is involved in checking if a matrix is totally positive?

Eg. $n = 4$:

$$\#\text{minors} = \sum_{k=1}^{n} \binom{n}{k}^2 = \qquad\qquad \approx$$

by using Stirling's approximation

$$n! \approx \sqrt{2\pi n}\frac{n^n}{e^n}$$

2×2 **case**

The matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has **five** minors: $a, b, c, d, \Delta = ad - bc$.

If $b, c, d, \Delta = ad - bc > 0$ then

$$a = \frac{\Delta + bc}{d} > 0$$

so it is sufficient to check **four** minors.

Theorem (Fekete, 1913)

A matrix is totally positive if each of its **solid minors** is positive.

Theorem (Gasca and Peña, 1992)

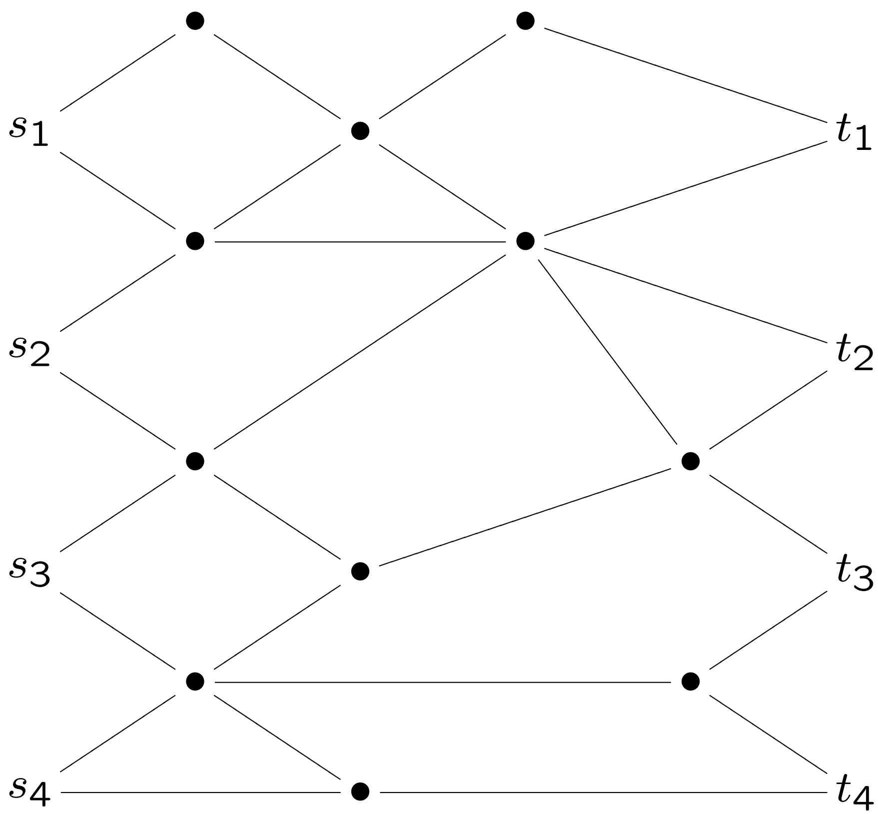
A matrix is totally positive if each of its **initial minors** is positive.

Theorem (Gasca and Peña, 1992)

A totally nonnegative matrix is totally positive if each of its **corner minors** is positive.

$$\begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix}$$

Planar networks Consider an directed graph with no directed cycles, n sources and n sinks.



$M = \left(m_{ij}\right)$ where m_{ij} is the number of paths from source s_i to sink t_j.

$$\begin{pmatrix} 5 & 6 & 3 & 0 \\ 4 & 7 & 4 & 0 \\ 1 & 4 & 4 & 2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

Edges directed left to right.

