POINCARÉ SERIES OF MULTI-FILTERED ALGEBRAS AND PARTITIVITY

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Abstract. We prove that if an algebra R over a field can be endowed with a pointed and locally finite $\mathbb{N}^n$-filtration such that the associated $\mathbb{N}^n$-graded algebra T is semi-commutative, then R is left and right finitely partitive. In order to do this, we consider a multi-variable Poincaré Series for every finitely generated graded T-module and show that this Poincaré Series is a rational function. Our methods apply to some iterated Ore extensions such as quantum matrices and quantum Weyl algebras as well as to the quantized enveloping algebra of $\mathfrak{sl}(\nu+1)$.

Introduction

The notion of a multi-filtration has been used recently in [3] to obtain the exactness of the Gelfand-Kirillov dimension of certain associative algebras over a field that have no "obvious" filtration with a simpler associated $\mathbb{N}$-graded algebra. A typical situation that arises is when the $\mathbb{N}^n$-graded algebra, $G_{\mathcal{F}}(R)$, associated to a multifiltration $\mathcal{F}$ on an algebra R, is semi-commutative. This is the case for the quantum coordinate ring, $\mathcal{O}_q(M_n(k))$, of quantum $n \times n$ matrices over k, or the quantized universal enveloping algebras $\breve{U}_q(\mathfrak{sl}(n+1))$. In this paper, we show that for these algebras the Gelfand-Kirillov dimension is left and right partitive, in addition to being exact.

On the way, we develop some tools which may be of independent interest. In Section 1, we consider a natural notion of a multi-variable Poincaré Series for finitely generated modules over a locally finite pointed $\mathbb{N}^n$-graded algebra. In the case that the algebra is semi-commutative, then we show that the Poincaré series is a rational function of n variables, Theorem 1.5. This is then applied to $G_{\mathcal{F}}(R)$ in Section 2, in conjunction with the main result of [3] to obtain the result that R is a left and right partitive algebra whenever the $\mathbb{N}^n$-filtration is finite dimensional and the associated multigraded algebra is semicommutative, Theorem 2.8. In order to achieve these results, we introduce a notion of multiplicity for $\mathbb{N}^n$-filtered modules, see 1.7 and 2.6.

1. Poincaré series

In this note, all algebras are associative with identity. By $\mathbb{N}^n$ we will denote the free abelian monoid with n generators. Let $R = \oplus_{\alpha \in \mathbb{N}^n} R_\alpha$ be an $\mathbb{N}^n$-graded algebra over a field

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k. Assume that R is **pointed** and **locally finite**, in the sense that $R_{\mathbf{0}} = k$ and that R_α is a finite-dimensional k–vector space for every $\alpha \in \mathbb{N}^n$. (Note that many authors used **connected** instead of pointed). If $r \in R_\alpha$ then we say that $\deg(r) = \alpha$. Every finitely generated $\mathbb{N}^n$–graded left R–module $M = \oplus_{\alpha \in \mathbb{N}^n} M_\alpha$ is locally finite; that is, $\dim_k M_\alpha < \infty$ for every multi-index α. Consider the function $d_M : \mathbb{N}^n \to \mathbb{N}$ given by $d_M(\alpha) = \dim_k M_\alpha$.

Definition 1.1. The **multi-variable Poincaré series** of a finitely generated $\mathbb{N}^n$-graded left R–module M is given by

$$P(M; \underline{t}) = \sum_{\alpha \in \mathbb{N}^n} d_M(\alpha) \underline{t}^\alpha$$

where $\underline{t}^\alpha = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$, for $t_1, \dots, t_n$ commuting indeterminates over $\mathbb{Z}$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$.

The aim of this section is to prove that the Poincaré series is a rational function in the variables $t_1, \dots, t_n$ for certain non-commutative algebras that resemble affine commutative graded algebras.

Definition 1.2. We will say that two elements x, y of the algebra R are **semi-commuting** if there is $0 \neq q \in k$ such that $yx = qxy$. The graded k-algebra R is called **semi-commutative** if R is generated as k–algebra by a finite set of homogeneous semi-commuting elements.

Note that a consequence of the definition is that such an algebra R is noetherian, since it can be presented as a factor algebra of an iterated Ore extension of the base field k. The aim of this section is to prove the following non-commutative version of the Hilbert-Serre Theorem.

Theorem. *Let M be a finitely generated $\mathbb{N}^n$–graded left module over a semi-commutative $\mathbb{N}^n$–graded k-algebra R. Let $x_1, \dots, x_s$ be a set of semi-commuting homogeneous generators of R, with $\deg(x_j) = \mu^j \in \mathbb{N}^n$. There exists a polynomial $f(\underline{t}) \in \mathbb{Z}[t_1, \dots, t_n]$ such that*

$$P(M; \underline{t}) = \frac{f(\underline{t})}{\prod_{j=1}^{s} (1 - \underline{t}^{\mu^j})}$$

To give a proof of this theorem, we need some notation. For each non-empty subset J of $\{1, \dots, n\}$, let G_J denote the submonoid of $\mathbb{N}^n$ whose elements are the multi-indices α such that $\alpha_j = 0$ for every $j \in J$. Observe that G_J is isomorphic to the monoid $\mathbb{N}^{n-d}$, where d is the cardinal of J. Define

$$R^{[J]} = \bigoplus_{\alpha \in G_J} R_\alpha.$$

It is immediate that R is a G_J–graded subalgebra of R.

