Dimension Theory of Noetherian Rings

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Introduction

This article is a report of three lectures given on "Dimension Theory of Noetherian Rings", at the meeting "Infinite Length Modules", held in Bielefeld, from 7th to 11th September 1998. I would like to thank the organisers for the opportunity to present these lectures, and for the invitation to write up the lectures for this volume. I was asked to present an introduction to the uses of dimension theory in noetherian rings for an audience of algebraists who were not specialists in the area of noetherian rings. In view of this, the material presented here is not original work, nor is it necessarily the most important work in this area. Rather, the results and examples were chosen to illustrate typical uses of dimension theory. Since most of the audience consisted of experts in the Representation theory of Artin rings and finite dimensional algebras, I decided to concentrate on two dimension functions, Krull dimension and Gelfand-Kirillov dimension, which are generalisations of the notions of artinian and finite dimensional, respectively.

1 Noetherian rings and modules

We start with a reminder of some of the basic definitions and results in the area of noetherian ring theory. A good reference for most of the notions introduced is the book "Noncommutative Noetherian Rings", by J C McConnell and J C Robson, [**?**].

Definition A ring R is **right noetherian** if the following three equivalent conditions hold.

1. Each right ideal of R is finitely generated as a right module over R.

2. For each ascending chain of right ideals

$$I_1 \subseteq I_2 \subseteq \dots I_n \subseteq \dots \subseteq R$$

   there exists an integer N such that $I_N = I_{N+i}$, for each $i \geq 0$.

3. Each nonempty set of right ideals of R contains a maximal element.

The ring R is **noetherian** if R is both left and right noetherian.

There is a similar definition for the idea of a noetherian module, and it is easy to check that a finitely generated right module over a right noetherian ring is a right noetherian module.

In commutative ring theory, the notion of a prime ideal is one of the key ideas. In the noncommutative theory, it is important, but has to be defined in terms of ideals rather than elements, since otherwise there would be too few prime ideals in an arbitrary ring to have much influence on the structure of rings.

An ideal P of R is a **prime** ideal of R if, whenever A, B are ideals of R with $AB \subseteq P$ then either $A \subseteq P$ or $B \subseteq P$.

A ring R is a **prime ring** if 0 is a prime ideal.

Example The ring of $n \times n$ matrices, for $n > 1$, over a field or commutative integral domain, is a prime ring that is not a domain.

The cornerstone of the whole theory of noetherian rings is Goldie's theorem which describes the structure of noetherian prime rings in terms of the existence of rings of fractions, and links the noetherian theory to artinian theory via the Artin-Wedderburn theorem. An artinian ring that is prime is semisimple artinian. The key point of departure from the theory of semisimple artinian rings is the realization that although it is not true that, in a noetherian prime ring, every right ideal has a complement direct summand;

nevertheless, introducing the notion of an essential extension is enough to deal with this deficiency.

A submodule N of a module M is said to be an **essential submodule** of M if and only if $N \cap L$ is nonzero whenever L is a nonzero submodule of M; in this situation, M is also said to be an **essential extension** of N.

Specializing to the ring R, we obtain the notion of an essential right ideal: a right ideal E of R is an **essential right ideal** of R if and only if $E \cap I \neq 0$, whenever I is a nonzero right ideal of R.

Goldie's Theorem A prime noetherian ring R has a ring of fractions $Q = Q(R)$ which is a simple artinian ring. Consequently, invoking the Artin-Wedderburn, R has a ring of fractions that is a full matrix ring over a division ring.

The key point of Goldie's Theorem is that a right ideal E of R is essential if and only if E contains a non-zerodivisor c of R. Nonzero divisors are often called **regular** elements.

Exercise Show that every nonzero two-sided ideal of a prime noetherian ring is essential as a right ideal (and hence contains a regular element).

A module M for which every nonzero submodule is essential is called a **uniform** module. Thus, in a uniform module, the intersection of any two nonzero submodules is nonzero. It is easy to check that any noetherian module M contains a uniform submodule, and in fact, there is a finite direct sum of uniform submodules $U_1 \oplus \ldots \oplus U_n$ which is an essential submodule of M. The number of summands n in such a direct sum of uniform submodules is an invariant of the module called the **uniform dimension** of M, or, alternatively, the **Goldie rank** of M, denoted by $\text{u.dim}(R)$. If R is a prime noetherian ring with simple artinian quotient ring $Q = M_n(D)$, for some division ring D, then $\text{u.dim}(R_R) = \text{u.dim}(Q_Q) = n$.

Goldie's Theorem admits an extension to the case of semiprime noetherian rings. A ring R is **semiprime** if it has no nonzero nilpotent ideals. If R is a noetherian semiprime ring then there are a finite number of minimal prime ideals, $P_1, \ldots, P_n$, such that $P_1 \cap \ldots \cap P_n = 0$.

Goldie's Theorem for semiprime rings A semiprime noetherian ring R has a ring of fractions $Q = Q(R)$ which is a semisimple artinian ring.

Example The **Weyl Algebra** is the algebra A generated over a field k by two indeterminates x, y satisfying the commutation relation $xy - yx = 1$.

The Weyl Algebra is a noetherian domain. If the field k has characteristic zero then the Weyl Algebra is a simple ring; that is, the only two-sided ideals are 0 and R. However, the Weyl Algebra is not a division ring.

In characteristic zero, the Weyl Algebra has no nonzero finite dimensional representations. Otherwise, we would have matrices A and B satisfying the relation $AB - BA = I$, and a trace argument gives a contradiction (the left hand side has trace zero, the right hand side has trace n if we are in $n \times n$ matrices).

