Orbits for the adjoint coaction on quantum matrices

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Abstract

Conjugation coactions of the quantum general linear group on the algebra of quantum matrices have been introduced in an earlier paper and the coinvariants have been determined. In this paper the notion of orbit is considered via co-orbit maps associated with $\mathbb{C}$-points of the space of quantum matrices, mapping the coordinate ring of quantum matrices into the coordinate ring of the quantum general linear group. The co-orbit maps are calculated explicitly for 2×2 quantum matrices. For quantum matrices of arbitrary size, it is shown that when the deformation parameter is transcendental over the base field, then the kernel of the co-orbit map associated with a $\mathbb{C}$-point ξ is a right ideal generated by coinvariants, provided that the classical adjoint orbit of ξ is maximal. If ξ is diagonal with pairwise different eigenvalues, then the image of the co-orbit map coincides with the subalgebra of coinvariants with respect to the left coaction of the diagonal quantum subgroup of the quantum general linear group.

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1 Introduction

Consider $\mathcal{O}(M_q)$, the coordinate ring of $N \times N$ quantum matrices over $\mathbb{C}$, where q is a non-zero element of $\mathbb{C}$. Denote by $\mathcal{O}(GL_q)$ the coordinate ring of the quantum $GL(N, \mathbb{C})$ (see [12]). Define the map

$$\beta : \mathcal{O}(GL_q) \to \mathcal{O}(GL_q) \otimes \mathcal{O}(GL_q), \quad \beta(h) = \sum h_2 \otimes S(h_1)h_3, \tag{1}$$

where we use Sweedler's convention for the Hopf algebra $\mathcal{O}(GL_q)$ (and S denotes the antipode).

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Similarly, set

$$\alpha : \mathcal{O}(GL_q) \to \mathcal{O}(GL_q) \otimes \mathcal{O}(GL_q), \quad \alpha(h) = \sum h_2 \otimes h_3 S(h_1).$$

Both α and β are right coactions of the Hopf algebra $\mathcal{O}(GL_q)$ on itself. Being the formal dual of the right adjoint action, β is usually called the right *adjoint coaction* of $\mathcal{O}(GL_q)$. Obviously $\mathcal{O}(M_q)$ is a subcomodule of $\mathcal{O}(GL_q)$ with respect to α and β, and we use the same symbols to denote the restrictions $\alpha : \mathcal{O}(M_q) \to \mathcal{O}(M_q) \otimes \mathcal{O}(GL_q)$, $\beta : \mathcal{O}(M_q) \to \mathcal{O}(M_q) \otimes \mathcal{O}(GL_q)$. These coactions can be viewed as quantum analogues of the adjoint action of $GL(N, \mathbb{C})$ on its Lie algebra $M(N, \mathbb{C})$. The coinvariants with respect to these coactions were described in [4].

Our aim here is to find a counterpart in the quantum setting of the notion of orbits of the classical adjoint action. We summarize the results concerning β, the case of α being similar. There is a natural way to associate with a "$\mathbb{C}$-point ξ of the space of $N \times N$ quantum matrices" a morphism $\beta^\xi : \mathcal{O}(M_q) \to \mathcal{O}(GL_q)$ of right $\mathcal{O}(GL_q)$-comodules (the right coaction on $\mathcal{O}(M_q)$ is given by β, whereas the right coaction on $\mathcal{O}(GL_q)$ comes from the comultiplication of $\mathcal{O}(GL_q)$). When q is specialized to 1, the kernel of β^ξ is the vanishing ideal of the closure of the orbit of ξ, so the image of β^ξ can be identified with the coordinate ring of the closure of the orbit of ξ. The q-deformed map β^ξ is not an algebra homomorphism since the β^ξ is not an algebra homomorphism (see [4]). However, it turns out that the kernel of β^ξ contains the right ideal of $\mathcal{O}(M_q)$ generated by the elements $\tau_i - \tau_i(\xi)$, $i = 1, \ldots, N$, where $\tau_1, \ldots, \tau_N$ are the basic coinvariants for β introduced in [4]. When ξ is diagonal, the image of β^ξ is contained in the quantum quotient space $\mathcal{O}(D \backslash GL_q)$, the subalgebra of coinvariants of the left coaction of the diagonal quantum subgroup $\mathcal{O}(D)$ on $\mathcal{O}(GL_q)$ (note that in the classical case $q = 1$, D is the stabilizer of ξ, provided that ξ has pairwise different diagonal entries). In the special case when $N = 2$ and q is not a root of unity, we are able to show that although β^ξ is not an algebra homomorphism, it has further nice properties. Namely, if ξ is diagonal, and ξ is not a scalar power of any of the non-negative powers of $\mathrm{diag}(q^2, 1)$, then β^ξ is surjective onto $\mathcal{O}(D \setminus GL_q)$. Moreover, for such ξ, the kernel of β^ξ is the right ideal of $\mathcal{O}(M_q)$ generated by the elements $\tau_1 - \tau_1(\xi)$, $\tau_2 - \tau_2(\xi)$, where τ_1, τ_2 are the basic β-coinvariants from [4]; that is, $\tau_1 = q^{-2}x_{11} + q^{-4}x_{22}$, a weighted trace, and $\tau_2 = x_{11}x_{22} - qx_{12}x_{21}$, the quantum determinant. When ξ is a scalar multiple of some non-negative power of $\mathrm{diag}(q^2, 1)$, then the kernel of β^ξ is larger, and the image is finite dimensional. In the final section we change the setup, and treat q as an indeterminate over a subfield of $\mathbb{C}$. In this generic situation, both the kernel and the image of the co-orbit map coincide with the subset predicted by the classical theory, for a $\xi \in M(N, \mathbb{C})$ whose classical adjoint orbit is maximal. To be more precise, the kernel of the co-orbit map is a right ideal generated by coinvariants, and if ξ is diagonal, then the image of the co-orbit map is the subalgebra of coinvariants with respect to the left coaction of the diagonal quantum subgroup. The present work can be viewed as a quantum version of some of the basic results of Kostant [7] on the classical adjoint orbits of the general linear group acting on its Lie algebra (which we use in our proof).

Recall that $\mathcal{O}(M_q)$ is the $\mathbb{C}$-algebra generated by N^2 indeterminates x_{ij}, for $i = 1, \dots, N$, subject to the following relations.

$$\begin{aligned} x_{ij}x_{il} &= qx_{il}x_{ij}, \\ x_{ij}x_{kj} &= qx_{kj}x_{ij}, \\ x_{il}x_{kj} &= x_{kj}x_{il}, \\ x_{ij}x_{kl} - x_{kl}x_{ij} &= (q-q^{-1})x_{il}x_{kj}, \end{aligned} \tag{2}$$

for $1 \leq i < k \leq N$ and $1 \leq j < l \leq N$, where $q \in \mathbb{C}^*$. The algebra $\mathcal{O}(M_q)$ is an iterated Ore extension, and so a noetherian domain. The *quantum determinant*, $\det_q$, is the element

$$\det_q := \sum_{\sigma \in S_N} (-q)^{l(\sigma)} x_{1,\sigma(1)} \dots x_{N,\sigma(N)}.$$

It is known that $\det_q$ is a central element in $\mathcal{O}(M_q)$ (see [10, Theorem 4.6.1]), and by adjoining its inverse we get the *quantum general linear group*

$$\mathcal{O}(GL_q) := \mathcal{O}(M_q)[\det_q^{-1}].$$

The algebra $\mathcal{O}(GL_q)$ is a Hopf algebra with *comultiplication* Δ and *antipode* S. Recall that $\Delta(x_{ij}) = \sum_{k=1}^{N} x_{ik} \otimes x_{kj}$. We shall need the explicit form of S only in the special case $N = 2$, when $S(x_{11}) = x_{22}/\det_q$, $S(x_{22}) = x_{11}/\det_q$, $S(x_{12}) = -q^{-1}x_{12}/\det_q$, $S(x_{21}) = -qx_{21}/\det_q$. The reader should be aware that in many papers the rôles of q and q^{-1} are interchanged, and so one has to be careful in translating results from one paper to another. In the special case $q = 1$, the algebra $\mathcal{O}(M_q)$ becomes the coordinate ring $\mathcal{O}(M)$ of $M(N, \mathbb{C})$, and $\mathcal{O}(GL_q)$ becomes the coordinate ring $\mathcal{O}(GL)$ of $GL(N, \mathbb{C})$.

2 Orbits of classical points

Start from the classical situation when we are given a (right) action

$$X \times G \to X, \quad (x, g) \mapsto xg$$

of an affine algebraic group G on an affine algebraic variety X. The orbit of some $x \in X$ is the image of the composition of the morphisms

$$G \to X \times G \to X, \quad g \mapsto (x, g) \mapsto xg.$$

Passing to coordinate rings, the action is encoded in the comorphism $\mu : \mathcal{O}(X) \to \mathcal{O}(X) \otimes \mathcal{O}(G)$, which makes $\mathcal{O}(X)$ a right $\mathcal{O}(G)$-comodule algebra. The point x corresponds to the $\mathbb{C}$-algebra homomorphism $\mathrm{ev}_x : \mathcal{O}(X) \to \mathbb{C}$, $f \mapsto f(x)$. The comorphism of the orbit map $G \to xG \subset X$, $g \mapsto xg$ is the composition

$$\mu_x : \mathcal{O}(X) \xrightarrow{\mu} \mathcal{O}(X) \otimes \mathcal{O}(G) \xrightarrow{\mathrm{ev}_x \otimes \mathrm{id}} \mathcal{O}(G).$$

The kernel of μ_x is the ideal of the (Zariski) closure of the orbit xG, and so the image of μ_x can be identified with the coordinate ring of the closure of xG. Note that $\mu_x : \mathcal{O}(X) \to \mathcal{O}(G)$ is an algebra homomorphism, as well as a morphism of right $\mathcal{O}(G)$-comodules (where the right coaction of $\mathcal{O}(G)$ on itself comes from the action of G on itself by right translations).