Lemma 1.3. *The graded algebra $R^{[J]}$ is generated by the set $\{x_j : \deg(x_j) \in G_J\}$. In particular, $R^{[J]}$ is a semi-commutative graded algebra.*

Proof. Pick $r \in R^{[J]}$ an homogeneous element with $\deg(r) = \alpha \in G_J$. Since R is generated by the semi-commuting elements $x_1, \ldots, x_s$, we can write $r = \sum_{\beta \in \mathbb{N}^s} a_\beta x_1^{\beta_1} \cdots x_s^{\beta_s}$ for some $a_\beta \in k$. But the x_i's are homogeneous, so that for every $\beta \in \mathbb{N}^s$ such that $a_\beta \neq 0$, we have that $\alpha = \deg(x_1^{\beta_1} \cdots x_s^{\beta_s}) = \sum_{i=1}^s \beta_i \deg(x_i)$. Now, $\beta_i \neq 0$ does imply $\deg(x_i) \in G_J$, as desired.

$\square$

1.4. For every $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, define $\alpha | J = (a_1, \ldots, a_n) \in G_J$ by requiring that $a_i = 0$ if $i \in J$ and $a_i = \alpha_i$ if $i \notin J$. Every $\mathbb{N}^n$–graded left R–module $M = \oplus_{\alpha \in \mathbb{N}^n} M_\alpha$ is a left $R^{[J]}$–module, by restriction of scalars. For every $\gamma \in G_J$, consider the vector subspace of M given by $M_{(\gamma)} = \oplus_{\alpha | J = \gamma} M_\alpha$. The decomposition $M = \oplus_{\gamma \in G_J} M_{(\gamma)}$ is a G_J–grading for the $R^{[J]}$–module M.
For each $\mu \in \mathbb{N}^n$ set $S_{J,\mu} = \{\alpha \in \mathbb{N}^n \mid \alpha_j < \mu_j \ \forall j \in J\}$. The vector space

$$M^{[J],\mu} = \bigoplus_{\alpha \in S_{J,\mu}} M_\alpha$$

is a submodule of the left $R^{[J]}$–module M (we consider $M^{[J],\mu} = 0$ when $S_{J,\mu}$ is empty). It is not difficult to check that $M^{[J],\mu} = \oplus_{\gamma \in G_J} (M^{[J],\mu} \cap M_{(\gamma)})$, which implies that $M^{[J],\mu}$ is a G_J–graded $R^{[J]}$–submodule of M.

Lemma. *If M is a finitely generated $\mathbb{N}^n$–graded left R–module, then $M^{[J],\mu}$ is a finitely generated G_J–graded left $R^{[J]}$–module.*

Proof. It is easy to prove that $(L + N)^{[J],\mu} = L^{[J],\mu} + N^{[J],\mu}$ for L, N any graded R–submodules of M; and so we can assume that M is cyclic, generated by an homogeneous element m, say. If $M^{[J],\mu} \neq 0$ then there is a nonzero homogeneous r in R such that $0 \neq rm \in M^{[J],\mu}$. Therefore $\deg(r) + \deg(m) = \deg(rm)$, and, thus, $\deg(m)_j < \mu_j$ for every $j \in J$. This implies that $m \in M^{[J],\mu}$. Since R is generated as an algebra by homogeneous semi-commuting elements $x_1, \ldots, x_s$ it follows that the left R–module $Rm = M^{[J],\mu}$ is spanned as vector space over k by the subset Y consisting of the elements of the form $x_1^{b_1} \cdots x_s^{b_s} m$ such that $\sum_{k=1}^s b_k \deg(x_k)_j + \deg(m)_j < \mu_j$ for every $j \in J$. Let X be the subset of Y consisting of those $x_1^{b_1} \cdots x_s^{b_s} m$ such that $b_i = 0$ for every $x_i \in R^{[J]}$. As the elements x_i are semi-commuting, it follows that X generates M as a left $R^{[J]}$–module. It is clear that X is a finite set. $\square$

We are now ready to give a proof of the theorem.