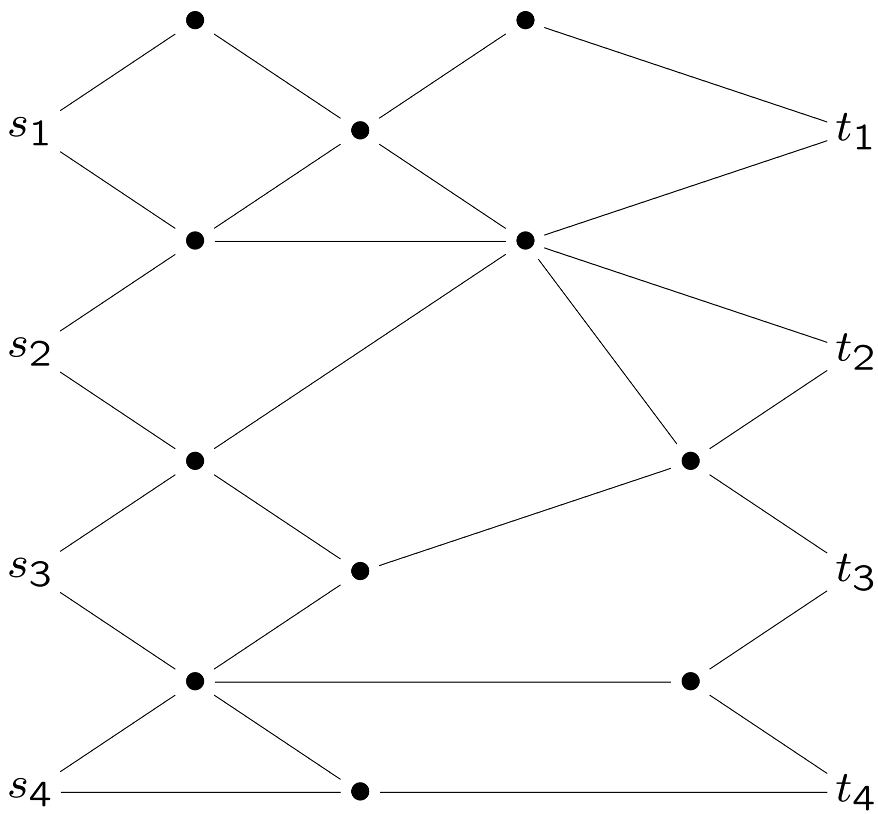
(Skandera: Introductory notes on total positivity)

Notation The minor formed by using rows from a set I and columns from a set J is denoted by $[I \mid J]$.

Theorem (Lindström)
The path matrix of any planar network is totally non-negative. In fact, the minor $[I \mid J]$ is equal to the number of families of non-intersecting paths from sources indexed by I and sinks indexed by J.

If we allow weights on paths then even more is true.

Theorem
Every totally non-negative matrix is the weighted path matrix of some planar network.



Edges directed left to right.

$M = \left(m_{ij}\right)$ where m_{ij} is the number of paths from source s_i to sink t_j.

$$\begin{pmatrix} 5 & 6 & 3 & 0 \\ 4 & 7 & 4 & 0 \\ 1 & 4 & 4 & 2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

Let $\mathcal{M}_{m,p}^{\mathsf{tnn}}$ be the set of totally non-negative $m \times p$ real matrices.

Let Z be a subset of minors. The **cell** S_Z^o is the set of matrices in $\mathcal{M}_{m,p}^{\mathsf{tnn}}$ for which the minors in Z are zero (and those not in Z are nonzero).

Some cells may be empty. The space $\mathcal{M}_{m,p}^{\mathsf{tnn}}$ is partitioned by the non-empty cells.

A trivial example In $\mathcal{M}_{2,1}^{\mathsf{tnn}}$ every cell is non-empty. There are 4 cells:

$$S^\circ_{\{\emptyset\}} = \{\begin{pmatrix} x \\ y \end{pmatrix} \mid x, y > 0\} \quad S^\circ_{\{[1,1]\}} = \{\begin{pmatrix} 0 \\ y \end{pmatrix} \mid y > 0\}$$

$$S^\circ_{\{[2,1]\}} = \{\begin{pmatrix} x \\ 0 \end{pmatrix} \mid x > 0\} \quad S^\circ_{\{[1,1],[2,1]\}} = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$$

Example In $\mathcal{M}_2^{\text{tnn}}$ the cell $S^{\circ}_{\{[2,2]\}}$ is empty.

For, suppose that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is tnn and $d = 0$.

Then $a, b, c \geq 0$ and also $ad - bc \geq 0$.

Thus, $-bc \geq 0$ and hence $bc = 0$ so that $b = 0$ or $c = 0$.

Note This is meant to jog your memory. Recall the proof that a prime in $\mathcal{O}_q(\mathcal{M}_2)$ that contains d must contain either b or c!

Exercise There are 14 non-empty cells in $\mathcal{M}_2^{\text{tnn}}$.

Postnikov (arXiv:math/0609764) defines **Le-diagrams**: an $m \times p$ array with entries either 0 or 1 is said to be a **Le-diagram** if it satisfies the following rule: if there is a 0 in a given square then either each square to the left is also filled with 0 or each square above is also filled with 0.

An example and a non-example of a Le-diagram on a 5×5 array

1	1	0	1	0
0	0	0	1	0
1	1	1	1	0
0	0	0	1	0
1	1	1	1	0

1	1	0	1	0
0	0	1	0	1
1	1	1	0	1
0	0	1	1	1
1	1	1	1	1

Note Le-diagrams are Cauchon diagrams with 0 = black and 1 = white!