Now assume in addition that the orbit of x is closed in the Zariski topology of X, and denote by H the stabilizer of x. Then H acts by left translation on G, and the orbit map $G \to xG$ factors through the quotient variety $H \setminus G$, and induces an isomorphism of the G-varieties $H \setminus G \cong xG$. In terms of coordinate rings this means that $\mathrm{im}(\mu_x)$ is the subalgebra of $\mathcal{O}(H)$-coinvariants in $\mathcal{O}(G)$ (the action of H on G by left translations induces a left coaction of $\mathcal{O}(H)$ on $\mathcal{O}(G)$).

Furthermore, if G is reductive, then $\mathcal{O}(X)^G$, the algebra of polynomial invariants (which coincides with the algebra of μ-coinvariants) is finitely generated as an algebra, say $f_1, \dots, f_s$ is a generating system. Assume in addition that the orbit xG is closed and maximal; that is, x is not contained in the closure of another orbit (note that for the adjoint action of $GL(N, \mathbb{C})$ on its Lie algebra $M(N, \mathbb{C})$ this holds for the general x). Then $\ker(\mu_x)$ can be described in terms of polynomial invariants. In this case the common zero locus of $f_1 - f_1(x), \dots, f_s - f_s(x)$ is the orbit xG, so $\ker(\mu_x)$ is the radical of the ideal generated by $f_1 - f_1(x), \dots, f_s - f_s(x)$.

Recall that

$$\beta = (\mathrm{id} \otimes p) \circ (\tau \otimes \mathrm{id}) \circ (S \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta^{(2)} : \mathcal{O}(GL_q) \to \mathcal{O}(GL_q) \otimes \mathcal{O}(GL_q),$$

where $\Delta^{(2)} = (\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$, $\tau(h \otimes k) = k \otimes h$, and $p : \mathcal{O}(GL_q) \otimes \mathcal{O}(GL_q) \to \mathcal{O}(GL_q)$ is the multiplication map. The corresponding formula for α is

$$\alpha = (\mathrm{id} \otimes p) \circ \tau_{(132)} \circ (S \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta^{(2)},$$

where $\tau_{(132)}(h \otimes k \otimes l) = k \otimes l \otimes h$. The above discussion on algebraic group actions motivates the following definition. Take a surjective $\mathbb{C}$-algebra homomorphism $\mathrm{ev}_\xi : \mathcal{O}(M_q) \to \mathbb{C}$. The ξ in the notation refers to the $N \times N$ matrix with complex entries obtained by evaluating the given homomorphism on the generators x_{ij} of $\mathcal{O}(M_q)$. Note that for $\xi \in M(N, \mathbb{C})$ there is a corresponding homomorphism ev_ξ if and only if ξ is a "quantum matrix" in the sense of [8]; that is, if the entries of ξ satisfy the relations defining $\mathcal{O}(M_q)$. We say that ξ is a $\mathbb{C}$-*point of* M_q in this case. If $q \neq 1$, then ξ is a $\mathbb{C}$-point of M_q if and only if there is at most one non-zero entry in each column and each row of ξ, and $\xi_{ij}\xi_{kl} \neq 0$ with $i < k$ implies that $j < l$. For example, a diagonal ξ satisfies this condition.

Definition 2.1 *Let ξ be a $\mathbb{C}$-point of M_q. The co-orbit map β^ξ of ξ with respect to the right coaction β is the composition* $(\mathrm{ev}_\xi \otimes \mathrm{id}) \circ \beta$, *that is,*

$$\beta^\xi : \mathcal{O}(M_q) \xrightarrow{\beta} \mathcal{O}(M_q) \otimes \mathcal{O}(GL_q) \xrightarrow{\mathrm{ev}_\xi \otimes \mathrm{id}} \mathcal{O}(GL_q).$$

In the same manner, α^ξ is defined to be $(\mathrm{ev}_\xi \otimes \mathrm{id}) \circ \alpha$.

The above discussion suggests that the kernel of the co-orbit map should contain information about the "embedding of the orbit into M_q", and the image of the co-orbit map should determine the isomorphism type of (the closure of) the "orbit" as a quantum space. The main point of this paper is that we treat the co-orbit map (and not only its image) as our central object, and demonstrate that it has certain nice properties, even though we started with a coaction which was not an algebra homomorphism. In particular, the coinvariants are used to study the kernel of the co-orbit map.

Remark 2.2 Note that Definition 2.1 makes sense if β is replaced by any right coaction ν of a Hopf algebra $\mathcal{A}$ on an associative $\mathbb{C}$-algebra R, and ev_ξ is an algebra homomorphism of R to $\mathbb{C}$. Proposition 2.3 below clearly holds in this general setting. The map ν^ξ (and Proposition 2.3) appears in [2], [3] in the situation when R is an $\mathcal{A}$- comodule algebra. In this case ν^ξ is an algebra homomorphism as well, and $\mathrm{im}(\nu^\xi)$ is a right coideal subalgebra of $\mathcal{A}$. A right coideal subalgebra of a quantum group is called a *quantum homogeneous space*, see [6, 11.6.1]. So if $\mathcal{A}$ is a quantum group and R is a quantum $\mathcal{A}$-space (that is, R is an $\mathcal{A}$-comodule algebra), then $\mathrm{im}(\nu^\xi)$ is a quantum homogeneous space. Moreover, any quantum homogeneous $\mathcal{A}$-space can be obtained as the image of ν^ξ for some classical point ξ in some quantum $\mathcal{A}$-space, see [3, Proposition 1.1] or [2, Proposition 3.2].

Proposition 2.3 *The co-orbit map $\beta^\xi : \mathcal{O}(M_q) \to \mathcal{O}(GL_q)$ is a morphism of right $\mathcal{O}(GL_q)$-comodules (given by β and Δ, respectively). Similarly, α^ξ is a morphism of $\mathcal{O}(GL_q)$-comodules (given by α and Δ, respectively).*

Proof. The claim is the equality $\Delta \circ \beta^\xi = (\beta^\xi \otimes \mathrm{id}) \circ \beta$. Observe that $\Delta \circ (\mathrm{ev}_\xi \otimes \mathrm{id}) = (\mathrm{ev}_\xi \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta)$, because both are equal to $\mathrm{ev}_\xi \otimes \Delta : \mathcal{O}(M_q) \otimes \mathcal{O}(GL_q) \to \mathcal{O}(GL_q) \otimes \mathcal{O}(GL_q)$. Since β is a right coaction we have $(\mathrm{id} \otimes \Delta) \circ \beta = (\beta \otimes \mathrm{id}) \circ \beta$. Using these two equalities one gets

$$\begin{aligned}
\Delta \circ \beta^\xi &= \Delta \circ (\mathrm{ev}_\xi \otimes \mathrm{id}) \circ \beta \\
&= (\mathrm{ev}_\xi \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) \circ \beta \\
&= (\mathrm{ev}_\xi \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\beta \otimes \mathrm{id}) \circ \beta \\
&= (\beta^\xi \otimes \mathrm{id}) \circ \beta.
\end{aligned}$$

□

The spaces of coinvariants

$$\begin{aligned}
\mathcal{O}(M_q)^{\alpha-GL_q} &:= \{f \in \mathcal{O}(M_q) \mid \alpha(f) = f \otimes 1\} \\
\mathcal{O}(M_q)^{\beta-GL_q} &:= \{f \in \mathcal{O}(M_q) \mid \beta(f) = f \otimes 1\}
\end{aligned}$$

are studied in [4]. It is shown there that although α and β are not algebra homomorphisms, nevertheless

$$\alpha(fh) = \alpha(f)\alpha(h) \quad \text{if} \quad h \in \mathcal{O}(M_q)^{\alpha-GL_q} \tag{3}$$

and similarly

$$\beta(fh) = \beta(f)\beta(h) \quad \text{if} \quad f \in \mathcal{O}(M_q)^{\beta-GL_q}. \tag{4}$$

It follows that $\mathcal{O}(M_q)^{\alpha-GL_q}$ and $\mathcal{O}(M_q)^{\beta-GL_q}$ are subalgebras. Let us recall their generators. Fix an integer t with $1 \leq t \leq N$. Let I and J be subsets of $\{1,\dots,N\}$ with $|I| = |J| = t$. The subalgebra of $\mathcal{O}(M_q)$ generated by x_{ij} with $i \in I$ and $j \in J$ can be regarded as an algebra of $t \times t$ quantum matrices, and so we can calculate its quantum determinant - this is a $t \times t$ *quantum minor* and we denote it by $[I|J]$. The quantum minor $[I|I]$ is said to be a *principal quantum minor*. We denote the sum of all the principal quantum minors of a given size i by σ_i. Note that $\sigma_1 = x_{11} + \dots + x_{NN}$ and that $\sigma_N = \det_q$. It is shown in [4] that the σ_i are α-coinvariants that pairwise commute, and if q is not a root of unity, then $\mathcal{O}(M_q)^{\alpha-GL_q} = \mathbb{C}[\sigma_1,\dots,\sigma_N]$, an N-variable commutative polynomial subalgebra of $\mathcal{O}(M_q)$. For the coaction β we have that the *weighted sums of principal minors* $\tau_i := \sum_I q^{-2w(I)}[I|I]$ (here $w(I)$ denotes the sum of the elements of I, and the summation ranges over all subsets I of size i), $i = 1,\dots,N$, are pairwise commuting β-coinvariants. Moreover, assuming again that q is not a root of unity, $\mathcal{O}(M_q)^{\beta-GL_q} = \mathbb{C}[\tau_1,\dots,\tau_N]$, the N-variable commutative polynomial subalgebra generated by the τ_i.

We shall write $f(\xi)$ for $\mathrm{ev}_\xi(f)$, where $f \in \mathcal{O}(M_q)$ and $\xi \in M_q(\mathbb{C})$.