Exercise Prove that the first Weyl algebra, A, over a field of characteristic zero, is simple (consider $[x, -], [y, -]$, to see that the ideal generated by any nonzero element contains a unit).

Here is an interesting result about modules over noetherian nonartinian rings that has no counterpart in the artinian theory, and which has far-reaching consequences in the noetherian theory.

Eisenbud-Robson Let M be a module of finite length over a simple noetherian ring R that is not artinian. Then M is cyclic.

Sketch of Proof We argue by induction on the length of the module. If M has length one then M is simple, and certainly cyclic. Suppose that M has length n and that modules of length less than n over R are cyclic. Choose a simple submodule B of M, and any $b \in B$. Then $B = bR$, and $M/bR = \bar{a}R$ for some $a \in M$, by the inductive hypothesis. Thus, $M = aR + bR$, with bR simple. Now, $aR \cong R/\mathrm{Ann}(a)$, so $\mathrm{Ann}(a) \neq 0$, since R is not artinian. Choose $0 \neq x$ with $ax = 0$.

Now,

$$bRx = 0 \Rightarrow bRxR = 0 \Rightarrow bR = 0 \Rightarrow b = 0,$$

a contradiction; so $brx \neq 0$, for some r.

Then

$$(a + br)R \supseteq (a + br)xR = brxR = bR$$

and

$$a = (a + br) - br \in (a + br)R,$$

and so $M = (a + br)R$. $\square$

2 Dimension functions

In commutative noetherian ring theory, one of the most powerful dimensions is the **Krull dimension** which measures the longest possible chain of prime ideals in the ring. In the case of Krull dimension zero, the ring is artinian, and if the ring is a finitely generated algebra over a field k then it is finite dimensional as a k vector space, see, for example, [**?**, 13.10.3].

We will try to find dimension functions for noncommutative rings that will replace the commutative Krull dimension. The obvious generalization is the **classical Krull dimension**, which again just measures chains of prime ideals. This works when there are lots of two-sided ideals (eg. for polynomial identity rings, cf. [**?**, 6.4.8 and Chapter 13]), but would give every simple ring dimension zero, and this is not much use!

We will ask for dimensions defined for noncommutative noetherian rings and modules, which correspond to the commutative setting in dimension zero, and are sufficiently delicate to enable progress to be made. First, what kinds of properties do we want?

A **dimension function**, d, defined on noetherian modules over noetherian rings, assigns a value $d(M_R)$ to each noetherian R-module M. Assume that $d(M)$ is either -1 (when $M = 0$) or in $\mathbb{R}^{\geq 0} \cup \infty$, or an ordinal number.

Let us try to explore the structure of noetherian rings by using a putative dimension function d, and see what desireable properties we would like to hold for the function d. We will use the following general strategy as a model for our attack. In a noetherian ring, the nil radical N is nilpotent, the ring R/N is semiprime (that is has no nilpotent ideals) and R/N has a semisimple artinian ring of quotients, by Goldie's Theorem. In a general problem, one hopes to be able to factor out N and then solve the problem in the resulting semiprime ring R by exploiting the existence of a semisimple artinian quotient ring. We will try to follow a similar strategy using d. Right ideals with dimension less than the dimension of R will be considered small; we factor out such right ideals and are then in a ring with all right ideals having the same dimension as that of the ring. We hope to be able to say something about the structure of such rings (for example, that they have rings of fractions). However, for all this to work, we need d to have certain desireable properties.

The following are some desireable properties that we might hope that the function d satisfies.

Suppose that $d(R) = \alpha$. Look at the sum I of all right ideals E of R with $d(E) < \alpha$. We would like $d(I) < \alpha$, and that each nonzero right ideal A of R/I has $d(A) = \alpha$. This will hold, provided that d is **exact** in the sense given below.

Now assume we have factored out this ideal I so that each nonzero right ideal A of R has $d(A) = d(R) = \alpha$. Can we prove that R has a ring of quotients? Let c be an element of R. Suppose that $d(R/cR) < \alpha$. If $xc = 0$ then we have R/cR maps onto xR and so $d(R/xR) < \alpha$, forcing $xR = 0$, and so $x = 0$, and the left annihilator of c is zero. With a suitable symmetry hypothesis we would also know that the right annihilator is zero, so that c is a regular element. Thus, we will have that $d(R/cR) < \alpha$ implies that c is regular. The torsion property defined below is sufficient to give the converse to this. We then have a way of passing between "large" right ideals I (ie. $d(R/I) < d(R)$) and regular elements. The strategy of proof of Goldie's Theorem can then be used to produce a ring of fractions.

A dimension function d is **exact** if whenever

$$0 \to A \to B \to C \to 0$$

is a short exact sequence of modules then

$$d(B) = \max\{d(A), d(C)\}.$$

Exactness ensures that the dimension of submodules and quotient modules are no bigger than the original modules, and enables arguments to be constructed by passing from a module to suitable submodules or quotients.

A dimension function d is **finitely partitive** if for each finitely generated module M there is an integer $n = n(M)$ such that for every chain

$$M \supseteq M_1 \supseteq M_2 \supseteq \ldots \supseteq M_m$$

with $d(M_i/M_{i+1}) = d(M)$, one has $m \leq n$.

Partitivity enables us to use artinian type techniques in proofs.

A dimension function d has the **torsion property** if whenever c is a regular element of R then $d(R/cR) + 1 \leq d(R)$.