• **Postnikov (arXiv:math/0609764)** There is a bijection between Le-diagrams on an $m \times p$ array and non-empty cells S_Z° in $\mathcal{M}_{m,p}^{\text{tnn}}$.

For 2×2 **matrices**, this says that there is a bijection between Cauchon/Le-diagrams on 2×2 arrays and non-empty cells in $\mathcal{M}_2^{\text{tnn}}$.

2×2 Le-diagrams

1	1
1	1

0	1
1	1

1	0
1	1

1	1
0	1

1	1
1	0

0	0
1	1

0	1
0	1

0	1
1	0

1	0
0	1

1	0
1	0

1	1
0	0

0	0
0	1

0	0
1	0

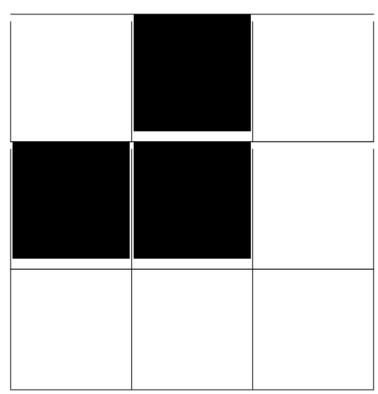
0	1
0	0

1	0
0	0

0	0
0	0

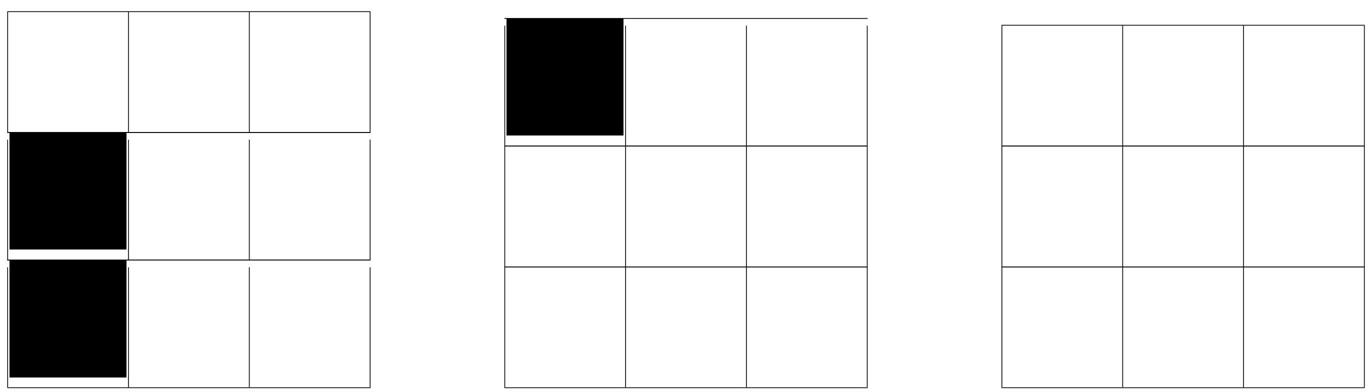
Postnikov's Algorithm starts with a Cauchon/Le-Diagram and produces a planar network from which one generates a totally non-negative matrix which defines a non-empty cell.

Example



$$\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

Perform Postnikov's algorithm on the following examples:



The Grand Unifying Theory

Reminder

- Cauchon diagrams.

- Restricted permutations.

- $\mathcal{H}$-primes: generated by families of q-minors.

- (Closure of) $\mathcal{H}$-orbits of leaves: defined by the vanishing of families of minors.

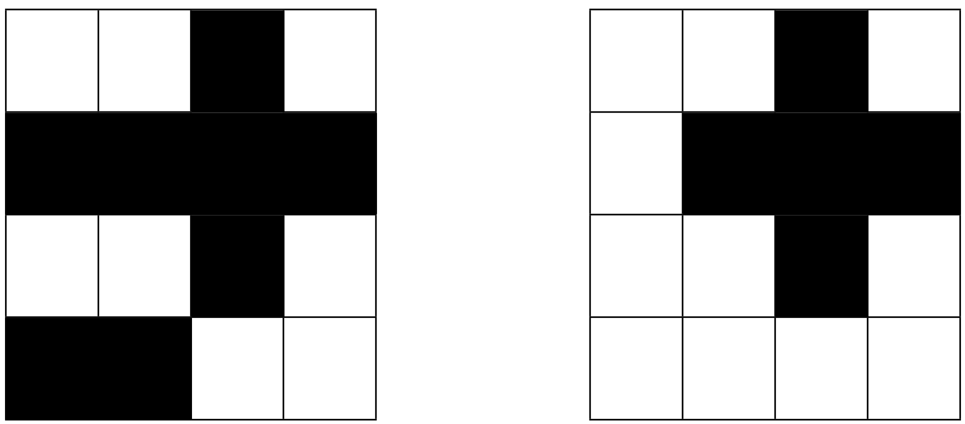
- TNN cells: defined by vanishing of families of minors.