Proposition 2.4 *The kernel of β^ξ contains the right ideals*

$$\sum_{f \in \mathcal{O}(M_q)^{\beta-GL_q}} (f - f(\xi))\mathcal{O}(M_q) \supseteq \sum_{i=1}^{N} (\tau_i - \tau_i(\xi))\mathcal{O}(M_q).$$

If q is not a root of unity (or $q = 1$), then the latter two right ideals are equal. The kernel of α^ξ contains the left ideals

$$\sum_{f \in \mathcal{O}(M_q)^{\alpha-GL_q}} \mathcal{O}(M_q)(f - f(\xi)) \supseteq \sum_{i=1}^{N} \mathcal{O}(M_q)(\sigma_i - \sigma_i(\xi)).$$

If q is not a root of unity (or $q = 1$), then the latter two left ideals are equal.

Proof. If $f \in \mathcal{O}(M_q)^{\beta-GL_q}$, then $\beta^\xi(f) = (\mathrm{ev}_\xi \otimes \mathrm{id})(\beta(f)) = (\mathrm{ev}_\xi \otimes \mathrm{id})(f \otimes 1) = f(\xi) \cdot 1 \in \mathcal{O}(M_q)$, hence $f - f(\xi) \in \ker(\beta^\xi)$. By formula (4) then $(f - f(\xi))\mathcal{O}(M_q) \subseteq \ker(\beta^\xi)$. Note that τ_i are β-coinvariants, and if q is not a root of unity, they generate $\mathcal{O}(M_q)^{\beta-GL_q}$ by [4]. Hence in the latter case the elements $\tau_i - \tau_i(\xi)$ generate the same right ideal within $\mathcal{O}(M_q)^{\beta-GL_q}$ as all the $f - f(\xi)$ with $f \in \mathcal{O}(M_q)^{\beta-GL_q}$.

The same argument works for α. $\square$

Remark 2.5 More is known in the classical case $q = 1$. If the adjoint orbit of $\xi \in M(N,\mathbb{C})$ is maximal, then $\ker(\beta^\xi)$, the vanishing ideal of the closure of the orbit of ξ, is generated by $\tau_i - \tau_i(\xi)$, $(i = 1,\dots,N)$; this is proved in [7, Theorem 10], considering more generally the adjoint action on reductive Lie algebras.

The coordinate Hopf algebra $\mathcal{O}(D)$ of the diagonal subgroup D of $GL(N,\mathbb{C})$ is the commutative algebra $\mathbb{C}[t_1,t_1^{-1},\dots,t_N,t_N^{-1}]$ of Laurent polynomials with comultiplication $\Delta(t_i)=t_i\otimes t_i$ and counit $\varepsilon(t_i)=1$. There exists a surjective Hopf algebra homomorphism $\pi_D:\mathcal{O}(GL_q)\to\mathcal{O}(D)$ determined by $\pi_D(x_{ij})=\delta_{ij}t_i$, $i,j=1,\dots,N$. Therefore we say that D is a quantum subgroup of GL_q, called the *diagonal subgroup*.

There is a natural left coaction $\lambda_D:=(\pi_D\otimes\mathrm{id})\circ\Delta$ of $\mathcal{O}(D)$ on $\mathcal{O}(GL_q)$. The subset of λ_D-coinvariants is denoted by

$$\mathcal{O}(D\setminus GL_q)=\{f\in\mathcal{O}(GL_q)\mid\lambda_D(f)=1\otimes f\}.$$

This is a subalgebra (since $\mathcal{O}(GL_q)$ is a comodule algebra with respect to λ_D), as well as a right $\mathcal{O}(GL_q)$-subcomodule (since λ_D commutes with the right coaction of $\mathcal{O}(GL_q)$ on itself). So $\mathcal{O}(D\setminus GL_q)$ is *the coordinate algebra of a quantum homogeneous space* in the sense of [6, 11.6.1.]. This quantum homogeneous space is often called the *quantum quotient space* $D\setminus G_q$, because it arises as the subalgebra of coinvariants with respect to the coaction of a quantum subgroup.

Recall that in the classical case $q=1$, the subgroup D is the stabilizer of ξ, provided that ξ is a diagonal matrix with pairwise different eigenvalues, and the orbit of a diagonal ξ is closed in $\mathcal{O}(M(N,\mathbb{C}))$. Hence for such ξ, the coordinate ring of the orbit of ξ is $\mathcal{O}(D\setminus G_1)$. A partial analogue of this holds for any q.

Proposition 2.6 *Let ξ be a diagonal $N\times N$ matrix with complex entries. Then the image of β^ξ is contained in $\mathcal{O}(D\setminus GL_q)$. Similarly we have* $\mathrm{im}(\alpha^\xi)\subseteq\mathcal{O}(D\setminus GL_q)$.

Proof. The restriction of π_D to $\mathcal{O}(M_q)$ is denoted by π_C, it maps $\mathcal{O}(M_q)$ onto $\mathcal{O}(C)=\mathbb{C}[t_1,\dots,t_N]$. By our assumption on ξ the homomorphism $\mathrm{ev}_\xi:\mathcal{O}(M_q)\to\mathbb{C}$ factors through π_C, we write ev_ξ also for the homomorphism $\mathcal{O}(C)\to\mathbb{C}$, $t_i\mapsto\xi_{ii}$.

By definition β^ξ is the composition

$$\beta^\xi:\ \mathcal{O}(M_q)\xrightarrow{\beta}\mathcal{O}(M_q)\otimes\mathcal{O}(GL_q)\xrightarrow{\pi_C\otimes\mathrm{id}}\mathcal{O}(C)\otimes\mathcal{O}(GL_q)\xrightarrow{\mathrm{ev}_\xi\otimes\mathrm{id}}\mathcal{O}(GL_q), \tag{5}$$

and λ_D is the composition

$$\lambda_D:\ \mathcal{O}(GL_q)\xrightarrow{\Delta}\mathcal{O}(GL_q)\otimes\mathcal{O}(GL_q)\xrightarrow{\pi_D\otimes\mathrm{id}}\mathcal{O}(D)\otimes\mathcal{O}(GL_q).$$

By linearity of $(\pi_D\otimes\mathrm{id})\otimes\Delta$ we have

$$(\pi_D\otimes\mathrm{id})\circ\Delta\circ(\mathrm{ev}_\xi\otimes\mathrm{id})=(\mathrm{ev}_\xi\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\pi_D\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta)$$

as maps $\mathcal{O}(C)\otimes\mathcal{O}(GL_q)\to\mathcal{O}(D)\otimes\mathcal{O}(GL_q)$. Thus

$$(\pi_D\otimes\mathrm{id})\circ\Delta\circ(\mathrm{ev}_\xi\otimes\mathrm{id})\circ(\pi_C\otimes\mathrm{id})=(\mathrm{ev}_\xi\otimes\mathrm{id}\otimes\mathrm{id})\circ(\pi_C\otimes\pi_D\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta)$$

as maps $\mathcal{O}(M_q)\otimes\mathcal{O}(GL_q)\to\mathcal{O}(D)\otimes\mathcal{O}(GL_q)$. Therefore we have

$$\begin{aligned}\lambda_D\circ\beta^\xi&=(\mathrm{ev}_\xi\otimes\mathrm{id}\otimes\mathrm{id})\circ(\pi_C\otimes\pi_D\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta)\circ\beta\\&=(\mathrm{ev}_\xi\otimes\mathrm{id}\otimes\mathrm{id})\circ(\pi_C\otimes\pi_D\otimes\mathrm{id})\circ(\beta\otimes\mathrm{id})\circ\beta\end{aligned}$$

(in the second equality we use the fact that $\mathcal{O}(M_q)$ is a right $\mathcal{O}(GL_q)$-comodule under β). Note that the formula (1) makes sense if we replace $\mathcal{O}(GL_q)$ by any Hopf algebra, in particular, it defines a right coaction $\beta^D : \mathcal{O}(D) \to \mathcal{O}(D) \otimes \mathcal{O}(D)$. Since π_D is a Hopf algebra homomorphism, we have $\beta^D \circ \pi_D = (\pi_D \otimes \pi_D) \circ \beta$. On the other hand, β^D is equal to the trivial corepresentation $\mathrm{id} \otimes 1$, since it is the comorphism of the conjugation action of D on itself, which is trivial. It follows that $(\pi_C \otimes \pi_D) \circ \beta = (\mathrm{id} \otimes 1) \circ \pi_C : \mathcal{O}(M_q) \to \mathcal{O}(C) \otimes \mathcal{O}(D)$, and hence $\lambda_D \circ \beta^\xi$ can be written as the composition

$$\mathcal{O}(M_q) \stackrel{(\pi_C \otimes \mathrm{id})\circ\beta}{\longrightarrow} \mathcal{O}(C) \otimes \mathcal{O}(GL_q) \stackrel{(\mathrm{id}\otimes 1)\otimes\mathrm{id}}{\longrightarrow} \tag{6}$$

$$\mathcal{O}(C) \otimes \mathcal{O}(D) \otimes \mathcal{O}(GL_q) \stackrel{\mathrm{ev}_\xi \otimes \mathrm{id} \otimes \mathrm{id}}{\longrightarrow} \mathcal{O}(D) \otimes \mathcal{O}(GL_q).$$

In order to prove our Proposition we need to show that for all $f \in \mathcal{O}(M_q)$ the equality $\lambda_D(\beta^\xi(f)) = 1 \otimes \beta^\xi(f)$ holds. Write $(\pi_D \otimes \mathrm{id})(\beta(f))$ as $\sum_i a_i \otimes b_i$, where $a_i \in \mathcal{O}(C)$ and $b_i \in \mathcal{O}(GL_q)$. Then $\beta^\xi(f) = \sum_i \mathrm{ev}_\xi(a_i) b_i$ by (5). Therefore, using (6), we have $\lambda_D(\beta^\xi(f)) = \sum_i \mathrm{ev}_\xi(a_i) 1 \otimes b_i = 1 \otimes \sum_i \mathrm{ev}_\xi(a_i) b_i = 1 \otimes \beta^\xi(f)$. □

3 Co-orbit maps of 2×2 quantum matrices

In the special case $N = 2$ and q not a root of unity (or $q = 1$) we are able to refine Propositions 2.4 and 2.6. Throughout this section we assume that $N = 2$, $\mathcal{O}(M_q)$, $\mathcal{O}(GL_q)$ denote the coordinate rings of 2×2 quantum matrices and quantum $GL(2)$, and $\mathcal{O}(D) = \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$. We will see in the proof of Lemma 3.4 that as an algebra $\mathcal{O}(D \setminus GL_q)$ is generated by $x_{11}x_{21}/\det_q$, $x_{12}x_{21}/\det_q$, $x_{12}x_{22}/\det_q$. We write $\mathcal{O}(D \setminus GL_q)^{\leq m}$ for the subspace spanned by products of length at most m in these generators.