A dimension function d **separates primes** if whenever $P \subsetneqq Q$ are prime ideals of R then

$$d(R/Q) + 1 \leq d(R/P).$$

Prime separation follows from the torsion property. One use of these properties is to establish isomorphisms. For example, suppose that R and S are noetherian domains and we have an epimorphism θ from R onto S which we suspect should be an isomorphism. If θ is not an isomorphism then $d(S) = d(R/\ker(\theta)) < d(R)$, and conversely; so if we can show R and S have the same dimension, then the isomorphism is established.

A dimension function d is **symmetric** if whenever A and B are noetherian rings in the class of rings under consideration and M is an A-$B-$bimodule that is finitely generated on each side. Then

$$d({}_AM) = d(M_B).$$

Many proofs in noncommutative rings require one to pass from left to right and vice-versa. Symmetry retains control of dimensions in this situation. In fact, in this discussion, we only need a weak symmetry condition on subfactors of ${}_RR_R$.

Suppose that d is an exact, symmetric dimension function on a class noetherian rings and assume the torsion property. Let $d(R) = \alpha$. By exactness, there is a unique maximal ideal A with $d(A) < d(R)$, and in R/A every nonzero right ideal has dimension equal to α. Such a ring is said to be **α-homogeneous**. By symmetry, each nonzero left ideal has dimension equal to α.

A further technical condition is needed to prove the existence of a ring of quotients. A dimension function d is **ideal invariant** if whenever M is a noetherian module and T is a two-sided ideal of R then $d(M \otimes T) \leq d(M)$; in particular, if I is a right ideal of R, then ideal invariance implies that $d(T/IT) \leq d(R/I)$.

Theorem Suppose that d is an exact, symmetric, ideal invariant dimension function on a class noetherian rings, and that R is α-homogeneous. Assume the torsion property. Then R has an artinian ring of quotients.

Sketch of Proof By Small's Theorem, [**?**, Corollary 4.1.4], it is enough to show that an element c that is a regular element modulo the nilpotent radical N of R is also a regular element in R. Let c be such an element.

By exactness, $d(R/N) = d(R) = \alpha$, and by the torsion property, $d(R/cR + N) < \alpha$. We aim to show that $d(R/cR) < \alpha$. If $N^m = 0$ then $R/cR = R/(cR + N^m)$, so it is enough to show that $d(R/(cR + N^s) < \alpha$, for each s. We prove this by induction, the case $s = 1$ being given above. Thus, assume that $d(R/(cR + N^{s-1}) < \alpha$. By exactness, it is enough to show that $d((cR + N)/(cR + N^s) < \alpha$. However, $(cR + N)/(cR + N^s) \cong N/(N \cap (cR + N^s))$ is a homomorphic image of $N/(cR + N^s) = N/(cR + N^{s-1})N$ and this latter module has Krull dimension less than α, by the inductive hypothesis and the fact that N is ideal invariant. Thus, $d((cR + N)/(cR + N^s)) < \alpha$, and $d(R/(cR + N^s)) < \alpha$, as required.

Now suppose that $xc = 0$. Then xR is a homomorphic image of R/cR; and so $d(xR) \leq d(R/cR) < \alpha$. Hence, $x = 0$, since R is homogeneous.

By a symmetric argument, $cx = 0 \Rightarrow x = 0$; so c is a regular element. Hence, R has an artinian ring of quotients, by Small's Theorem. □

3 Krull dimension

The following are suitable references for the basic material on noncommutative Krull dimension:
R Rentschler and P Gabriel, *Sur la dimension des anneaux et ensembles ordonnés*, [**?**];
R Gordon and J C Robson, *Krull dimension*, [**?**];
J C McConnell and J C Robson, *Noncommutative noetherian rings*, [**?**].

Krull dimension is a measure of how far a module deviates from being artinian. The definition is inductive, starting with artinian modules as the base case.

Krull dimension of a noetherian module.

Kdim(0) $:= -1(\text{or } -\infty)$

Kdim$(M) := 0$ if and only if M is artinian.

Kdim$(M) := 1$ if and only if in any chain

$$M \supseteq M_1 \supseteq M_2 \supseteq \dots$$

almost all factors have Kdim$(M_i/M_{i+1}) < 1$, we then continue inductively for all ordinals.

In general, we define $\mathrm{Kdim}(M) = \alpha$ if and only if $\mathrm{Kdim}(M) \not< \alpha$ but in any chain

$$M \supseteq M_1 \supseteq M_2 \supseteq \dots$$

all but finitely many factors have Krull dimension less than α.

An alternative way to look at this definition is via **quotient categories**: Let $\mathbb{A}$ be the category of noetherian R-modules, and $\mathbb{C}$ the subcategory of all modules with Krull dimension less than α. Let T be the natural map $\mathbb{A} \to \mathbb{A}/\mathbb{C}$. Then $\mathrm{Kdim}(M) = \alpha$ if and only if $T(A)$ is an artinian object of $\mathbb{A}/\mathbb{C}$. For a discussion of this approach, see [**?**].

Example Let G be a finitely generated abelian group (that is, a $\mathbb{Z}$-module), say,

$$G \cong \text{Finite} \oplus \{\mathbb{Z}^n\}.$$

In this case, the **torsionfree rank**, $\mathrm{Rank}(G)$, is n. Note that $\mathrm{Kdim}(G) = 0$ iff G artinian iff G is finite iff $\mathrm{Rank}(G) = 0$.