- A family of minors is *admissible* if the associated TNN cell is non-empty.

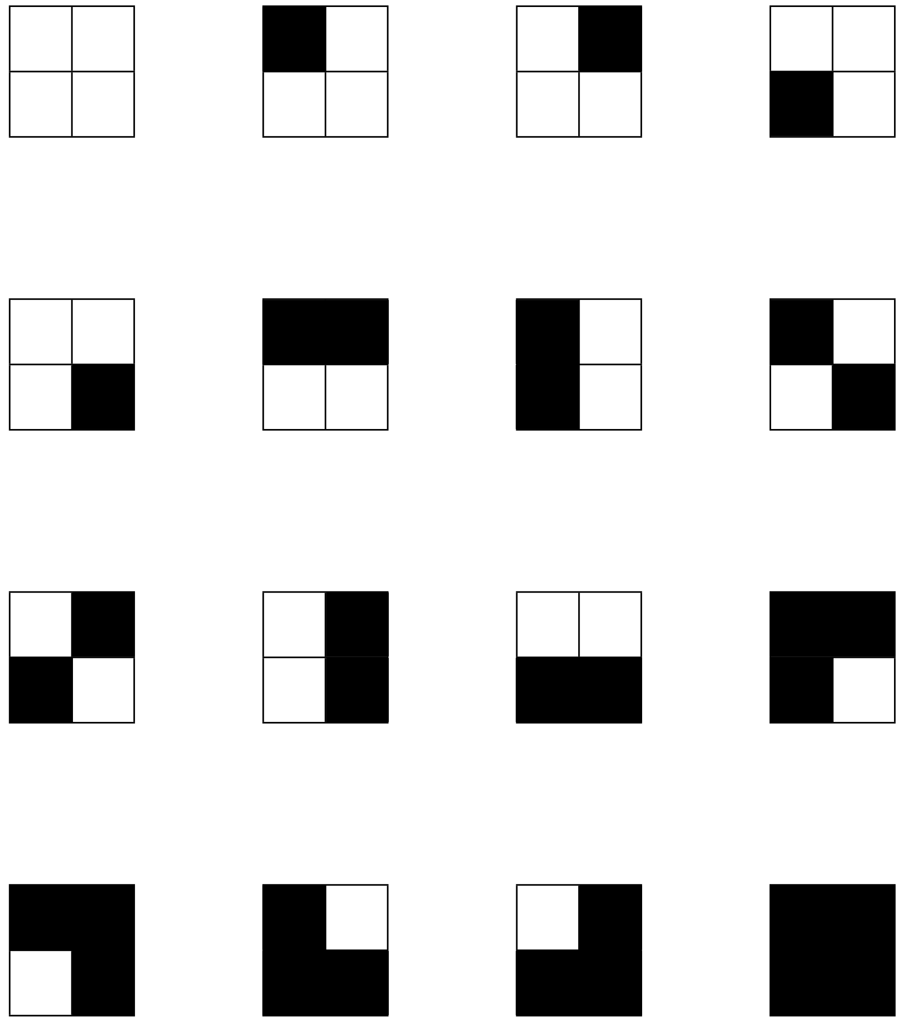
- In the 2x2 case, we get the same families of (quantum) minors. What about the general case?

Cauchon Diagrams

A **Cauchon Diagram** on an $m \times p$ array is an $m \times p$ array of squares filled either black or white such that if a square is coloured black then either each square to the left is coloured black, or each square above is coloured black. Here are an example and a non-example