Proposition 3.1 *Assume that q is not a root of unity (or $q = 1$), and take* $\xi := \begin{bmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{bmatrix}$.

(i) If $\xi_1 - q^{2k}\xi_2 \neq 0$ for $k = 0, 1, 2, \ldots,$ then $\mathrm{im}(\beta^\xi) = \mathcal{O}(D \setminus GL_q)$.

(ii) If $\xi_1 - \xi_2, \xi_1 - q^2\xi_2, \ldots, \xi_1 - q^{2m-2}\xi_2$ are non-zero and $\xi_1 - q^{2m}\xi_2 = 0$, then $\mathrm{im}(\beta^\xi) = \mathcal{O}(D \setminus GL_q)^{\leq m}$.

(iii) If $\xi_1 - q^{2k}\xi_2 \neq 0$ for $k = 0, 1, 2, \ldots,$ then $\ker(\beta^\xi) = \sum_{i=1}^2 (\tau_i - \tau_i(\xi))\mathcal{O}(M_q)$.

(iv) If $\xi_1 - \xi_2, \xi_1 - q^2\xi_2, \ldots, \xi_1 - q^{2m-2}\xi_2$ are non-zero and $\xi_1 - q^{2m}\xi_2 = 0$, then $\ker(\beta^\xi)$ is the sum of $\sum_{i=1}^2 (\tau_i - \tau_i(\xi))\mathcal{O}(M_q)$ and the $\mathcal{O}(GL_q)$-comodule generated by $\{x_{21}^k \mid k > m\}$.

(i') If $\xi_1 - q^{-2k}\xi_2 \neq 0$ for $k = 1, 2, \ldots,$ then $\mathrm{im}(\alpha^\xi) = \mathcal{O}(D \setminus GL_q)$.

(ii') If $\xi_1 - q^{-2}\xi_2, \ldots, \xi_1 - q^{-2m}\xi_2$ are nonzero and $\xi_1 - q^{-2m-2}\xi_2 = 0$, then $\mathrm{im}(\alpha^\xi) = \mathcal{O}(D \setminus GL_q)^{\leq m}$.

(iii') If $\xi_1 - q^{-2k}\xi_2 \neq 0$ for $k = 1, 2, \ldots,$ then $\ker(\alpha^\xi) = \sum_{i=1}^2 \mathcal{O}(M_q)(\sigma_i - \sigma_i(\xi))$.

(iv') If $\xi_1 - q^{-2}\xi_2, \dots, \xi_1 - q^{-2m}\xi_2$ are nonzero and $\xi_1 - q^{-2m-2}\xi_2 = 0$, then $\ker(\alpha^\xi)$ is the sum of $\sum_{i=1}^2 \mathcal{O}(M_q)(\sigma_i - \sigma_i(\xi))$ and the $\mathcal{O}(GL_q)$-comodule generated by $\{x_{21}^k \mid k > m\}$.

As an immediate corollary we obtain the following.

Theorem 3.2 *Assume that q is not a root of unity (or $q = 1$) and take a diagonal complex matrix ξ which is not a scalar multiple of any integral power of $\begin{bmatrix} q^2 & 0 \\ 0 & 1 \end{bmatrix}$. Then both β^ξ and α^ξ map $\mathcal{O}(M_q)$ surjectively onto $\mathcal{O}(D \setminus GL_q)$. In particular, $\operatorname{im}(\beta^\xi) = \operatorname{im}(\alpha^\xi)$ is a quantum homogeneous space. Moreover, the kernel of β^ξ is the right ideal of $\mathcal{O}(M_q)$ generated by the β-coinvariants $\tau_1 - \tau_1(\xi)$, $\tau_2 - \tau_2(\xi)$. The kernel of α^ξ is the left ideal of $\mathcal{O}(M_q)$ generated by $\sigma_1 - \sigma_1(\xi)$, $\sigma_2 - \sigma_2(\xi)$.*

Remark 3.3 In some sense the map β^ξ (respectively α^ξ) is the best possible we can hope for. Suppose that $\varphi : \mathcal{O}(M_q) \to \mathcal{O}(D \setminus GL_q)$ is a map which is both a $\mathbb{C}$-algebra homomorphism and intertwines between the $\mathcal{O}(GL_q)$-corepresentations β and Δ. Then $\ker \varphi$ is a completely prime two-sided ideal in $\mathcal{O}(M_q)$, since $\mathcal{O}(D \setminus GL_q)$ is a domain. We shall see below that the trivial $\mathcal{O}(GL_q)$-corepresentation appears with multiplicity 1 in $\mathcal{O}(D \setminus GL_q)$. Therefore $\ker(\varphi)$ must contain $\tau_1 - \lambda_1$ and $\tau_2 - \lambda_2$ with some $\lambda_1, \lambda_2 \in \mathbb{C}$. Elementary commutator computations (exploiting the fact that τ_1 is non-central) show that $\varphi(x_{12}) = \varphi(x_{21}) = 0$, $\varphi(x_{11}) \in \mathbb{C}$, and $\varphi(x_{22}) \in \mathbb{C}$. In particular, $\operatorname{im}(\varphi) = \mathbb{C}$ for any φ compatible with both the algebra and comodule structures.

It will be convenient for the proof of Proposition 3.1 to pass from $\mathcal{O}(GL_q)$ to $\mathcal{O}(SL_q)$, the Hopf algebra quotient of $\mathcal{O}(GL_q)$ modulo the ideal generated by $\det_q - 1$. We keep the symbol Δ to denote the comultiplication in $\mathcal{O}(SL_q)$. Denote by π the natural epimorphism $\pi : \mathcal{O}(GL_q) \to \mathcal{O}(SL_q)$, and $a := \pi(x_{11})$, $b := \pi(x_{12})$, $c := \pi(x_{21})$, $d := \pi(x_{22})$. The group K of diagonal matrices in $SL(2, \mathbb{C})$ is a quantum subgroup of $\mathcal{O}(SL_q)$ with surjective Hopf algebra homomorphism $\pi_K : \mathcal{O}(SL_q) \to \mathcal{O}(K) = \mathbb{C}[z^{\pm 1}]$ given by $\pi_K(a) = z$, $\pi_K(b) = 0 = \pi_K(c)$, $\pi_K(d) = z^{-1}$. There is a natural left coaction $\lambda_K := (\pi_K \otimes \mathrm{id}) \circ \Delta$ and a right coaction $\rho_K := (\mathrm{id} \otimes \pi_K) \circ \Delta$ of $\mathcal{O}(K)$ on $\mathcal{O}(SL_q)$. The subalgebra of λ_K-coinvariants is denoted by $\mathcal{O}(K \setminus SL_q)$.

We shall use the corepresentation theory of $\mathcal{O}(SL_q)$, the material below can be found for example in [6, 4.2]. We assume that q is not a root of unity (or $q = 1$). Then $\mathcal{O}(SL_q)$ is cosemisimple, so any $\mathcal{O}(SL_q)$-comodule decomposes as the direct sum of irreducible subcomodules. The degree n homogeneous component P_n ($n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$) of the subalgebra $\mathbb{C}\langle a, b\rangle$ of $\mathcal{O}(SL_q)$ is an $n+1$-dimensional right $\mathcal{O}(SL_q)$-subcomodule. Denote the corresponding corepresentation by $T_{n/2}$. Then $T_l : P_{2l} \to P_{2l} \otimes \mathcal{O}(SL_q)$, $l \in \frac{1}{2}\mathbb{N}_0$, is a complete list of irreducible corepresentations of $\mathcal{O}(SL_q)$ up to isomorphism, see for example [6, 4.2.1]. We adopt the following notation: if $T : V \to V \otimes \mathcal{O}(SL_q)$ is a corepresentation of $\mathcal{O}(SL_q)$, then write $V[n] := \{v \in V \mid (\mathrm{id} \otimes \pi_K) \circ T(v) = v \otimes z^n\}$, where $n \in \mathbb{Z}$. Since any $\mathcal{O}(K)$-comodule decomposes as a direct sum of one-dimensional $\mathcal{O}(K)$-subcomodules, the comodule V decomposes as $\bigoplus_{n\in\mathbb{Z}} V[n]$. Assume that V is finite dimensional. By the *character* $\chi(T)$ of T we mean $\pi_K(f)$, where $f \in \mathcal{O}(SL_q)$ is the sum of the diagonal matrix elements of T with repect to an arbitrarily chosen basis of V. In other words, $\chi(T) = \sum_{n\in\mathbb{Z}} \dim_{\mathbb{C}}(V[n])z^n \in \mathcal{O}(K)$. (Note that our use of the word 'character' slightly differs from that of [6, 11.2.2], where f is called the character of T, and $\chi(T) = \pi_K(f)$ the character of the $\mathcal{O}(K)$-corepresentation $(\mathrm{id} \otimes \pi_K) \circ T$.) For the irreducible $\mathcal{O}(SL_q)$-corepresentations T_l we have that

$$\dim(P_{2l}[n]) = \begin{cases} 1, \text{ if } n = 2l, 2l-2, 2l-4, \dots, -2l \\ 0, \text{ otherwise,} \end{cases}$$

so $\chi(T_l) = z^l + z^{l-2} + \cdots + z^{-l}$. In particular, the set $\{\chi(T_l) \mid l \in \frac{1}{2}\mathbb{N}_0\}$ is linearly independent, and the multiplicity of T_j as a direct summand of T equals the coefficient of $\chi(T_j)$ in $\chi(T)$, expressed as a linear combination of the elements $\chi(T_l)$.