Theorem 1.5. *Let M be a finitely generated $\mathbb{N}^n$–graded left module over a semi-commutative $\mathbb{N}^n$–graded k-algebra R. Let $x_1, \ldots, x_s$ be a set of semi-commuting homogeneous generators of R, with $\deg(x_j) = \mu^j \in \mathbb{N}^n$ for $j = 1, \ldots, s$. There exists a polynomial $f(\underline{t}) \in \mathbb{Z}[t_1, \ldots, t_n]$ such that*

$$P(M; \underline{t}) = \frac{f(\underline{t})}{\prod_{j=1}^s (1 - \underline{t}^{\mu^j})}$$

Proof. We proceed by induction on s. For any n and for $s = 0$, the algebra R is just k and the module M is finite-dimensional, so that the Poincaré series is a polynomial. Now, let $s > 0$. Assume that the result is true for any n and $s' < s$. Write $\mu = \deg(x_s)$ and consider the subalgebra B of R generated by $x_1, \dots x_{s-1}$. Since the semi-commuting generators of B are homogeneous, it is clear that B is $\mathbb{N}^n$-graded in a straightforward way, and, in fact, the inductive hypothesis applies to B, since B has fewer than s semi-commuting generators. Now, for an $\mathbb{N}^n$–graded left R–module M consider the k–linear graded map $M \to M(\mu)$ given by multiplication by x_s. Here, $M(\mu)$ denotes the μ–shifting of M. The fact that x_s semi-commutes with the rest of the generators of R gives that the kernel K and the image L of this mapping are R–submodules of M and $M(\mu)$, respectively. Hence,

$$P(M(\mu); \underline{t}) - P(M; \underline{t}) = P(C; \underline{t}) - P(K; \underline{t})$$

where C is the cokernel of our mapping $M \to M(\mu)$. Now, K and C are finitely generated graded left R–modules, since R is a noetherian algebra, and, they are finitely generated graded left B–modules, since x_s annihilates both of them. By induction on s it follows that

$$P(C; \underline{t}) - P(K; \underline{t}) = \frac{g(\underline{t})}{\prod_{j=1}^{s-1}(1 - \underline{t}^{\mu_j})}$$

for some polynomial $g(\underline{t}) \in \mathbb{Z}[t_1, \dots, t_n]$. Call this rational function $q(\underline{t})$; so that

$$P(M(\mu); \underline{t}) = P(M; \underline{t}) + q(\underline{t}).$$

On the other hand,

$$\begin{aligned}
(1) \quad P(M(\mu); \underline{t}) &= \sum_{\beta \in \mathbb{N}^n} d_{M(\mu)}(\beta) \underline{t}^\beta \\
&= \sum_{\beta \in \mathbb{N}^n} d_M(\beta + \mu) \underline{t}^\beta = \sum_{\beta + \mu} d_M(\beta + \mu) \underline{t}^{\beta + \mu} \underline{t}^{-\mu} \\
&= \underline{t}^{-\mu} \left(\sum_{\alpha \in \mathbb{N}^n} d_M(\alpha) \underline{t}^\alpha - \sum_{\alpha \notin \mu + \mathbb{N}^n} d_M(\alpha) \underline{t}^\alpha \right) \\
&= \underline{t}^{-\mu} \left(P(M; \underline{t}) - P(M; \mu, \underline{t}) \right),
\end{aligned}$$

where

$$P(M; \mu, \underline{t}) = \sum_{\alpha \notin \mu + \mathbb{N}^n} d_M(\alpha) \underline{t}^\alpha.$$

By comparing equation (1) with the earlier formula for $P(M(\mu); \underline{t})$, we see that

$$(1 - \underline{t}^\mu) P(M; \underline{t}) = \underline{t}^\mu q(\underline{t}) + P(M; \mu, \underline{t}). \qquad (2)$$

It remains to show that $P(M;\mu,\underline{t})$ can be written as a rational function of the required form; that is, $P(M;\mu,\underline{t})$ can be written as a rational function with denominator $\prod_{j=1}^{s-1}(1-\underline{t}^{\mu^j})$. From the formula

$$\sum_{J\subseteq\{1,\dots,n\}}(-1)^{|J|}=0,$$

where $|J|$ denotes the cardinal of J, we get

$$P(M;\mu,\underline{t})=\sum_{\emptyset\neq J\subseteq\{1,\dots,n\}}(-1)^{|J|+1}P(M^{[J],\mu};\underline{t}) \tag{3}$$

Consider a fixed nonempty subset J, and suppose that $x_s\in R^{[J]}$. Then $\mu_i=0$, for each $i\in J$, where $\mu=(\mu_1,\dots,\mu_n)=\deg(x_s)$. Recall that

$$M^{[J],\mu}=\bigoplus_{\alpha\in S_{J,\mu}}M_\alpha,$$

where

$$S_{J,\mu}=\{\alpha\in\mathbb{N}^n\mid\alpha_j<\mu_j\ \forall j\in J\}.$$

Now, $\alpha\in S_{J,\mu}$ means that $\alpha_i<\mu_i$, for each $i\in J$. Thus, $\alpha_i<0$, for each $i\in J$; and so $M^{[J],\mu}=0$. Hence, $P(M^{[J],\mu};\underline{t})=0$, whenever $x_s\in R^{[J]}$, and the only nonzero contributions to the right hand side of (3) occur when $x_s\notin R^{[J]}$. In these cases, the inductive hypothesis applies, and $P(M^{[J],\mu};\underline{t})$ can be written as a rational function with denominator $\prod_{j=1}^{s-1}(1-\underline{t}^{\mu^j})$, since $R^{[J]}$ is generated by a subset of $\{x_1,\dots,x_{s-1}\}$, by Lemma 1.3 and $M^{[J],\mu}$ is a finitely generated $R^{[J]}$-module, by Lemma 1.4. This fact, together with (2), proves the theorem. ☐