2×2 Cauchon Diagrams

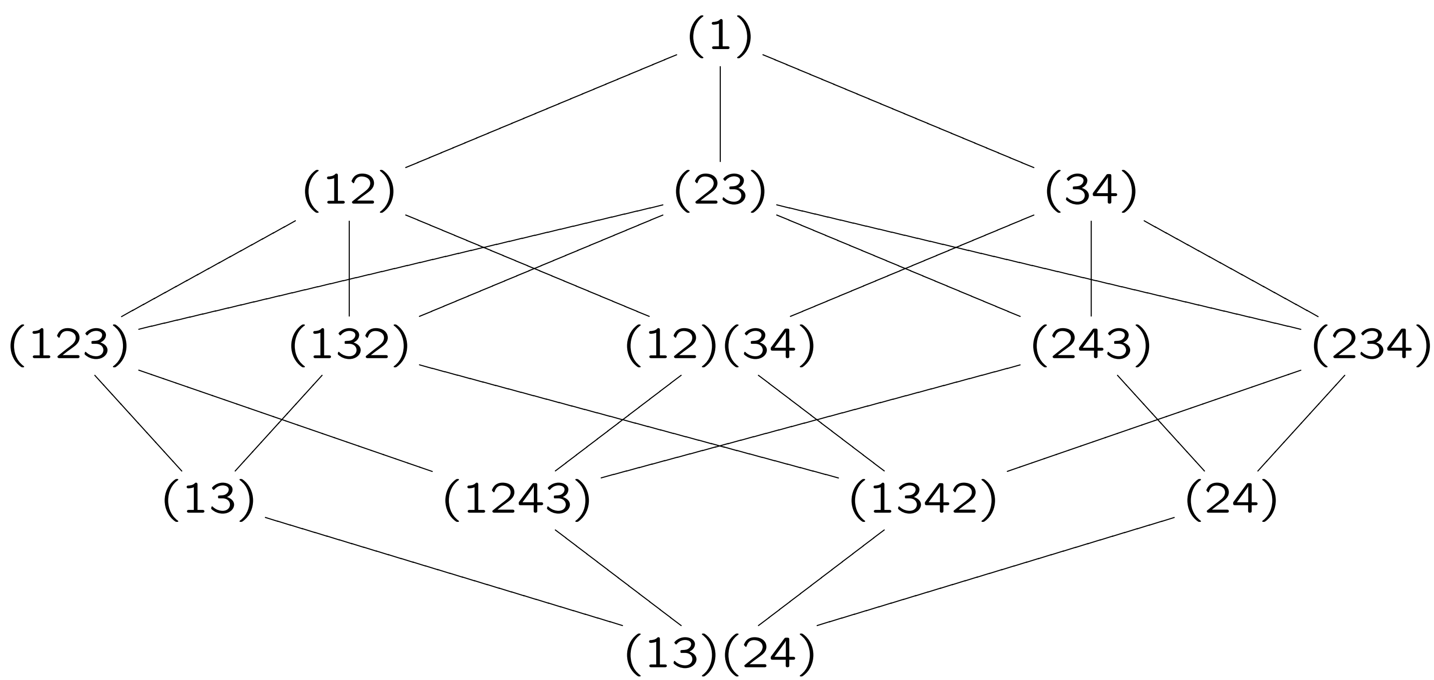


Restricted permutations

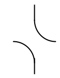
$w \in S_{m+p}$ with

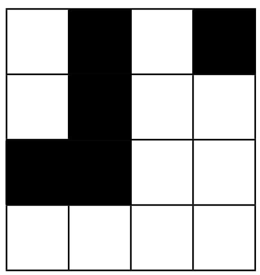
$$-p \leq w(i) - i \leq m \text{ for all } i = 1, 2, \ldots, m+p.$$

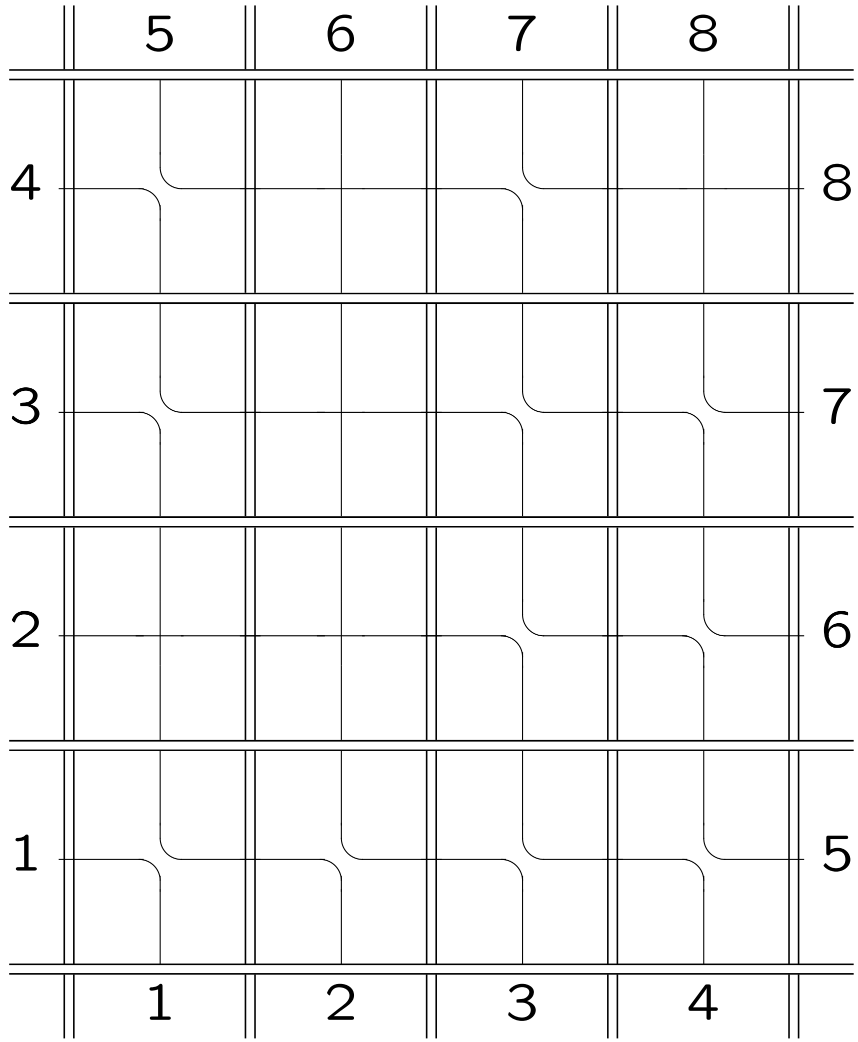
When $m = p = 2$, there are 14 of them.