If $H \leq G$ then

$$\mathrm{Rank}(G) = \mathrm{Rank}(H) + \mathrm{Rank}(G/H);$$

and so if

$$G \geq G_1 \geq G_2 \geq \dots$$

is a descending chain of subgroups of G then at most n factors G_i/G_{i+1} can be infinite, so all but finitely many factors are finite; that is, have Krull dimension less than 1. $\square$

Elementary properties of Krull dimension

- $\mathrm{Kdim}(R) := \mathrm{Kdim}(R_R)$
- Every noetherian module has a Krull dimension
- If $N \subseteq M$ then

$$\mathrm{Kdim}(M) = \max\{\mathrm{Kdim}(N), \mathrm{Kdim}(M/N)\}$$

- If c is a regular element of R then

$$\mathrm{Kdim}(R/cR) + 1 \leq \mathrm{Kdim}(R)$$

- $\mathrm{Kdim}(R[x]) = \mathrm{Kdim}(R) + 1$

The proofs of these basic properties can be found in [**?**, Chapter 6]. In the next example, we need a generalization of the last property in the list to the case of skew polynomial rings: if δ is a derivation of the ring R then $\mathrm{Kdim}(R) \leq \mathrm{Kdim}(R[x;\delta]) \leq \mathrm{Kdim}(R)+1$, while if R is artinian then $\mathrm{Kdim}(R[x;\delta]) = 1$. [**?**, 6.5.4]. In general, the behaviour of Krull dimension under extensions of rings is very difficult to calculate; see the paper by Vladimir Bavula, [**?**], in these proceedings for more information about this.

Example Let $A = k[x,y]$, with $xy - yx = 1$ be the first Weyl algebra over a field of characteristic zero. Then $\mathrm{Kdim}(A) = 1$.

Sketch of proof. The Weyl algebra is not artinian; so $\mathrm{Kdim}(A) \geq 1$. The sets $k[x]\backslash 0$ and $k[y]\backslash 0$ are Ore sets in A, and so the localizations $B = k(x)[y]$ and $C = k(y)[x]$ can be constructed. Both of these rings are skew polynomial algebras, for example, $k(x)[y] = k(x)[y;\delta]$, where $\delta(f) = df/dx$, for $f \in k(x)$. Set $R := B \oplus C$. Note that R is a flat extension of A, since B and C are both localizations of A. Since Krull dimension is defined on lattices of right ideals, and since right ideals in a localization are extensions of right ideals in the original ring, it is obvious that $\mathrm{Kdim}(A) \geq \max\{\mathrm{Kdim}(B), \mathrm{Kdim}(C)\}$. It is easy to establish that $\mathrm{Kdim}(R) = \max\{\mathrm{Kdim}(B), \mathrm{Kdim}(C)\} = 1$.

Consider $A \hookrightarrow R$ by $a \mapsto a \oplus a$. We will show that R is a faithfully flat extension of A. Since this then gives an injection of the lattice of right ideals of A into the lattice of right ideals of R, we deduce that $\mathrm{Kdim}(A) \leq \mathrm{Kdim}(R) = 1$. In order to establish faithful flatness, it is enough to show that if I is a right ideal of A with $IR = R$ then $I = A$, [**?**, 7.2.3].

Suppose that I is a right ideal of A such that $IR = R$. Then $IB = B$, and $IC = C$, and so there exist $0 \neq f(x) \in I$ and $0 \neq g(y) \in I$, and it follows that A/I is finite dimensional over k, a contradiction unless $I = A$.

Hence, R is a faithfully flat extension of A and so $\mathrm{Kdim}(A) \leq \mathrm{Kdim}(R) = 1$.

$\square$

A result that applies to right ideals of the Weyl algebra can now be proved.

Theorem If R is a simple noetherian ring, and $\mathrm{Kdim}(R) = 1$, then each right ideal of R can be generated by 2 elements.

Sketch of proof. Assume R is a domain, to simplify the argument. Let I be a nonzero right ideal of R, and choose $0 \neq c \in I$. Then

$$\mathrm{Kdim}(I/cR) \leq \mathrm{Kdim}(R/cR) < \mathrm{Kdim}(R) = 1,$$

so that either $\mathrm{Kdim}(I/cR) = -1$ and $I = cR$, or $\mathrm{Kdim}(I/cR) = 0$.

In the latter case, I/cR is an artinian module over a simple nonartinian ring, and so is cylic, generated by the coset $b + cR$, say. Hence, $I = bR + cR$. $\square$

Corollary Every right ideal of the Weyl algebra can be generated by 2 elements.

In fact, this is usually known as a $1\frac{1}{2}$**-generator** result, since there is a free choice of the first generator, among the regular elements in the right ideal.

If M is a noetherian module with the property that $\mathrm{Kdim}(M/N) < \mathrm{Kdim}(M)$, for each nonzero submodule N of M then M is said to be a **critical module**. Critical modules are generalizations of simple modules.

It is not difficult to establish that every noetherian module contains a critical submodule.

If M is a noetherian module with $\mathrm{Kdim}(M) = \alpha$, then there is an integer n, called the α**-length** of M such that in any chain,

$$M \supseteq M_1 \supseteq M_2 \supseteq \dots$$

at most n factors can have $\mathrm{Kdim}(M_i/M_{i+1}) = \alpha$.

A general strategy for proving results in noetherian rings, using Krull dimension is to use artinian type methods, arguing on the α-length of a module, and then to replace simple modules by critical modules.

Example Levitski's Theorem states that a nil right ideal in a noetherian ring is nilpotent. Here is a proof of this fundamental result, using an artinian perspective and Krull dimension techniques.