The quantum quotient space $\mathcal{O}(S^2_{q,\infty}) := \mathcal{O}(K \setminus SL_q)$ is a well studied object, belonging to the one-parameter family of *quantum* 2*-spheres* introduced in [11]. The facts we need about $\mathcal{O}(S^2_{q,\infty})$ can be found in [6, 4.5]. As a subalgebra of $\mathcal{O}(SL_q)$, the quantum 2-sphere $\mathcal{O}(S^2_{q,\infty})$ is generated by the elements

$$x_{-1} := (1+q^2)^{1/2}ac, \quad x_0 := 1 + (q+q^{-1})bc, \quad x_{+1} := (1+q^2)^{1/2}db.$$

The right $\mathcal{O}(SL_q)$-corepresentation on $\mathcal{O}(S^2_{q,\infty})$ is isomorphic to $T_0 \oplus T_1 \oplus T_2 \oplus \cdots$. The linear span of the products of length at most n in the generators x_{-1}, x_0, x_{+1}, denoted by W^n, is a right $\mathcal{O}(SL_q)$-subcomodule. The corepresentation on W^n is isomorphic to $\bigoplus_{l=0}^n T_l$. Moreover, the simple subcomodule of W^n corresponding to the summand T_n is the subcomodule generated by x_{-1}^n, because $W^n[2n] = \mathbb{C}x_{-1}^n$ and $W^n[d] = 0$ if $d > 2n$. In particular, as an $\mathcal{O}(SL_q)$-comodule $\mathcal{O}(S^2_{q,\infty})$ is generated by the powers of x_{-1}.

The reason that we are able to switch from $\mathcal{O}(GL_q)$ to $\mathcal{O}(SL_q)$ is in the following lemma:

Lemma 3.4 *The restriction of π maps $\mathcal{O}(D \setminus GL_q)$ isomorphically onto $\mathcal{O}(K \setminus SL_q)$.*

Proof. The coaction λ_D defines a $\mathbb{Z} \oplus \mathbb{Z}$-grading on $\mathcal{O}(GL_q)$: for $(m,n) \in \mathbb{Z}^2$ set

$$\mathcal{O}(GL_q)_{(m,n)} := \{f \in \mathcal{O}(GL_q) \mid \lambda_D(f) = t_1^m t_2^n \otimes f\},$$

then $\mathcal{O}(GL_q) = \bigoplus_{(m,n)\in\mathbb{Z}^2} \mathcal{O}(GL_q)_{(m,n)}$. By definition $\mathcal{O}(D \setminus GL_q) = \mathcal{O}(GL_q)_{(0,0)}$. For the algebra generators of $\mathcal{O}(GL_q)$ we have $x_{11}, x_{12} \in \mathcal{O}(GL_q)_{(1,0)}$, $x_{21}, x_{22} \in \mathcal{O}(GL_q)_{(0,1)}$, and $\det_q^{-1} \in \mathcal{O}(GL_q)_{(-1,-1)}$. Therefore $\mathcal{O}(D \setminus GL_q)$ contains $x_{11}x_{21}/\det_q$, $x_{12}x_{21}/\det_q$, $x_{12}x_{22}/\det_q$. These elements are mapped via π onto a system of algebra generators of $\mathcal{O}(K \setminus SL_q)$.

Since $\det_q - 1$ is not homogeneous with respect to this $\mathbb{Z}^2$-grading and $\mathcal{O}(GL_q)$ is a domain, we have that $\ker(\pi) = (\det_q - 1)\mathcal{O}(GL_q)$ does not contain any homogeneous elements. In particular, the kernel of π is disjoint from $\mathcal{O}(D \setminus GL_q)$. $\square$

The $\mathcal{O}(GL_q)$-coaction β is replaced by the $\mathcal{O}(SL_q)$-coaction

$$\psi := (\mathrm{id} \otimes \pi) \circ \beta : \mathcal{O}(M_q) \to \mathcal{O}(M_q) \otimes \mathcal{O}(SL_q),$$

and α is replaced by

$$\varphi := (\mathrm{id} \otimes \pi) \circ \alpha : \mathcal{O}(M_q) \to \mathcal{O}(M_q) \otimes \mathcal{O}(SL_q).$$

For $\xi \in M_q(\mathbb{C})$ the maps $\psi^\xi, \varphi^\xi : \mathcal{O}(M_q) \to \mathcal{O}(SL_q)$ are defined in an obvious way: $\psi^\xi = (\mathrm{ev}_\xi \otimes \mathrm{id}) \circ \psi = \pi \circ \beta^\xi$ and $\varphi^\xi = (\mathrm{ev}_\xi \otimes \mathrm{id}) \circ \alpha = \pi \circ \alpha^\xi$. Propositions 2.3 and 2.6 combined with Lemma 3.4 translate to the following. The map ψ^ξ intertwines between the $\mathcal{O}(SL_q)$-corepresentations ψ and Δ, whereas φ^ξ intertwines between the $\mathcal{O}(SL_q)$-corepresentations φ and Δ. We have that $\ker(\psi^\xi) = \ker(\beta^\xi)$ and $\ker(\varphi^\xi) = \ker(\alpha^\xi)$. If ξ is diagonal then $\mathrm{im}(\psi^\xi) \subseteq \mathcal{O}(K \setminus SL_q)$ with equality if and only if $\mathrm{im}(\beta^\xi) = \mathcal{O}(D \setminus GL_q)$, and $\mathrm{im}(\varphi^\xi) \subseteq \mathcal{O}(K \setminus SL_q)$ with equality if and only if $\mathrm{im}(\alpha^\xi) = \mathcal{O}(D \setminus GL_q)$. Moreover, $\mathrm{im}(\beta^\xi) = (\mathcal{O}(D \setminus GL_q))^n$ if and only if $\mathrm{im}(\psi^\xi) = W^n$, and $\mathrm{im}(\alpha^\xi) = (\mathcal{O}(D \setminus GL_q))^n$ if and only if $\mathrm{im}(\varphi^\xi) = W^n$.

Proof of Proposition 3.1.

The kernel of ψ^ξ contains $\sum_{i=1}^2 (\tau_i - \tau_i(\xi))\mathcal{O}(M_q)$ by Proposition 2.4, therefore ψ^ξ factors through the composition of the natural homomorphisms

$$\mathcal{O}(M_q) \xrightarrow{\mu} C \xrightarrow{\eta} B,$$

where $C := \mathcal{O}(M_q)/(\tau_2 - \tau_2(\xi))\mathcal{O}(M_q)$, $B := C/uC$ with $u := \mu(\tau_1 - \tau_1(\xi))$. Set $y_{ij} := \mu(x_{ij})$. Note that $\tau_2 - \tau_2(\xi) = \det_q - \xi_1\xi_2$ is a central element in $\mathcal{O}(M_q)$, so C is a quotient algebra of $\mathcal{O}(M_q)$. It is a domain by [5]. Since $\tau_2 - \tau_2(\xi)$ is a ψ-coinvariant, $(\tau_2 - \tau_2(\xi))\mathcal{O}(M_q)$ is a subcomodule of $\mathcal{O}(M_q)$ by (4), hence C is a factorcomodule of $\mathcal{O}(M_q)$; the corresponding corepresentation is denoted by ψ_C. Similarly, B is a factorcomodule of C, with corepresentation ψ_B. Our next aim is to show that

$$\psi_B \cong T_0 \oplus T_1 \oplus T_2 \oplus \cdots, \tag{7}$$

and moreover that, $\eta(y_{21})^r$ generates the simple subcomodule of B corresponding to T_r for $r \in \mathbb{N}_0$.

Denote by $\mathcal{O}(M_q)^{\leq r}$ the linear span of monomials of degree $\leq r$ in the generators x_{ij}, and $C^r := \mu(\mathcal{O}(M_q)^{\leq r})$, $B^r := \eta(C^r)$. Note that $\mathcal{O}(M_q)^{\leq r}$, C^r, D^r are subcomodules. The symbols ψ_{C^r}, ψ_{B^r} stand for the obvious subcorepresentations of ψ_C, ψ_B.

For $r \in \mathbb{N}_0$, a vector space basis of C^r is

$$\{y_{11}^{i+1} y_{12}^j y_{21}^k,\ y_{12}^l y_{21}^m y_{22}^n \mid i,j,k,l,m,n \in \mathbb{N}_0,\ i+j+k \leq r-1,\ l+m+n \leq r\} \tag{8}$$

(by the same argument as the corresponding result for $\mathcal{O}(SL_q)$ is proved in [6, 4.1.5]). From (8) we see that the character of the corepresentation ψ_{C^r} is

$$\chi(\psi_{C^r}) = \sum_{i+j+k \leq r-1} z^{2(k-j)} + \sum_{l+m+n \leq r} z^{2(m-l)}. \tag{9}$$

(This character does not depend on q or ξ.) In particular, $C^r[n] = 0$ if $n > 2r$, and $C^r[2r] = \mathbb{C}y_{21}^r$, implying that the corepresentation ψ_{C^r} has a unique irreducible summand isomorphic to T_r, and y_{21}^r generates the corresponding simple subcomodule of C^r. It follows that $\eta(y_{21}^r)$ is either zero, or it generates a simple subcomodule of B, on which the given corepresentation is isomorphic to T_r.

In order to show that $\eta(y_{21}^r)$ is non-zero, we introduce a $\mathbb{Z}$-grading deg on $\mathcal{O}(M_q)$ as follows: $\deg(x_{11}) := 1$, $\deg(x_{22}) := -1$, $\deg(x_{12}) := 0$, $\deg(x_{21}) := 0$. It is easy to check that deg extends to an algebra-grading of $\mathcal{O}(M_q)$. Moreover, $\ker(\mu)$ is a homogeneous ideal, so C inherits the grading. Note that u is non-homogeneous, and $\deg(y_{21}^r) = 0$. Since C is a domain, any non-zero multiple of u is non-homogeneous. Thus y_{21}^r is not contained in uC. In other words, $\eta(y_{21}^r)$ is non-zero for any $r \in \mathbb{N}_0$.