Restricted permutations versus Cauchon diagrams

Replace $\blacksquare$ by $+$ and $\square$ by .

$C =$ 

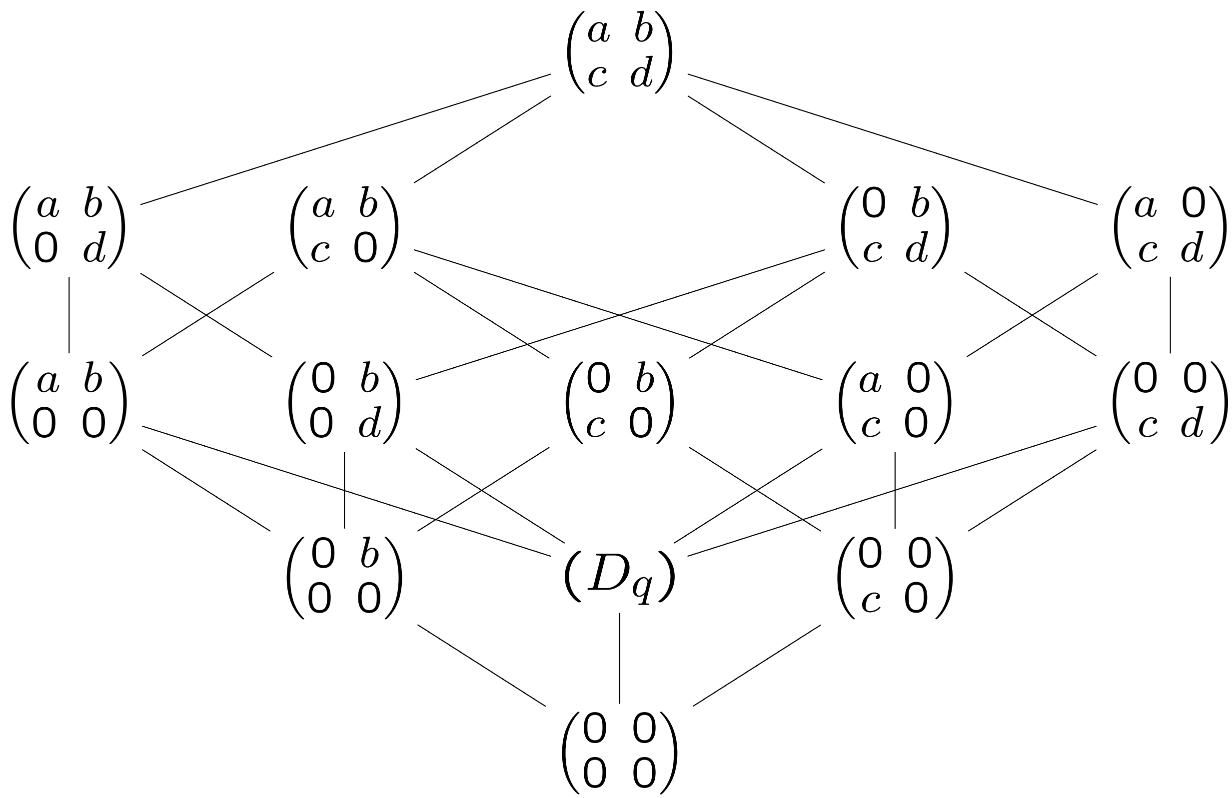


Generators of $\mathcal{H}$-primes in quantum matrices.