Sketch of proof. Note that if I is a nil right ideal that is not nilpotent and N is a nilpotent ideal, then $(I + N)/N$ is a nonzero nil right ideal of R/N. By using this observation, we may first reduce to the case that R has no nilpotent ideals, and then we need to show that any nil right ideal is equal to 0. Suppose not, and choose a nonzero nil right ideal I with $\alpha = \mathrm{Kdim}(I)$ as small as possible. Since I contains a critical right ideal which will also be nil and have Krull dimension α, by the minimality of α, we may assume that I itself is critical. If $I^2 = 0$, then $(RI)^2 = 0$, and we have a nonzero nilpotent ideal, a contradiction. Thus, $I^2 \neq 0$ and we may choose $0 \neq c \in I$, with $cI \neq 0$.

Consider the homomorphism

$$\theta : I \to I, \qquad i \mapsto ci.$$

Note that $\ker(\theta) \neq 0$, since $c^n = 0$, for some n.

Thus,

$$\mathrm{Kdim}(cI) = \mathrm{Kdim}(I/\ker(\theta)) < \alpha,$$

since I is critical; and so $cI = 0$, by the minimality of α, a contradiction. $\square$

A famous result of Stafford starts with generalization to arbitrary noetherian modules, using Krull dimension, of the Eisenbud-Robson result for modules of finite length given earlier.

In order to prove Stafford's result, we use the idea of **noetherian induction**. Suppose one wants to prove that all modules in some class $\mathcal{C}$, which is closed under taking factors, have a certain property P. Let M be a module in $\mathcal{C}$. If M does not have property P, then the set $\mathcal{S}$ of submodules N of M such that M/N does not have property P is not empty, since $0 \in \mathcal{S}$. By the noetherian property, we may choose a maximal member $N \in \mathcal{S}$. Then M/N does not have property P, but M/A has property P for all submodules A of M which properly contain N. Thus, M/N does not have property P, but all of its proper factors do have property P. In using noetherian induction, we replace the original M by M/N, so that we may assume that M does not have the desired property P, but each of its proper factors does have property P. We then aim to produce a contradiction.

Lemma Let M be a nonzero noetherian module over a simple noetherian ring R, and suppose that $\mathrm{Kdim}(M) < \mathrm{Kdim}(R)$. Then, there exists $d \in M$ such that $\mathrm{Kdim}(M/dR) < \mathrm{Kdim}(M)$.

Sketch Proof By noetherian induction, we may assume that the result holds in all proper factors of M. Let B be any critical submodule of M and choose any $0 \neq b \in B$. Then $0 \neq bR$ is also critical, and by noetherian induction, the result holds in $\bar{M} = M/bR$. Thus, there is an element $a \in M$ such that $\mathrm{Kdim}(\bar{M}/\bar{a}R) < \mathrm{Kdim}(\bar{M}) \leq \mathrm{Kdim}(M)$. It follows that

$$\mathrm{Kdim}(M/aR + bR) < \mathrm{Kdim}(M)$$

with bR critical.

Suppose that we can find a submodule dR of $aR+bR$ such that $\mathrm{Kdim}(aR+bR/dR) < \mathrm{Kdim}(M)$. Then $\mathrm{Kdim}(M/dR) < \mathrm{Kdim}(M)$, since $\mathrm{Kdim}(M/aR+bR) < \mathrm{Kdim}(M)$. Thus, we may assume that $M = aR+bR$.

Note that

$$\begin{aligned}\mathrm{Kdim}(R/\mathrm{Ann}(a)) &= \mathrm{Kdim}(aR)\\ &\leq \mathrm{Kdim}(M) < \mathrm{Kdim}(R);\end{aligned}$$

and so $\mathrm{Ann}(a) \neq 0$. Choose $0 \neq x$ with $ax = 0$.

As in the Eisenbud-Robson argument,

$$bRx = 0 \Rightarrow bRxR = 0 \Rightarrow bR = 0 \Rightarrow b = 0,$$

a contradiction; so $brx \neq 0$, for some r.

Set $d = a + br$. Then $a = d - br \in dR + bR$ and $b \in dR + bR$, so that $M = aR+bR \subseteq dR+bR$ and it follows that $M = dR+bR$. The element $dx = (a+br)x = brx \neq 0$ is in $dR \cap bR$, and so this submodule is nonzero. Thus, $\mathrm{Kdim}(bR/(dR \cap bR)) < \mathrm{Kdim}(bR) \leq \mathrm{Kdim}(M)$, since bR is critical. Hence, $\mathrm{Kdim}(M/dR) = \mathrm{Kdim}(dR+bR/dR) = \mathrm{Kdim}(bR/(dR \cap bR)) < \mathrm{Kdim}(M)$, as required.

Theorem (Stafford) Let R be a simple noetherian ring with $\mathrm{Kdim}(R) = n$. Then every right ideal of R can be generated by $n+1$ elements.

Sketch of proof. First, we show that it is enough to establish the result for essential right ideals. Suppose that I is a right ideal that is not essential. Then choose a nonzero right ideal J maximal such that $I \cap J = 0$. It is easy to check that $E := I \oplus J$ is essential right ideal, and any generating set for E projects onto a generating set for I.

Thus, we assume that I is an essential right ideal. Choose a regular element $c \in I$ and set $M = I/cR$. Note that

$$\mathrm{Kdim}(M) \leq \mathrm{Kdim}(R/cR) < \mathrm{Kdim}(R).$$

Now apply the previous result to M reducing Krull dimension by at least one each time we factor out an element. $\square$

Notice that in Eisenbud-Robson, and Stafford's result above, the generator $(a+br)R$ arises in a "stable" way from $aR+bR$. By developing these ideas much further, Stafford obtains noncommutative versions of Serre's theorem on free summands, and Bass cancellation:

SERRE'S THEOREM If M is a 'big' module, then M has a free summand.