1.6. For each multi-index $\alpha\in\mathbb{N}^n$, we set $|\alpha|:=\alpha_1+\cdots+\alpha_n$. Define, for $m\in\mathbb{N}$, $R_m=\oplus_{|\alpha|=m}R_\alpha$. The vector subspace decomposition $R=\oplus_{m\in\mathbb{N}}R_m$ is an $\mathbb{N}$–grading of R. Moreover, if M is an $\mathbb{N}^n$–graded left R–module, then M is an $\mathbb{N}$–graded left R–module by putting $M_m=\oplus_{|\alpha|=m}M_\alpha$ for every natural number m. If M is finitely generated as a left R–module, then it makes sense to define the classical Poincaré series of M with respect to the grading $M=\oplus_{m\in\mathbb{N}}M_m$ by

$$P_c(M;t)=\sum_{m\in\mathbb{N}}\dim_k M_m t^m$$

But it is clear that $P_c(M;t)=P(M;t,\dots,t)$, so that Theorem 1.5 implies that this series is rational of the form

$$P_c(M;t)=\frac{f(t)}{\prod_{i=1}^{s}(1-t^{|\mu^i|})}, \tag{4}$$

where $f(t)=f(t,\dots,t)\in\mathbb{Z}[t]$.

1.7. Consider, for the graded R–module $M = \oplus_{m\in\mathbb{N}} M_m$, the mapping $m \mapsto g(m) = \sum_{l=0}^{m} \dim_k(M_l)$. By [6, Lemma I.2.5], the Gelfand-Kirillov dimension of M is the degree of growth of the function $g(m)$. Now, the series $P(t) = \sum_{m\in\mathbb{N}} g(m)t^m = P_c(M;t)/(1-t)$. By (4) and [6, Lemma III.1.3] there exists a natural number ν and there exist functions $c_l : \mathbb{Z}/\nu\mathbb{Z} \to \mathbb{Q}$ $(l = 0, 1, \dots, d)$ with c_d constant and positive such that $g(m) = \sum_{l=0}^{d} c_l(m+\nu\mathbb{Z})m^l$, for m big enough. In fact, ν is the least common multiple of $|\mu^1|, \dots, |\mu^s|$, so that it does not depends on the module M. Moreover, $d+1$ is the order of the pole of $P(t)$ at $t = 1$. In particular, we obtain the following well-known result.

Proposition 1.8. *The Gelfand-Kirillov dimension of any finitely generated $\mathbb{N}^n$–graded left R–module is the order of the pole at $t = 1$ of the Poincaré series $P_c(M;t)$.*

1.9. Keep the notation of 1.7. Put $h(m) = g(\nu m)$, for every $m \in \mathbb{N}$. The function h is a polynomial of degree d with rational coefficients, for sufficiently large m. Moreover, the leading coefficient of $h(m)$ is $\nu^d c_d$. By [5, Lemma 1.5], $d!\nu^d c_d$ is a positive integer.

Definition. The positive integer $e(M) = d!\nu^d c_d$ will be called the **multiplicity** of the $\mathbb{N}^n$–graded left R–module M.

2. Finitely partitive algebras

In this section, R will denote an algebra over the field k. We do not assume that R is graded nor semi-commutative. We recall the definitions of multi-filtered algebras and modules, in the sense of [3].

Definition 2.1. An **admissible order** '$\leq$' on $(\mathbb{N}^n, +)$ is a total order such that,
(a) $\mathbf{0} = (0, \dots, 0) \leq \alpha$ for every $\alpha \in \mathbb{N}^n$.
(b) For all $\alpha, \beta, \gamma \in \mathbb{N}^n$ with $\alpha \leq \beta$ it follows $\alpha + \gamma \leq \beta + \gamma$.

Remark 2.2. By Dickson's Lemma (see, e.g., [2, Corollary 4.48]), admissible orders on $\mathbb{N}^n$ are **good** orders; that is, any non empty subset of $\mathbb{N}^n$ has a first element.

¿From now on, fix an admissible order '$\leq$' on $(\mathbb{N}^n, +)$.

Definition 2.3. A **multi-filtration** on R is a family $\mathcal{F} = \{F_\gamma(R) | \gamma \in \mathbb{N}^n\}$ of vector subspaces of R satisfying the following axioms.

1. $F_\gamma(R) \subseteq F_\delta(R)$ if $\gamma \leq \delta$.
2. $F_\gamma(R)F_\delta(R) \subseteq F_{\gamma+\delta}(R)$.
3. $\bigcup_{\alpha\in\mathbb{N}^n} F_\alpha(R) = R$.
4. $1 \in F_{\mathbf{0}}(R)$.

In what follows, we assume that the algebra R is endowed with a fixed multi-filtration.

Definition 2.4. A **multi-filtration** on a left R–module M is a family $\mathcal{F} = \{F_\gamma(M) | \gamma \in \mathbb{N}^n\}$ of vector subspaces of M satisfying the following axioms.