Theorem (Launois): Assume that q is transcendental.
Then $\mathcal{H}$-primes of $\mathcal{O}_q(\mathcal{M}(m,p))$ are generated by quantum minors.

Question: which families of quantum minors?

The following 14 $\mathcal{H}$-invariant ideals are all prime and these are the only $\mathcal{H}$-prime ideals in $\mathcal{O}_q(\mathcal{M}_2)$.



Matrix Poisson varities

$\mathcal{H}$-orbits of symplectic leaves are algebraic, and are defined by rank conditions. In other words, they are defined by the vanishing and non-vanishing of some families of minors.

Question: which families of minors?

Totally nonnegative cells

Totally nonnegative cells are defined by the vanishing of families of minors. Some of the TNN cells are empty.

We denote by S_Z^0 the TNN cell associated to the family of minors Z.

A family of minors is *admissible* if the corresponding TNN cell is non-empty.

Question: what are the admissible families of minors?

Conjecture

Let Z_q be a family of quantum minors, and Z be the corresponding family of minors.

$\langle Z_q \rangle$ is a $\mathcal{H}$-prime ideal iff the cell S_Z^0 is non-empty.

An algorithm to rule them all

Deleting derivations algorithm:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} a - bd^{-1}c & b \\ c & d \end{pmatrix}$$

Restoration algorithm:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} a + bd^{-1}c & b \\ c & d \end{pmatrix}$$

An algorithm to rule them all

If $M = (x_{i,\alpha}) \in \mathcal{M}_{m,p}(K)$, then we set

$$f_{j,\beta}(M) = (x'_{i,\alpha}) \in \mathcal{M}_{m,p}(K),$$

where

$$x'_{i,\alpha} := \begin{cases} x_{i,\alpha} + x_{i,\beta} x_{j,\beta}^{-1} x_{j,\alpha} & \text{if } x_{j,\beta} \neq 0,\ i < j \text{ and } \alpha < \beta \\ x_{i,\alpha} & \text{otherwise.} \end{cases}$$

We set $M^{(j,\beta)} := f_{j,\beta} \circ \cdots \circ f_{1,2} \circ f_{1,1}(M)$.

An example

Set $M = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Then

$$M^{(2,2)} = M^{(2,1)} = M^{(1,3)} = M^{(1,2)} = M^{(1,1)} = M,$$

$$M^{(3,1)} = M^{(2,3)} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad M^{(3,2)} = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and

$$M^{(3,3)} = \begin{pmatrix} 3 & 2 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Exercise. Is this matrix TNN?

Exercises

Perform the restoration algorithm for each of the following matrices and compute the minors of the resulting matrices. Are the resulting matrices TNN?

1. $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

2. $M = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

3. $M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

TNN Matrices and restoration algorithm

Theorem (Goodearl-Launois-Lenagan 2009).

• If the entries of M are nonnegative and its zeros form a Cauchon diagram, then $M^{(m,p)}$ is TNN.

• Let M be a matrix with real entries. We can apply the deleting derivation algorithm to M. Let N denote the resulting matrix.

Then M is TNN iff the matrix N is nonnegative and its zeros form a Cauchon diagram.

Exercise. Use the deleting derivation algorithm to test whether the following matrices are TNN:

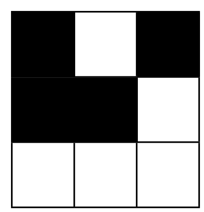
$$M_1 = \begin{pmatrix} 11 & 7 & 4 & 1 \\ 7 & 5 & 3 & 1 \\ 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} 7 & 5 & 4 & 1 \\ 6 & 5 & 3 & 1 \\ 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Another example

Let C be a Cauchon diagram and $T = (t_{i,\alpha})$ with $t_{i,\alpha} = 0$ iff (i, α) is a black box of C.

We set $T_C := f_{m,p} \circ \cdots \circ f_{1,2} \circ f_{1,1}(T)$.

Here $m = p = 3$ and

$C =$ 

We set $T = \begin{pmatrix} 0 & t_{1,2} & 0 \\ 0 & 0 & t_{2,3} \\ t_{3,1} & t_{3,2} & t_{3,3} \end{pmatrix}$ and $T^{(j,\beta)} := f_{j,\beta} \circ \cdots \circ f_{1,1}(T)$.