Consequently, ψ_{B^r} contains a subcorepresentation isomorphic to $T_0 \oplus T_1 \oplus \cdots T_r$, on the subcomodule generated by $1, \eta(y_{21}), \ldots, \eta(y_{21}^r)$. On the other hand, $\ker(\eta) \cap C^r = uC \cap C^r$ clearly contains uC^{r-1}, so B^r is a factorcomodule of C^r/uC^{r-1}. Since u is a coinvariant and C is a domain, by (4) we get that the corepresentation on uC^{r-1} is isomorphic to $\psi_{C^{r-1}}$, hence the character of the corepresentation on C^r/uC^{r-1} is $\chi(\psi_{C^r}) - \chi(\psi_{C^{r-1}})$. Using (9) we get that

$$\begin{aligned}
\chi(\psi_{C^r}) - \chi(\psi_{C^{r-1}}) &= \sum_{i+j+k=r-1} z^{2(k-j)} + \sum_{l+m+n=r} z^{2(m-l)} \\
&= \sum_{j+k\leq r-1} z^{2(k-j)} + \sum_{l+m\leq r} z^{2(m-l)} \\
&= \sum_{s=0}^{r} \chi(T_s).
\end{aligned} \tag{10}$$

So the corepresentation on C^r/uC^{r-1} is also isomorphic to $\bigoplus_{s=0}^r T_r$. It follows that $C^r \cap uC = uC^{r-1}$, and $\psi_{B^r} \cong \bigoplus_{s=0}^r T_r$. Moreover, the subcomodule corresponding to T_s is generated by $\eta(y_{21}^s)$. This shows (7), since $B = \bigcup_{r=0}^\infty B^r$.

As an immediate consequence of (7) and the fact that $\psi^\xi = \nu \circ \eta \circ \mu$ for some morphism $\nu : B \to \mathcal{O}(S_{q,\infty}^2)$ of comodules, we obtain that the image of ψ^ξ is the subcomodule of $\mathcal{O}(S_{q,\infty}^2)$ generated by $\{\nu(\eta(y_{21}^n)) \mid n \in \mathbb{N}_0\} = \{\psi^\xi(x_{21}^n) \mid n \in \mathbb{N}_0\}$. Moreover, the kernel of ψ^ξ is $(\eta \circ \mu)^{-1}(\ker(\nu))$. By (7) B is the direct sum of pairwise non-isomorphic simple subcomodules, and so any subcomodule is a direct sum of some of these simple subcomodules; in particular, $\ker(\nu)$ is the $\mathcal{O}(SL_q)$-subcomodule of B generated by $\{\eta(y_{21}^n) \mid \nu(\eta(y_{21}^n)) = 0\}$. Therefore $(\eta \circ \mu)^{-1}(\ker(\nu))$ is the sum of $\ker(\eta \circ \mu) = \sum_{i=1,2}(\tau_i - \tau_i(\xi))\mathcal{O}(M_q)$ and the subcomodule of $\mathcal{O}(M_q)$ generated by $\{x_{21}^n \mid \psi^\xi(x_{21}^n) = 0\}$. So (i), (ii), (iii), and (iv) follow from the fact that $W^n \leq \mathcal{O}(S_{q,\infty}^2)$ is generated by $\{x_{-1}^r \mid r = 0, 1, \ldots, n\}$ as an $\mathcal{O}(SL_q)$-comodule, and from the evaluation of $\psi^\xi(x_{21}^n)$ given below.

Since ψ^ξ is not an algebra homomorphism, it is helpful to write it as the composition $\psi^\xi = p \circ \Psi$, where Ψ is the map

$$\Psi := ((S \circ \pi) \otimes \mathrm{ev}_\xi \otimes \pi) \circ \Delta^{(2)} : \mathcal{O}(M_q) \to \mathcal{O}(SL_q) \otimes \mathcal{O}(SL_q),$$

and $p : \mathcal{O}(SL_q) \otimes \mathcal{O}(SL_q) \to \mathcal{O}(SL_q)$ is the multiplication map in $\mathcal{O}(SL_q)$. Introduce a new multiplication $*$ on $\mathcal{O}(SL_q) \otimes \mathcal{O}(SL_q)$ given by $(x_1 \otimes y_1) * (x_2 \otimes y_2) = x_2 x_1 \otimes y_1 y_2$. Observe that Ψ is an algebra homomorphism from $\mathcal{O}(M_q)$ to $(\mathcal{O}(SL_q) \otimes \mathcal{O}(SL_q), *)$. We have that

$$\Psi \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} \xi_1 d \otimes a - q^{-1}\xi_2 b \otimes c & \xi_1 d \otimes b - q^{-1}\xi_2 b \otimes d \\ -q\xi_1 c \otimes a + \xi_2 a \otimes c & -q\xi_1 c \otimes b + \xi_2 a \otimes d \end{bmatrix}. \tag{11}$$

By induction on n we show that

$$\psi^\xi(x_{21}^n) = (-q)^n(\xi_1 - \xi_2)(\xi_1 - q^2\xi_2)\cdots(\xi_1 - q^{2n-2}\xi_2)c^n a^n. \tag{12}$$

Indeed, (12) holds for $n = 1$ by (11). Assume that (12) holds for n. Then

$$\begin{aligned} \psi^\xi(x_{21}^{n+1}) &= p(\Psi(x_{21}^{n+1})) = p(\Psi(x_{21}^n) * \Psi(x_{21})) \\ &= p(\Psi(x_{21}^n) * (-q\xi_1 c \otimes a + \xi_2 a \otimes c)) \\ &= -q\xi_1 c p(\Psi(x_{21}^n))a + \xi_2 a p(\Psi(x_{21}^n))c \\ &= (-q\xi_1 + q^{2n+1}\xi_2)c\psi^\xi(x_{21}^n)a, \end{aligned}$$

and formula (12) easily follows for $n+1$. Hence if $\prod_{i=0}^{n-1}(\xi_1 - q^{2i}\xi_2) \neq 0$, then $\psi^\xi(x_{21}^n)$ is a non-zero scalar multiple of x_{-1}^n, and if $\prod_{i=0}^{n-1}(\xi_1 - q^{2i}\xi_2) = 0$, then $\psi^\xi(x_{21}^n) = 0$.

The same arguments show (i'), (ii'), (iii'), (iv'), by using the fact that

$$\varphi^\xi(x_{21}^n) = (-q)^n(\xi_1 - q^{-2}\xi_2)(\xi_1 - q^{-4}\xi_2)\cdots(\xi_1 - q^{-2n}\xi_2)a^n c^n.$$

□

4 Co-orbit maps for non-diagonal 2×2 matrices

In this section we still restrict to the case $N = 2$. The kernel and the image of the co-orbit map can be described also when ξ is a non-diagonal $\mathbb{C}$-point of M_q.

Proposition 4.1 *Let $0 \neq q \in \mathbb{C}$, and take $\xi := \begin{bmatrix} 0 & \xi_1 \\ 0 & 0 \end{bmatrix}$, where $\xi_1 \neq 0$.*

(i) The image of β^ξ is the subalgebra of $\mathcal{O}(GL_q)$ generated by x_{21}^2/det_q, $x_{21}x_{22}/det_q$, x_{22}^2/det_q. In particular, $\mathrm{im}(\beta^\xi)$ is a quantum homogeneous space.

(ii) The kernel of β^ξ is the right ideal of $\mathcal{O}(M_q)$ generated by τ_1 and τ_2.

(i') $\mathrm{im}(\alpha^\xi) = \mathbb{C}\langle x_{21}^2/det_q, x_{21}x_{22}/det_q, x_{22}^2/det_q\rangle$.

(ii') $\ker(\alpha^\xi) = \sum_{i=1}^2 \mathcal{O}(M_q)\sigma_i$.

Again it is more convenient to switch from the $\mathcal{O}(GL_q)$-coactions to the corresponding $\mathcal{O}(SL_q)$-coactions.

Lemma 4.2 *The restriction of π maps $\mathbb{C}\langle x_{21}^2/det_q, x_{21}x_{22}/det_q, x_{22}^2/det_q\rangle$ isomorphically onto $\mathbb{C}\langle c^2, cd, d^2\rangle$.*

Proof. The subalgebras $\mathbb{C}\langle x_{21}/\det_q, x_{22}/\det_q\rangle$ of $\mathcal{O}(GL_q)$ and $\mathbb{C}\langle c, d\rangle$ of $\mathcal{O}(SL_q)$ are isomorphic to the coordinate ring of the quantum plane: the generators q-commute, and satisfy no more relations, as can be seen from standard basis arguments. The two algebras in our statement correspond to the subalgebra of the coordinate ring of the quantum plane generated by its quadratic elements, or in other words, the sum of the even degree homogeneous components. $\square$

Proof of Proposition 4.1. As before, $\psi = (\mathrm{id} \otimes \pi) \circ \beta$ and $\psi^\xi = (\mathrm{ev}_\xi \otimes \mathrm{id}) \circ \psi$. The statements about β^ξ translate by Lemma 4.2 to the assertions $\ker(\psi^\xi) = \sum_{i=1}^2 \tau_i \mathcal{O}(M_q)$ and $\mathrm{im}(\psi^\xi) = \mathbb{C}\langle c^2, cd, d^2\rangle$.