1. $F_\gamma(M) \subseteq F_\delta(M)$ if $\gamma \leq \delta$.
2. $F_\gamma(R)F_\delta(M) \subseteq F_{\gamma+\delta}(M)$.

3. $\bigcup_{\alpha\in\mathbb{N}^n} F_\alpha(M) = M$.

The multi-filtered algebra R has an **associated $\mathbb{N}^n$–graded algebra** defined as follows.

Given a multi-filtered left R–module M, and $\gamma \in \mathbb{N}^n$, write

$$V_\gamma(M) = \bigcup_{\gamma'<\gamma} F_{\gamma'}(M).$$

It is understood that $V_{\mathbf{0}} = \{0\}$. Consider the vector space

$$G_\gamma(M) = F_\gamma(M)/V_\gamma(M),$$

and define

$$G(M) = G_{\mathcal{F}}(M) = \bigoplus_{\gamma\in\mathbb{N}^n} G_\gamma(M).$$

For $r + V_\gamma(R) \in G_\gamma(R)$ and $m + V_\delta(M) \in G_\delta(M)$, the product

$$(r + V_\gamma(R))(m + V_\delta(M)) = rm + V_{\gamma+\delta}(M).$$

is well-defined. Thus, if $M = R$, then we have a product in $G(R)$ which makes $G(R)$ into an $\mathbb{N}^n$–graded algebra; moreover, $G(M)$ becomes an $\mathbb{N}^n$–graded left $G(R)$–module.

Definition 2.5. Let $F_\alpha(R)$ be an $\mathbb{N}^n$–filtration over a k–algebra R. We say that this multi-filtration is **locally finite** if the vector subspaces $F_\alpha(R)$ are finite-dimensional and **pointed** if $F_{\mathbf{0}}(R) = k$.

2.6. Assume that R is endowed with a locally finite, pointed multi-filtration. This implies that $G(R)$ is pointed and locally finite. Moreover, suppose that $G(R)$ is a semi-commutative multi-graded algebra in the sense of Definition 1.2. It follows from [3, Theorems 2.5 and 2.10] that R is noetherian and that Gelfand-Kirillov dimension is exact on finitely generated modules. The aim of this section is to show that Gelfand-Kirillov dimension is finitely partitive (see definition 2.8). Let M be a left R–module endowed with an $\mathbb{N}^n$–filtration $\mathcal{F} = \{F_\alpha(M)\}$. Assume that the associated $\mathbb{N}^n$–graded left $G(R)$–module $G_{\mathcal{F}}(M)$ is finitely generated. It makes sense consider the Poincaré series $P(G_{\mathcal{F}}(M;\underline{t}))$. By 1.6, 1.7 and 1.9 we can associate to $G_{\mathcal{F}}(M)$ the multiplicity when considered as $\mathbb{N}$–graded module. Since, once the multi-filtration on R is fixed, this multiplicity depends on the multi-filtration $\mathcal{F}$ in M, we will denote it by $e_{\mathcal{F}}(M)$. We know that it is a positive integer.

Proposition 2.7. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of homomorphisms of R–modules. Assume given $\mathbb{N}^n$–filtrations $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ on M_1, M_2 and M_3, respectively, such that the homomorphisms in the short exact sequence are filtered and that the associated sequence of $\mathbb{N}^n$–graded $G(R)$–modules*

$$0 \to G_{\mathcal{F}_1}(M_1) \to G_{\mathcal{F}_2}(M_2) \to G_{\mathcal{F}_3}(M_3) \to 0 \tag{5}$$

is exact. If $G_{\mathcal{F}_2}(M_2)$ is finitely generated then

$$e_{\mathcal{F}_2}(M_2) = \begin{cases} e_{\mathcal{F}_1}(M_1) + e_{\mathcal{F}_3}(M_3) & \textit{if } GKdim(M_1) = GKdim(M_3) \\ e_{\mathcal{F}_1}(M_1) & \textit{if } GKdim(M_1) > GKdim(M_3) \\ e_{\mathcal{F}_3}(M_3) & \textit{if } GKdim(M_1) < GKdim(M_3) \end{cases}$$

Proof. The $G(R)$–modules $G_{\mathcal{F}_i}(M_i)$ are finitely generated since $G(R)$ is noetherian. The exactness of the sequence (5) gives

$$P(G_{\mathcal{F}_2}(M_2); \underline{t}) = P(G_{\mathcal{F}_1}(M_1); \underline{t}) + P(G_{\mathcal{F}_3}(M_3); \underline{t}) \tag{6}$$

Now consider the exact sequence (5) as a sequence of $\mathbb{N}$–graded $G(R)$–modules (see 1.6). If $g_i(m)$ denotes the growth function of $G_{\mathcal{F}_i}(M_i)$ (see 1.7), then, by (6), $g_2(m) = g_1(m) + g_3(m)$ for m big enough. By [6, Lemma III.1.3] (see also 1.7), there is a natural number ν and functions $c_l^i : \mathbb{Z}/\nu\mathbb{Z} \to \mathbb{Q}$, for $l = 0, 1, \ldots, d_i$ and $i = 1, 2, 3$ such that $g_i(m) = \sum_{l=0}^{d_i} c_l^i(m + \nu\mathbb{Z})m^l$, where $d_i = GKdim(G_{\mathcal{F}_i}(M_i))$. By [3, Theorems 2.8, 2.10] $GKdim(M_i) = d_i$ for $i = 1, 2, 3$ and $d_2 = \max\{d_1, d_3\}$. The result can be easily deduced now.