Recall that $f_{j,\beta}(x_{i,\alpha}) = (x'_{i,\alpha}) \in \mathcal{M}_{m,p}(K)$, where

$$x'_{i,\alpha} := \begin{cases} x_{i,\alpha} + x_{i,\beta} x_{j,\beta}^{-1} x_{j,\alpha} & \text{if } x_{j,\beta} \neq 0,\ i < j \text{ and } \alpha < \beta \\ x_{i,\alpha} & \text{otherwise.} \end{cases}$$

- $T^{(3,1)} = T^{(2,3)} = T^{(2,2)} = T^{(2,1)} = T^{(1,3)} = T^{(1,2)} = T.$

- $T^{(3,2)} = \begin{pmatrix} t_{1,2} t_{3,2}^{-1} t_{3,1} & t_{1,2} & 0 \\ 0 & 0 & t_{2,3} \\ t_{3,1} & t_{3,2} & t_{3,3} \end{pmatrix}.$

- $T_C = T^{(3,3)} = \begin{pmatrix} t_{1,2} t_{3,2}^{-1} t_{3,1} & t_{1,2} & 0 \\ t_{2,3} t_{3,3}^{-1} t_{3,1} & t_{2,3} t_{3,3}^{-1} t_{3,2} & t_{2,3} \\ t_{3,1} & t_{3,2} & t_{3,3} \end{pmatrix}.$

Results

• If $K = \mathbb{R}$ and T is nonnegative, then T_C is TNN.

• If $K = \mathbb{C}$ and the nonzero entries of T are algebraically independent, then the minors of T_C that are equal to zero are exactly those that vanish on the closure of a given $\mathcal{H}$-orbit of symplectic leaves.

• If $K = \mathbb{C}$ and the nonzero entries of T are the generators of a certain quantum affine space, then the quantum minors of T_C that are equal to zero are exactly those belonging to a given $\mathcal{H}$-prime in $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$.

• The families of (quantum) minors we get depend only on C in these three cases. And if we start from the same Cauchon diagram in these three cases, then we get exactly the same families.

Main Result

Theorem. (GLL) Let $\mathcal{F}$ be a family of minors in the coordinate ring of $\mathcal{M}_{m,p}(\mathbb{C})$, and let $\mathcal{F}_q$ be the corresponding family of quantum minors in $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$. Then the following are equivalent:

1. The totally nonnegative cell associated to $\mathcal{F}$ is non-empty.

2. $\mathcal{F}$ is the set of minors that vanish on the closure of a torus-orbit of symplectic leaves in $\mathcal{M}_{m,p}(\mathbb{C})$.

3. $\mathcal{F}_q$ is the set of quantum minors that belong to torus-invariant prime in $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$.

Consequences of the Main Result

The TNN cells are the traces of the $\mathcal{H}$-orbits of symplectic leaves on $\mathcal{M}_{m,p}^{\text{tnn}}$.

The sets of minors that vanish on the closure of a torus-orbit of symplectic leaves in $\mathcal{M}_{m,p}(\mathbb{C})$ can be explicitely described thanks to results of Fulton and Brown-Goodearl-Yakimov. So, as a consequence of the previous result, **the sets of minors that define non-empty totally nonnegative cells are explicitely described.**

On the other hand, when the deformation parameter q is transcendental over the rationals, then the torus-invariant primes in $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$ are generated by quantum minors, and so we deduce from the above result **explicit generating sets of quantum minors for the torus-invariant prime ideals of** $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$.

Explicit descriptions of the families of minors

For $w \in \mathcal{S}$, define $\mathcal{M}(w)$ to be the set of minors $[I|\Lambda]$, with $I \subseteq [\![1,m]\!]$ and $\Lambda \subseteq [\![1,p]\!]$, that satisfy at least one of the following conditions.

1. $I \not\leq w_\circ^m w(L)$ for all $L \subseteq [\![1,p]\!] \cap w^{-1}[\![1,m]\!]$ such that $|L| = |I|$ and $L \leq \Lambda$.

2. $m + \Lambda \not\leq w w_\circ^N(L)$ for all $L \subseteq [\![1,m]\!] \cap w_\circ^N w^{-1}[\![m+1,N]\!]$ such that $|L| = |\Lambda|$ and $L \leq I$.

3. There exist $1 \leq r \leq s \leq p$ and $\Lambda' \subseteq \Lambda \cap [\![r,s]\!]$ such that $|\Lambda'| > |[\![r,s]\!] \setminus w^{-1}[\![m+r,\, m+s]\!]|$.

4. There exist $1 \leq r \leq s \leq m$ and $I' \subseteq I \cap [\![r,s]\!]$ such that $|I'| > |w_\circ^N[\![r,s]\!] \setminus w^{-1}w_\circ^m[\![r,s]\!]|$.