As in the proof of Proposition 3.1, decompose ψ^ξ as $p \circ \Psi$, where $\Psi = ((S \circ \pi) \otimes \mathrm{ev}_\xi \otimes \pi) \circ \Delta^{(2)}$ and p is the multiplication map in $\mathcal{O}(M_q)$. By definition we have

$$\Psi \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} \xi_1 d \otimes c & \xi_1 d \otimes d \\ -q\xi_1 c \otimes c & -q\xi_1 c \otimes d \end{bmatrix}. \tag{13}$$

Since Ψ is an algebra homomorphism from $\mathcal{O}(M_q)$ to $(\mathcal{O}(SL_q) \otimes \mathcal{O}(SL_q), *)$, it is easy to see from the images of the generators that $\mathrm{im}(\Psi)$ is contained in $\sum_{k=0}^\infty \mathbb{C}\langle c, d\rangle_k \otimes \mathbb{C}\langle c, d\rangle_k$, where $\mathbb{C}\langle c, d\rangle_k$ denotes the degree k component of $\mathbb{C}\langle c, d\rangle$. This space is mapped by p into $W := \mathbb{C}\langle c^2, cd, d^2\rangle$, showing that $\mathrm{im}(\beta^\xi) \subseteq W$. We have

$$\psi^\xi(x_{21}^n) = p \circ \Psi(x_{21}^n) = p(\Psi(x_{21})^n) = (-1)^n q^n \xi^n c^{2n},$$

implying that $\mathrm{im}(\psi^\xi)$ contains c^{2n} for all $n \in \mathbb{N}_0$. Similarly, $\psi^\xi(x_{21}^i x_{12}^j x_{11}^k)$ is a non-zero scalar multiple of $(c^2)^i (d^2)^j (dc)^k$, hence we obtain (i).

Set $C := \mathcal{O}(M_q)/\tau_2\mathcal{O}(M_q)$ and $B := C/\mu(\tau_1)C$, where $\mu : \mathcal{O}(M_q) \to C$ is the natural homomorphism. These are graded homomorphic images of $\mathcal{O}(M_q)$ (endowed with the usual grading). By Proposition 2.4 the linear map $\beta^\xi : \mathcal{O}(M_q) \to W$ factors through B, denote by $\eta : B \to W$ the induced surjective linear map. We shall show that η is an isomorphism (this is clearly equivalent to the assertion $\ker(\psi^\xi) = \sum_{i=1}^2 \tau_i \mathcal{O}(M_q)$). Define $\mathcal{O}(M_q)^{\le r}$ as the degree $\le r$ part of $\mathcal{O}(M_q)$, and write C^r,B^r, W^r for the image of $\mathcal{O}(M_q)^{\le r}$ in C, B, W. We know from [5] that C is a domain, and C^r has the basis (8). By homogeneity of $\mu(\tau_1)$ we have $C^r \cap \mu(\tau_1)C = \mu(\tau_1)C^{r-1}$. Hence $\dim(B^r) = \dim(C^r) - \dim(C^{r-1}) = (r+1)^2$ (the latter equality follows for example from (10)). Since $\dim(W^r) = (r+1)^2$ as well, the map $\eta\big|_{B^r} : B^r \to W^r$ is an isomorphism. This holds for all $r \in \mathbb{N}_0$, hence $\eta : B \to W$ is an isomorphism.

The same arguments work for α^ξ. $\square$

A similar result holds when $\xi = \begin{bmatrix} 0 & 0 \\ \xi_1 & 0 \end{bmatrix}$ is lower triangular.

Remark 4.3 $\mathbb{C}\langle c^2, cd, d^2\rangle$ and $\mathcal{O}(S^2_{q,\infty})$ are isomorphic as right $\mathcal{O}(SL_q)$-comodules. However, they are not isomorphic quantum homogeneous spaces, since they are not isomorphic as algebras. Indeed, $\mathcal{O}(S^2_{q,\infty})$ is generated by x_{-1} and x_{+1} as a $\mathbb{C}$-algebra with unity. Assume that $\mathbb{C}\langle c^2, cd, d^2\rangle$ is also generated by two elements. We may suppose that the generators are contained in the maximal ideal (c^2, cd, d^2). Then the images of the generators span the 3-dimensional $\mathbb{C}$-vector space $(c^2, cd, d^2)/(c^2, cd, d^2)^2$. This is a contradiction.

It is not difficult to check that $\mathcal{O}(SL_q)$ is a free left $\mathbb{C}\langle c^2, cd, d^2\rangle$-module. Therefore by [13, Theorem 1] the right coideal subalgebra $\mathbb{C}\langle c^2, cd, d^2\rangle$ of $\mathcal{O}(SL_q)$ can be realized as the space of coinvariants with respect to the left coaction of an appropriate quotient left $\mathcal{O}(SL_q)$-module coalgebra of $\mathcal{O}(SL_q)$ (called a left coisotropic quantum subgroup of $\mathcal{O}(SL_q)$ in [1]). This quotient left $\mathcal{O}(SL_q)$-module coalgebra can be viewed as the "stabilizer" of ξ.

5 The generic case

We return to quantum matrices of arbitrary size N, but assume that q is transcendental over the base field. It turns out that if the classical adjoint orbit of $\xi \in M(N, \mathbb{C})$ is of maximal dimension, then the kernel of the (quantized) co-orbit map coincides with the subset predicted by the classical theory, and is given in terms of coinvariants. For a diagonal ξ with pairwise different eigenvalues, the image of the co-orbit map also reflects the classical picture.

Let A be an arbitrary commutative ring and q a unit in A. We replace $\mathbb{C}$ by A in the constructions of Sections 1 and 2, and define the A-bialgebra $\mathcal{O}_A(M_q)$, the A-Hopf algebras $\mathcal{O}_A(GL_q)$, $\mathcal{O}_A(D)$, the right coaction $\beta_A : \mathcal{O}_A(M_q) \to \mathcal{O}_A(M_q) \otimes \mathcal{O}_A(GL_q)$, the β_A-coinvariants $\tau_1, \ldots, \tau_N \in \mathcal{O}_A(M_q)$. We say that an $N \times N$ matrix ξ is an *A-point of M_q*, if the entries of ξ are elements of A satisfying the relations (2). Such a ξ determines an A-algebra homomorphism $\mathrm{ev}_\xi : \mathcal{O}_A(M_q) \to A$, $x_{ij} \mapsto \xi_{ij}$, and the co-orbit map $\beta^\xi_A : \mathcal{O}_A(M_q) \to \mathcal{O}_A(GL_q)$ is defined as in Definition 2.1. The A-algebra $\mathcal{O}_A(M_q)$ is graded, the generators x_{ij} having degree 1. Write $\mathcal{O}_A(M_q)^{\leq d}$ for the subspace spanned by the homogeneous components of degree $\leq d$.

Let K be a subfield of $\mathbb{C}$, and let q be an indeterminate over K. We shall apply the above constructions in the case when $A = K(q)$, the field of rational functions in q, or when $A = K[q, q^{-1}]$, the ring of Laurent polynomials. When $(A, q) = (K, 1)$, then $\mathcal{O}_A(M_q)$ becomes $\mathcal{O}_K(M)$, the coordinate ring of the space of $N \times N$ matrices over K, and $\mathcal{O}_A(GL_q)$ becomes the coordinate ring $\mathcal{O}_K(GL)$ of the general linear group $GL(N, K)$.

Denote by $\eta : K[q, q^{-1}] \to K$ the K-algebra surjection mapping q to 1. The symbol η will stand also for the induced surjections $M(N, K[q, q^{-1}]) \to M(N, K)$, $\mathcal{O}_{K[q,q^{-1}]}(GL_q) \to \mathcal{O}_K(GL)$.

Lemma 5.1 *The elements $f_1, \dots, f_s \in \mathcal{O}_{K[q,q^{-1}]}(GL_q)$ are $K(q)$-linearly independent in $\mathcal{O}_{K(q)}(GL_q)$, if $\eta(f_1), \dots, \eta(f_s)$ are K-linearly independent in $\mathcal{O}_K(GL)$.*

Proof. Assume that $a_1 f_1 + \dots + a_s f_s = 0$ is a non-trivial relation with $a_i \in K(q)$. Multiplying by an appropriate element of $K[q]$ we may achieve that $a_i \in K[q, q^{-1}]$ for all i. Since $K[q, q^{-1}]$ is a unique factorization domain, cancelling an appropriate power of $q - 1$ we ensure that not all the a_i are contained in the ideal $\langle q - 1 \rangle$. Then apply the map η to get a non-trivial K-linear relation $\sum_{i=1}^{s} \eta(a_i)\eta(f_i) = 0$. $\square$

We fix a $K[q, q^{-1}]$-point ξ of M_q such that the centralizer of $\eta(\xi)$ in $M(N, K)$ has dimension N; that is, the adjoint orbit of $\eta(\xi)$ is of maximal dimension. For a fixed $d \in \mathbb{N}_0$ set

$$X^d := \mathcal{O}_{K(q)}(M_q)^{\leq d}, \quad X_0^d := \mathcal{O}_{K[q,q^{-1}]}(M_q)^{\leq d}, \quad X_1^d := \mathcal{O}_K(M)^{\leq d},$$

$$b^d := \beta^{\xi}_{K(q)}\big|_{X^d}, \quad b_0^d := \beta^{\xi}_{K[q,q^{-1}]}\big|_{X_0^d}, \quad b_1^d := \beta^{\eta(\xi)}_{K}\big|_{X_1^d},$$

$$Y^d := \operatorname{im}(b^d) \subset \mathcal{O}_{K(q)}(GL_q), \; Y_0^d := \operatorname{im}(b_0^d) \subset \mathcal{O}_{K[q,q^{-1}]}(GL_q), \; Y_1^d := \operatorname{im}(b_1^d) \subset \mathcal{O}_K(GL).$$

Then we have a commutative diagram

$$\begin{array}{ccccc} X^d & \supset & X_0^d & \xrightarrow{\eta} & X_1^d \\ \downarrow b^d & & \downarrow b_0^d & & \downarrow b_1^d \\ Y^d & \supset & Y_0^d & \xrightarrow{\eta} & Y_1^d. \end{array}$$

Denote by Z_1^d the kernel of b_1^d. It is the intersection of X_1^d and the ideal of $\mathcal{O}_K(M)$ generated by $\eta(\tau_i - \tau_i(\xi))$, $i = 1, \dots, N$, by the classical result [7, Theorem 10] (the base field is $\mathbb{C}$ in this paper; since $\beta^{\eta(\xi)}$ is obtained from $\beta^{\eta(\xi)}_K$ by extending scalars, and since the τ_i are defined over the field of rational numbers, the cited results of [7] hold over any subfield K of $\mathbb{C}$). Moreover, by Lemma 5.2 below, Z_1^d has a K-basis of the form

$$\Lambda_1 := \bigcup_{i=1}^{N} \{\eta(\tau_i - \tau_i(\xi))w \mid w \in \Gamma^i\},$$

where Γ^i is an appropriate set of monomials in the variables x_{kl} of degree $\leq d - i$, for $i = 1, \dots, N$.