□

Definition 2.8. Let R be an affine algebra over k of finite Gelfand-Kirillov dimension. The algebra R is said to be **left finitely partitive** if for every finitely generated left R–module M there is an integer $e > 0$ such that, for every strictly descending chain

$$M = M_0 \supset M_1 \supset \cdots \supset M_m \tag{7}$$

of submodules with $\mathrm{GKdim}(M_i/M_{i+1}) = \mathrm{GKdim}(M)$, it follows $m \leq e$.

Theorem 2.9. *Let R be an $\mathbb{N}^n$–filtered algebra such that $F_{\mathbf{0}}(R) = k$. Assume that the multi-filtration is finite and that $G(R)$ is a semi-commutative $\mathbb{N}^n$-graded algebra. Then R is left and right partitive.*

Proof. By [3, Corollary 2.12], the Gelfand-Kirillov dimension of R is a non-negative integer. Let $M = M_0$ be a nonzero finitely generated left R–module, and consider a descending chain as in (7). Define recursively a locally finite $\mathbb{N}^n$–filtration $\mathcal{F}_i = \{F_\alpha(M_i) \mid \alpha \in \mathbb{N}^n\}$ on M_i for every $i = 1, \ldots, m$ as follows. As the first step, the multi-filtration $\mathcal{F}_0$ can be given by $F_\alpha(M) := F_\alpha(R)U$, for U a finite-dimensional generating subspace of M. Now assume, for some $i \geq 0$, that $F_\alpha(M_i)$ has been defined for every $\alpha \in \mathbb{N}^n$. Define $F_\alpha(M_{i+1}) := M_{i+1} \cap F_\alpha(M_i)$. Analogously, we will define locally finite filtrations $\mathcal{G}_{i+1} = \{F_\alpha(M_i/M_{i+1}) \mid \alpha \in \mathbb{N}^n\}$, for each M_i/M_{i+1}, $0 \leq i < m$. This filtration is given by $F_\alpha(M_i/M_{i+1}) := (F_\alpha(M_i) + M_{i+1})/M_{i+1}$, for every $\alpha \in \mathbb{N}^n$ and for $0 \leq i < m$. For each $i \in \{0, \ldots, m-1\}$ the sequence of associated $\mathbb{N}^n$–graded $G_{\mathcal{F}}(R)$–modules

$$0 \to G_{\mathcal{F}_{i+1}}(M_{i+1}) \to G_{\mathcal{F}_i}(M_i) \to G_{\mathcal{G}_{i+1}}(M_i/M_{i+1}) \to 0$$

is exact. Moreover, since $G_{\mathcal{F}}(R)$ is a noetherian algebra and $G_{\mathcal{F}_0}(M)$ is a finitely generated left $G_{\mathcal{F}(R)}$–module, it follows that $G_{\mathcal{F}_i}(M_i)$ is finitely generated for every $i \in \{0, \ldots, m\}$. By Proposition 2.7, we have that the associated multiplicities satisfy

$$e_{\mathcal{F}_i}(M_i) = e_{\mathcal{F}_{i+1}}(M_{i+1}) + e_{\mathcal{G}_{i+1}}(M_i/M_{i+1})$$

for every $i < m$. Therefore

$$e_{\mathcal{F}_0}(M) = \sum_{i=0}^{m-1} e_{\mathcal{G}_{i+1}}(M_i/M_{i+1}) + e_{\mathcal{F}_m}(M_m)$$

The fact that the multiciplities of nonzero modules are positive integers implies that if we put $e = e_{\mathcal{F}_0}(M)$ then $m \leq e$. □

Remark 2.10. The Krull dimension of every module over a finitely partitive algebra is bounded by its Gelfand-Kirillov dimension, see, for example, [9, Corollary 8.4.9], and this last is an integer when R satisfies the hypotheses of Theorem 2.9.

3. Examples

This section is devoted to providing some non-trivial examples of multi-filtered algebras by finite filtrations having a pointed semi-commutative associated graded algebra.

Definition 3.1. An admissible order $\leq$ on $\mathbb{N}^n$ is called **locally finite** if the interval $[\mathbf{0}, \alpha] = \{\gamma \in \mathbb{N}^n : \gamma \leq \alpha\}$ is finite for every $\alpha \in \mathbb{N}^n$.

A source of examples is given by the following proposition.