BASS CANCELLATION If M is a 'big' module then

$$M \oplus R \cong N \oplus R \Rightarrow M \cong N.$$

In each case, 'big' is specified in terms of Krull dimension. Exact statements of these results can be found in [**?**, 11.7.13].

4 Gelfand-Kirillov dimension

Suitable references for Gelfand-Kirillov dimension are:

W Borho and H-P Kraft, *Über die Gelfand-Kirillov Dimension*, [**?**];
G R Krause and T H Lenagan, *Growth of algebras and Gelfand-Kirillov dimension*, [**?**];
J C McConnell and J C Robson, *Noncommutative noetherian rings*, [**?**, Chapter 8];
V A Ufnarovskij, *Combinatorial and asymptotic methods in algebra*, [**?**].

Let A be a finitely generated k-algebra and let V be a finite dimensional k-subspace that contains 1 and a set of algebra generators of A.

Then
$$k = V^0 \subseteq V \subseteq V^2 \subseteq V^3 \subseteq \ldots \subseteq \bigcup V^n = A.$$

The **growth function of A (relative to V)** is the function F given by

$$F(n) := \dim(V^n).$$

It is sometimes easier to calculate the first difference function $f(n)$, where

$$\begin{aligned} f(n) &:= F(n) - F(n-1) \\ &= \dim(V^n) - \dim(V^{n-1}) \\ &= \dim(V^n/V^{n-1}). \end{aligned}$$

(i) The algebra A is said to have **polynomially bounded growth** if there is a constant d such that $F(n) \leq n^d$, for $n \gg 0$.

(ii) The algebra A is said to have **exponential growth** if there is a constant $c > 1$ such that $F(n) \geq c^n$, for $n \gg 0$.

(iii) Otherwise, A is said to have **intermediate growth**.

Example $R = k[x_1, \ldots, x_d]$ a commutative **polynomial algebra**.

$$f(n) = \# \text{ distinct } x_1^{a_1} x_2^{a_2} \ldots x_d^{a_d}$$

with $\sum a_i = n$.

Thus

$$f(n) = \binom{n+d-1}{d-1},$$

which is a polynomial of degree $d-1$, and so

$$F(n) = f(0) + f(1) + \ldots + f(n) = \binom{n+d}{d}$$

is a polynomial of degree d, and the polynomial algebra has **polynomial growth**! □

Exercise Prove the statement above.

Example The free algebra $R = k\{x, y\}$

$f(n)$ is the number of words of length n, so

$$f(n) = 2^n.$$

The free algebra has **exponential growth**. □

Example Let $\mathfrak{g}$ be the infinite dimensional Lie algebra with basis $e_1, e_2, \ldots$ and relations

$$[e_i, e_j] = (i-j)e_{i+j}.$$

Let $\mathcal{U}$ be the universal enveloping algebra of $\mathfrak{g}$; so $\mathcal{U} = k[e_1, e_2, \ldots]$ with relations

$$e_i e_j - e_j e_i = (i-j)e_{i+j}.$$

Then, $\mathcal{U} = k[e_1, e_2]$, and, in fact, $\mathcal{U}$ is finitely presented, [**?**].

To calculate $f(n)$ we need to find

$$p(n) = \# \text{ partitions of } n$$

and

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$$

so the growth is **intermediate**. $\square$

(The methods we present here will not deal with algebras of intermediate growth. However, Petrogradsky, [**?**, **?**], has recently introduced a family of functions designed to measure different types of subexponential growth.)

The distinction between subexponential and exponential growth is important, as indicated by the following structure result for domains with subexponential growth.

Proposition (Jategaonkar) If R is a domain with subexponential growth then R has a division ring of fractions.

Sketch of Proof. The crucial point concerning the existence of the division ring of quotients is to see that left fractions can be re-written as right fractions; this is the content of the Ore condition for the existence of a ring of fractions.

Given $a, c \in R$, we need to be able to find $b, d \in R$ such that $c^{-1}a = bd^{-1}$, or, $ad = cb$ $(\neq 0)$; ie.

$$aR \cap cR \neq 0.$$

Now, $aR \cap cR = 0$ implies $k\{a, c\}$ is a free algebra, and so $k\{a, c\}$ has exponential growth and hence, so does R, a contradiction.$\square$

Exercise Prove $aR \cap cR = 0$ implies $k\{a, c\}$ is a free algebra.

In case (i), we define the **Gelfand-Kirillov dimension** of the algebra R by

$$\mathrm{GKdim}(R) = \overline{\lim} \left\{ \frac{\log(\dim(V^n))}{\log(n)} \right\}$$

where V is a finite dimensional subspace of R that contains 1 and a set of algebra generators of R.

Exercise Prove that the definition is independent of the choice of V, using the fact that if W is another such subspace then there are integers s, t with $V \subseteq W^s \subseteq V^t$.

An alternative definition of **Gelfand-Kirillov dimension** is

$$\mathrm{GKdim}(R) := \inf\{\gamma \in \mathbb{R} \mid \dim(V^n) \leq n^\gamma, n \gg 0\}$$

(Again, this is independent of V.)

Exercise Prove that the two definitions are equivalent.

In cases (ii) and (iii), (exponential and intermediate growth),

$$\mathrm{GKdim}(R) = \infty.$$

Example $R = k[x_1, \ldots, x_d]$ a commutative **polynomial algebra**.

$$f(n) = \#\text{ distinct } x_1^{a_1} x_2^{a_2} \ldots x_d^{a_d}$$

with $\sum a_i = n$.

Set $V := k + kx_1 + kx_2 + kx_n$. Then

$$\dim(V^n) = F(n) = f(0) + f(1) + \ldots + f(n) = \binom{n+d}{d}$$

is a polynomial of degree d, and

$$\mathrm{GKdim}(k[x_1, \ldots, x_d]) = d. \qquad \square$$

Example The Weyl algebra $R = k[x, y]$, with relation $xy - yx = 1$.

Let V be spanned by $1, x, y$. Start writing out the basis for R given by the powers of V.

$$1$$

$$x, \quad y$$

$$x^2, \quad xy, \quad yx, \quad y^2$$

$$x^3,\ x^2y,\ xyx,\ xy^2,\ yx^2,\ yxy,\ y^2x,\ y^3$$

$$\vdots$$

Notice that at any stage where there is a monomial with a y occuring before an x then the defining relation $xy - yx = 1$ can be used to show that the monomial is dependent on earlier monomials, and so does not contribute to the dimension count.

Thus the growth is exactly the same as the commutative polynomial ring $k[x, y]$; and so the Weyl Algebra has Gelfand-Kirillov dimension 2. $\square$

The Bergman Gap

$$\mathrm{GKdim}(R) = 0$$

iff $\quad V^n = V^{n+1} = \ldots$, for $n \gg 0$,
iff $\quad R$ is a finite dimensional algebra.

Otherwise, $\dim V^n \geq n$, for all n, and so

$$\mathrm{GKdim}(R) \geq 1.$$

Borho and Kraft showed that any real number in the interval $[2, \infty)$ occurs as a Gelfand-Kirillov dimension,

and

Bergman showed that there is **no** algebra R such that $1 < \mathrm{GKdim}(R) < 2$.

Example Let A be the algebra generated by x and y subject to the relations

$$yx^m yx^n y = 0,$$

for all $m, n \geq 0$; while

$$yx^n y = 0,$$

whenever $n \geq 1$ is a nonsquare.

Exercise
Show that the number of nonzero monomials of length n is $\mathcal{O}(n^{\frac{3}{2}})$; so that $\mathrm{GKdim}(A) = \frac{5}{2}$.

Let A be a k-algebra, and let M be a finitely generated right A-module. Let M_0 be a finite dimensional vector space such that $M = M_0 A$. We define the **Gelfand-Kirillov dimension** of M by

$$\mathrm{GKdim}(M) := \inf\{\gamma \in \mathbb{R} \mid \dim(M_0V^n) \leq n^\gamma, n \gg 0\}$$

Proposition Let c be a regular element of a finitely generated k-algebra A. Then

$$\mathrm{GKdim}(A/cA) + 1 \leq \mathrm{GKdim}(A).$$

Proof Set $I = cA$. Let V generate A, and $1, c \in V$. Then $\bar{V}$ generates $\bar{A} := A/I$. Let D_n be a complement of $I \cap V^n$ in V^n. Then, $\dim(\bar{V}^n) = \dim(D_n)$. However,

$$D_n \cap cA \subseteq D_n \cap (I \cap V^n) = 0,$$

and it follows that

$$D_n + cD_n + c^2D_n + \ldots + c^nD_n \qquad (*)$$

is a direct sum of k-subspaces of A contained in V^{2n}.

Suppose that $\mathrm{GKdim}(A) = \beta$, and let $\gamma > \beta$; so $\dim(V^n) \leq n^\gamma$, for $n \gg 0$. Then

$$n \times \dim(\bar{V}^n) \leq (2n)^\gamma,$$

for $n \gg 0$, since

$$D_n \oplus cD_n \oplus c^2D_n \oplus \ldots \oplus c^nD_n \subseteq V^{2n};$$

and so

$$\dim(\bar{V}^n) \leq 2^\gamma n^{\gamma-1},$$

for $n \gg 0$. The result then follows by using the definition of Gelfand-Kirillov dimension. $\square$

Exercise Prove $(*)$ is a direct sum.

Proposition Let A and B be finitely generated K-algebras, and let M be an A-$B-$bimodule that is finitely generated on each side. Then

$$\mathrm{GKdim}({}_AM) = \mathrm{GKdim}(M_B).$$

If, in addition, ${}_AM$ is faithful as an A-module, then

$$\mathrm{GKdim}({}_AM) = \mathrm{GKdim}(A).$$

Sketch of proof. Let F and V be finite dimensional subspaces of M and B, respectively, with $\cup FV^n = M$. Since ${}_AM$ is finitely generated, there exists a finite dimensional subspace $G \supseteq F$ such that $AG = M$. Obviously $FV^n \subseteq GV^n$ for all integers $n \geq 0$. Now GV is a finite dimensional subspace of M; so, since $AG = M \supseteq GV$, there exists a finite dimensional subspace W of A such that $WG \supseteq GV$. Then $FV^n \subseteq GV^n \subseteq W^nG$ for any n, and the claim follows from the definition of the Gelfand-Kirillov dimension for modules. □

Proposition Let A be a k-algebra, let M be a right A-module, and let ${}_AN_A$ be a bimodule, finitely generated on each side. Then

$$\mathrm{GKdim}(M \otimes_A N) \leq \mathrm{GKdim}(M).$$

Exercise Prove this

Gelfand-Kirillov dimension is known to be an integer for many classes of affine noetherian algebras. In fact, there are no known examples of such algebras with finite Gelfand-Kirillov dimension where the Gelfand-Kirillov dimension is not an integer.

Also, Gelfand-Kirillov dimension is known to be finitely partitive for many classes of affine noetherian algebras. In fact, there are no known examples of such algebras which are not finitely partitive.

Gelfand-Kirillov dimension is also known to be exact for many classes of affine noetherian algebras, and again, there are no known examples of such algebras which are not exact.