Proposition 3.2. [3, Proposition 3.2] *Let R be a k–algebra finitely generated by $x_1, \ldots, x_n$ and set $X^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ for all $\alpha \in \mathbb{N}^n$. If there is a locally finite admissible order '$\leq$' and nonzero scalars $q_{ji} \in K$ such that*

$$x_j x_i = q_{ji} x_i x_j + \sum_{\gamma < \epsilon_i + \epsilon_j} c_\gamma X^\gamma$$

where $\epsilon_i = (0, \ldots, \overset{(i)}{1}, \ldots, 0)$, then

1. *R is endowed with a locally finite multi-filtration given by*

$$F_\alpha(R) = \sum_{\beta \leq \alpha} k X^\beta$$

2. *$G(R)$ is a semi-commutative algebra with locally finite associated filtration. In particular, R is a finitely partitive algebra.*

This proposition applies, [3, Example 3.3], to any iterated Ore extension of the form $R = k[x_1][x_2; \sigma_2, \delta_2] \ldots [x_n; \sigma_n, \delta_n]$, satisfying $\sigma_j(x_i) = q_{ij} x_i$, for every $i < j \leq n$, where q_{ij}'s are nonzero scalars in k. Some examples of these iterated Ore extensions are the algebras $H(p, \lambda)$ defined in [1], which include the quantum coordinate algebras of $M_n(k)$; and the multiparameter quantized Weyl algebra $R = A_n^{Q,\Gamma}(k)$ introduced in [7] (see also [4]). The iterated differential operator algebras of [10] are also covered. It can be also shown [3, Example 3.4] that the quantized universal enveloping algebra $U_q(\mathfrak{sl}(\nu + 1))$, with $q^8 \neq 1$ can be endowed with a finite filtration with pointed semi-commutative associated graded algebra.

4. Filtrations induced by multifiltrations

Let $\mathcal{F}$ be a multifiltration on the k-algebra R. One might ask whether there is an $\mathbb{N}$-filtration $\mathcal{E}$ that can be induced from $\mathcal{F}$ in such a way that the associated graded algebras $G_{\mathcal{E}}(R)$ and $G_{\mathcal{F}}(R)$ would be isomorphic. If so, this would simplify some calculations. There is an obvious way to try to achieve this aim. Let R be an algebra endowed with an $\mathbb{N}^n$-filtration $\mathcal{F} = \{F_\alpha(R) \mid \alpha \in \mathbb{N}^n\}$. Let $T = G_{\mathcal{F}}(R) = \oplus_{\alpha \in \mathbb{N}^n} T_\alpha$ be the associated $\mathbb{N}^n$-graded algebra. For each nonzero $r \in R$, we define $\exp(r) = \min\{\alpha \in \mathbb{N}^n \mid r \in F_\alpha(R)\}$, and put $\mathrm{in}_{\mathcal{F}}(r) = r + V_{\exp(r)}(R)$, the **initial form** of r with respect to $\mathcal{F}$.

For each multi-index $\alpha \in \mathbb{N}^n$, we set $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If we set

$$R_m = \{r \in R \mid \mathrm{in}_{\mathcal{F}}(r) \in \oplus_{|k| \le m} T_k\},$$

for each $m \in \mathbb{N}$, then one might hope that $\{R_m\}$ defines a filtration on R. However, this is not the case in general, the problem being due to possible failure of the implication: $\alpha < \beta \Rightarrow |\alpha| \le |\beta|$, as the following example shows.

Consider the ordering, $<$, on $\mathbb{N}^2$, given by

$$(\alpha_1, \alpha_2) < (\beta_1, \beta_2) \Leftrightarrow \begin{cases} \alpha_1 + 2\alpha_2 < \beta_1 + 2\beta_2 \quad \text{or} \\ \alpha_1 + 2\alpha_2 = \beta_1 + 2\beta_2 \text{ and } \alpha_2 < \beta_2 \end{cases}.$$

Then $(3,1) < (0,3)$, while $|(3,1)| > |(0,3)|$.

A **weighted admissible order** on $\mathbb{N}^n$, is defined as follows, see, for example, [3]. Let $\mathbf{w} = (w_1, \ldots, w_n)$ be a vector in $\mathbb{R}^n$ with $w_i > 0$ for every $i = 1, \ldots, p$. Consider the usual dot product $\langle -, - \rangle$ on $\mathbb{R}^n$. By $<_{lex}$ we will denote the lexicographic order in $\mathbb{N}^n$ with $\epsilon_1 < \epsilon_2 < \cdots < \epsilon_n$. The binary relation in $\mathbb{N}^n$ defined for $\alpha, \beta \in \mathbb{N}^n$ as $\alpha <_{\mathbf{w}} \beta$ if $\langle \alpha, \mathbf{w} \rangle < \langle \beta, \mathbf{w} \rangle$ or $\langle \alpha, \mathbf{w} \rangle = \langle \beta, \mathbf{w} \rangle$ and $\alpha <_{lex} \beta$ is a locally finite admissible order in $\mathbb{N}^n$. We will say that $\mathbf{w}$ is the **weight vector** for the order $<_{\mathbf{w}}$.

Note that, in the situation of Proposition 3.2, locally finite admissible orders lead to locally finite multifiltrations.