However, there are examples, due to Bergman of affine PI algebras which are not exact.

Stephenson and Zhang have shown that a connected graded noetherian k-algebra cannot have exponential growth. It is an open problem whether intermediate growth is possible

The situation for noetherian rings or noetherian algebras with finite Gelfand-Kirillov dimension is summarized by the following table.

	GK	Kdim
Exact		$\surd$
Partitive		$\surd$
Symmetric	$\surd$	
Torsion property	$\surd$	$\surd$
Separates primes	$\surd$	$\surd$
Ideal invariant	$\surd$	

Filling in any of these gaps would be a welcome development, and (if positive) would lead to substantial progress.

For example, Jategaonkar proved symmetry of Krull dimension for fully bounded noetherian rings, and used this to prove Jacobson's conjecture for fullly bounded noetherian rings:

$$\bigcap_{i=1}^{\infty} J^n = 0.$$

(A noetherian ring R is **fully bounded noetherian** if in each prime factor ring of R every nonzero one-sided ideal contains a nonzero two-sided ideal. This guarantees that a fully bounded noetherian ring has many two-sided ideals; in fact, the Krull dimension and the classical Krull dimension coincide for fully bounded noetherian rings. The main examples of fully bounded noetherian rings are the noetherian polynomial identity rings.)

I showed that the artinian condition ($\mathrm{Kdim}(M) = 0$) is symmetric for noetherian bimodules and used this to prove Jacobson's conjecture for rings with Krull dimension one. A major question in the area is whether Krull dimension is symmetric on noetherian bimodules.

Incidentally, the symmetry of the artinian condition for noetherian bimodules has a nice consequence in representation theory: a finite length module which is a bimodule and artinian or noetherian as a module over its endomorphism ring, is endofinite, [**?**, Prop 4.5].

Substantial progress can often be made if a link can be found between one dimension function and another, since then desireable properties of both can be exploited.

In another paper in this volume, Vladimir Bavula, [?], will explore some of the connections between Gelfand-Kirillov dimension and Krull dimension, so I won't go into these.

However, connections between growth and homological conditions have also been particularly useful. Gabber used such a connection to establish catenarity for enveloping algebras of finite dimensional solvable Lie algebras. Goodearl and I have recently put Gabber's result into an abstract form which has enabled catenarity to be established for many classes of algebras.

The ring R is said to be **catenary** if, for any two prime ideals $P < Q$ of R, all saturated chains of prime ideals between P and Q have the same length.

Matsumura, [?, p31], says

"Practically all important (commutative) noetherian rings arising in applications are catenary. The first example of a non-catenary ring is due to Nagata 1956", see the back of Nagata's book on local rings, [?], for the example,

and on page 123

"The theorem that quotients of (commutative) Cohen-Macaulay rings are catenary is of great importance in dimension theory".

The abstract form of Gabber's Theorem proved by Goodearl and myself says informally:

Let R be an affine, noetherian, algebra over a field, with finite Gelfand-Kirillov dimension, good homological properties and a link between the Gelfand-Kirillov dimension and the homological properties. If R has lots of normal elements, then R is catenary.

In order to give a precise statement, we need to introduce some homological notions.

The algebra R is said to be **Auslander-Gorenstein** provided

(a) the injective dimension of R (as both a right and a left R-module) is finite, and

(b) for any integers $0 \leq i < j$ and any finitely generated (right or left) R-module M, we have

$$\mathrm{Ext}^i_R(N, R) = 0$$

for all R-submodules N of $\mathrm{Ext}^j_R(M, R)$.

The **grade** of a finitely generated R-module M is defined to be

$$j(M) := \inf\{j \geq 0 \mid \mathrm{Ext}^j_R(M,R) \neq 0\}.$$

The algebra R is called **Cohen-Macaulay** (or **CM**) if

$$j(M) + \mathrm{GKdim}(M) = \mathrm{GKdim}(R)$$

for all finitely generated R-modules M.

An element $x \in R$ is said to be a **normal** element of R if

$$xR = Rx.$$

If $P < Q$ are prime ideals of the noetherian k-algebra R with $\mathrm{height}(Q/P) = 1$, then

$$\mathrm{GKdim}(R/P) \geq \mathrm{GKdim}(R/Q) + 1.$$

Let $P < Q$ be prime ideals of a ring R such that $\mathrm{height}(Q/P) = 1$. We say that Q and P are **normally separated** if there exists an element $x \in Q \setminus P$ that is normal modulo P.

Theorem [**?**] Let R be Auslander-Gorenstein and Cohen-Macaulay with $\mathrm{GKdim}(R)$ finite, and let $P < Q$ be prime ideals of R with $\mathrm{height}(Q/P) = 1$. If Q and P are normally separated, then

$$\mathrm{GKdim}(R/P) = \mathrm{GKdim}(R/Q) + 1.$$

Let R be an affine, noetherian, Auslander-Gorenstein, Cohen-Macaulay algebra over a field, with finite Gelfand-Kirillov dimension. If $\mathrm{Spec}(R)$ is normally separated, then R is catenary.

STOP PRESS

In a recent pre-print, Yekuteili and Zhang, [**?**], have shown that results such as the previous one hold for rings with an Auslander Dualizing Complex, and they give criteria for the existence of Auslander Dualizing complexes which show that they occur quite frequently.

Letzter and Lorenz, [**?**], have just announced a proof that group algebras of polycylic-by-finite groups are catenary. Roseblade obtained this, up to a finite extension part, long ago, and recent developments in the understanding of the prime ideals of noetherian rings have now enabled the last step to be made. The proof uses Krull dimension techniques

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