Assume that we have a weighted ordering $\le_{\mathbf{w}}$ on $\mathbb{N}^n$, with $\mathbf{w} \in \mathbb{N}^n$, and let R be an algebra endowed with an $\mathbb{N}^n$-filtration $\mathcal{F} = \{F_\alpha(R) \mid \alpha \in \mathbb{N}^n\}$. Let $T = G_{\mathcal{F}}(R) = \oplus_{\alpha \in \mathbb{N}^n} T_\alpha$ be the associated $\mathbb{N}^n$-graded algebra. For each $m \in \mathbb{N}$, define $T_m = \oplus_{|\alpha|_{\mathbf{w}} = m} T_\alpha$, where $|\alpha|_{\mathbf{w}} = \langle \mathbf{w}, \alpha \rangle = \sum_i w_i \alpha_i$. Then $T = \oplus_{m \in \mathbb{N}} T_m$ is an $\mathbb{N}$-graded algebra. In general, $\exp(rs) \le_{\mathbf{w}} \exp(r) + \exp(s)$ and, in fact,

$$\exp(rs) <_{\mathbf{w}} \exp(r) + \exp(s) \Leftrightarrow rs \in V_{\exp(r)+\exp(s)},$$

for each $r, s \in R$. Observe that

$$|\exp(rs)|_{\mathbf{w}} \le_{\mathbf{w}} |\exp(r) + \exp(s)|_{\mathbf{w}} = |\exp(r)|_{\mathbf{w}} + |\exp(s)|_{\mathbf{w}}.$$

Now, we use the multifiltration $\mathcal{F}$ to induce an $\mathbb{N}$-filtration on R. If we set

$$R_m = \{r \in R \mid \mathrm{in}_{\mathcal{F}}(r) \in \oplus_{k \le m} T_k\} = \{r \in R \mid |\exp(r)|_{\mathbf{w}} \le m\},$$

for each $m \in \mathbb{N}$, then $\{R_m\}_{m\in\mathbb{N}}$ is a filtration on R, by virtue of the second description in the display above.

Denote by $S = \oplus_{m\in\mathbb{N}} S_m$ the associated graded algebra for this filtration; that is,

$$S_m = \frac{R_m}{R_{m-1}}.$$

Thus, for each $m \in \mathbb{N}$, we have

$$T_m = \bigoplus_{|\alpha|=m} \frac{F_\alpha}{V_\alpha}, \qquad \text{and} \qquad S_m = \frac{\bigcup_{|\alpha|_{\mathbf{w}}\le m} F_\alpha(R)}{\bigcup_{|\alpha|_{\mathbf{w}}\le m-1} F_\alpha(R)}.$$

Unfortunately, even in this setting, the new associated graded algebra S may fail to be semi-commutative, even if the associated multigraded algebra T is commutative. The following example shows that the graded algebras S and T are not necessarily isomorphic and illustrates the above point.

Example 4.1. Let R be the algebra over the field k generated by indeterminates X, Y, subject to the relation $XY = YX + X^2$. Thus, R is the co-ordinate ring of the Jordan plane. Consider the ordering, $\le_{grlex}$, on $\mathbb{N}^2$, given by

$$(\alpha_1, \alpha_2) \le_{grlex} (\beta_1, \beta_2) \Leftrightarrow \begin{cases} \alpha_1 + \alpha_2 < \beta_1 + \beta_2 \quad \text{or} \\ \alpha_1 + \alpha_2 = \beta_1 + \beta_2 \text{ and } \alpha_2 \le \beta_2 \end{cases}.$$

Thus, the ordering begins

$$(0,0) <_{grlex} (1,0) <_{grlex} (0,1) <_{grlex} (2,0) <_{grlex} (1,1) <_{grlex} (0,2) <_{grlex} \cdots.$$

In fact, the ordering $\le_{grlex}$ is the same as the ordering deteermined by the weighting $(1,1)$. The algebra R is $\mathbb{N}^2$-filtered by

$$F_\alpha(R) = \sum_{\beta \le_{grlex} \alpha} kx^{\beta_1} y^{\beta_2}$$

and $T = G_{\mathcal{F}}(R)$ is commutative; in fact, T is just the polynomial algebra in two variables over k.

However, we will show that the ring S is not commutative. Note that $S_0 = k$, while

$$S_1 = \frac{F_{(0,1)}(R)}{k} = \frac{k + kX + kY}{k}$$

and

$$S_2 = \frac{F_{(0,2)}(R)}{F_{(0,1)}(R)} = \frac{k + kX + kY + kX^2 + kXY + kY^2}{k + kX + kY}.$$

Set $x = X + S_0 \in S_1$ and $y = Y + S_0 \in S_1$. Then $yx = YX + S_1 = (XY + X^2) + S_1 = xy + x^2$, and $x^2 \neq 0$. Thus, S is not commutative. In fact, one can check that $S \cong R$.

In connection with this example, it is interesting to note that even beginning with a graded algebra (R is graded, since the defining relation is homogeneous), a multifiltration can be used to produce an associated graded algebra with nicer properties.

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