Lemma 5.2 *Let R be a finitely generated commutative polynomial algebra over K, endowed with the usual grading. For $r \in R$ write $r = \bar{r} + \hat{r}$, where $\bar{r}$ is the highest degree homogeneous component of r. Let $u_1, \dots, u_N$ be given elements of R, and assume that $\bar{u}_1, \dots, \bar{u}_N$ is a regular sequence in R. Then for all f in the ideal generated by $u_1, \dots, u_N$, there exist elements $f_i \in R$, $i = 1, \dots, N$, such that $f = \sum_{i=1}^{N} u_i f_i$, and* $\deg(u_i f_i) \leq \deg(f)$, *$i = 1, \dots, N$.*

Proof. Apply induction on N. The case $N = 1$ is trivial. Assume that the lemma is true for $N-1$. Write $f \in \sum_{i=1}^{N} u_i R$ as $f = \sum_{i=1}^{N} u_i h_i$, where $d := \max\{\deg(u_i h_i) \mid i = 1, \dots, N\}$ is minimal. If $d \le \deg(f)$, then we are done. Suppose that $d > \deg(f)$. Without loss of generality we may assume that $\deg(u_N h_N) = d$ because any permutation of $\bar{u}_1, \dots, \bar{u}_N$ is a regular sequence, see the corollary to [9, Theorem 16.3]. (We note that it is not essential for the proof to make this assumption, however, it simplifies the notation below.) Then

$$\sum_{i:\deg(u_i h_i)=\deg(u_N h_N)} \bar{u}_i \bar{h}_i = 0.$$

By assumption $\bar{u}_N$ is not a zero-divisor modulo $\sum_{i=1}^{N-1} \bar{u}_i R$, hence $\bar{h}_N = \sum_{i=1}^{N-1} \bar{u}_i g_i$ with appropriate homogeneous elements g_i, where $\deg(\bar{u}_i g_i) = \deg(\bar{h}_N)$, $i = 1, \dots, N-1$. Thus

$$\begin{aligned}
u_N h_N &= u_N(\bar{h}_N + \hat{h}_N) \\
&= u_N \hat{h}_N + \sum_{i=1}^{N-1} u_N \bar{u}_i g_i \\
&= u_N \hat{h}_N + \sum_{i=1}^{N-1} u_N (u_i - \hat{u}_i) g_i \\
&= \sum_{i=1}^{N-1} u_i (u_N g_i) + u_N(\hat{h}_N - \sum_{i=1}^{N-1} \hat{h}_i g_i).
\end{aligned}$$

Obviously $\deg(u_i u_N g_i) \le d$, $\deg(u_N \hat{u}_i g_i) < d$ for $i = 1, \dots, N-1$. Set $b_i := h_i + u_N g_i$ for $i = 1, \dots, N-1$, and $b_N := \hat{h}_N - \sum_{i=1}^{N-1} \hat{u}_i g_i$. We have $f = \sum_{i=1}^{N} u_i b_i$, with $\deg(u_i b_i) \le d$ for $i = 1, \dots, N-1$, and $\deg(u_N b_N) < \deg(u_N h_N) = d$. We claim that $\deg(\sum_{i=1}^{N-1} u_i b_i) < d$. Indeed, either $\deg(\sum_{i=1}^{N-1} u_i b_i) \le \deg(u_N b_N) < d$, or $\deg(\sum_{i=1}^{N-1} u_i b_i) > \deg(u_N b_N)$ implying $\deg(\sum_{i=1}^{N-1} u_i b_i) = \deg(f) < d$. By the induction hypothesis on N, there exist $c_1, \dots, c_{N-1} \in R$ with $\sum_{i=1}^{N-1} u_i b_i = \sum_{i=1}^{N-1} u_i c_i$, and $\deg(u_i c_i) \le \deg(\sum_{i=1}^{N-1} u_i b_i) < d$ for $i = 1, \dots, N-1$. Therefore

$$f = u_1 c_1 + \dots + u_{N-1} c_{N-1} + u_N b_N,$$

where each summand on the right hand side has degree $< d$. This contradicts to our assumption on the minimality of d. $\square$

Since $\eta(\tau_1), \dots, \eta(\tau_N)$ is a regular sequence in $\mathcal{O}_K(M)$ by [7, Theorem 10], we may apply Lemma 5.2 to conclude the existence of the basis Λ_1 in Z_1^d. Our next aim is to show that Λ_1 can be lifted to a basis of the kernel b^d. There is a $\mathbb{Z}^N$-grading on the spaces X^d, X_1^d, Y_d, Y_1^d, determined by a right coaction of $\mathcal{O}_A(D)$. This coaction is $(\mathrm{id} \otimes \pi_D) \circ \beta_A$ for X^d, X_1^d, and $(\mathrm{id} \otimes \pi_D) \circ \Delta$ for Y^d, Y_1^d. (Recall that a coaction $\varphi : V \to V \otimes \mathcal{O}_A(D)$ yields the direct sum decomposition $V = \bigoplus_{a \in \mathbb{Z}^N} \{v \in V \mid \varphi(v) = v \otimes t_1^{a_1} \cdots t_N^{a_N}\}$.) The maps b^d, b_1^d, η are compatible with the $\mathbb{Z}^N$-grading, because b^d, b_1^d are homomorphisms of $\mathcal{O}_A(GL_q)$-comodules, hence they are homomorphisms of $\mathcal{O}_A(D)$-comodules, whereas η

maps a monomial in $\mathcal{O}_{K[q,q^{-1}]}(GL_q)$ to formally the same monomial in $\mathcal{O}_K(GL)$. Write $\chi(V)$ for the Hilbert series of a finite dimensional $\mathbb{Z}^N$-graded vector space V, so $\chi(V)$ is an element of $\mathbb{Z}[t_1^{\pm 1}, \dots, t_N^{\pm 1}]$. In other words, $\chi(V)$ is the character of the corresponding corepresentation of $\mathcal{O}_A(D)$ on V. Introduce a partial order $\geq$ on $\mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ by defining $f \geq g$ if all coefficients of $f - g$ are non-negative. Obviously $W \leq V$ and $\chi(W) = \chi(V)$ imply $W = V$.

Denote by Z^d the intersection of X^d and the right ideal of $\mathcal{O}_{K(q)}(GL_q)$ generated by $\tau_i - \tau_i(\xi)$, $i = 1, \dots, N$. The same argument as in the proof of Proposition 2.4 shows that $Z^d \subseteq \ker(b^d)$.

Proposition 5.3 *For all $d \in \mathbb{N}_0$ we have the following:*

(i) the kernel of b^d is Z^d;

(ii) the Hilbert series $\chi(Y^d)$ equals $\chi(Y_1^d)$.

Proof. Choose a set Ω_1 of monomials of degree $\leq d$ in $\mathcal{O}_K(M)$ such that $\Lambda_1 \cup \Omega_1$ is a K-basis of X_1^d. Set

$$\Lambda := \{(\tau_i - \tau_i(\xi))\hat{w} \mid i = 1, \dots, N;\ w \in \Gamma^i\},$$

where $\hat{w}$ is a chosen monomial in X_0^d with $\eta(\hat{w}) = w$. The elements of Λ are multihomogeneous with respect to the $\mathbb{Z}^N$-grading introduced on X^d. Similarly, lift each $v \in \Omega_1$ to a monomial $\hat{v} \in X_0^d$ with $\eta(\hat{v}) = v$. We obtain the set $\Omega := \{\hat{v} \mid v \in \Omega_1\}$ of multihomogeneous elements in X^d.

As we noted above, Λ is a subset of $\ker(b^d)$, and it is $K(q)$-linearly independent by Lemma 5.1. Therefore

$$\chi(\ker(b^d)) \geq \chi(Z^d) \geq \chi(\mathrm{Span}_{K(q)}\{\Lambda\}) = \chi(\mathrm{Span}_K\{\Lambda_1\}) = \chi(Z_1^d). \tag{14}$$

Since $Z_1^d = \ker(b_1^d)$, we have that Ω_1 is mapped under b_1^d to a basis of Y_1^d. Again by Lemma 5.1 we have that $b^d(\Omega)$ is a $K(q)$-linearly independent subset of Y^d. Thus

$$\chi(Y^d) \geq \chi(\mathrm{Span}_{K(q)}\{b^d(\Omega)\}) = \chi(\mathrm{Span}_K\{b_1^d(\Omega_1)\}) = \chi(Y_1^d). \tag{15}$$

It is well known (see [6, 9.2.1 Proposition 6]) that there is a set of monomials in the variables x_{ij} which is a basis both for X^d and X_1^d. It follows that

$$\chi(Y^d) + \chi(\ker(b^d)) = \chi(X^d) = \chi(X_1^d) = \chi(Y_1^d) + \chi(Z_1^d). \tag{16}$$

Comparing (14), (15), (16) we obtain that all inequalities in (14) and (15) must be equalities. In particular, $\chi(Y^d) = \chi(Y_1^d)$ in (15) and $\chi(\ker(b^d)) = \chi(\mathrm{Span}_{K(q)}\{\Lambda\})$ in (14), implying $\ker(b^d) = Z^d = \mathrm{Span}_{K(q)}\{\Lambda\}$. $\square$

As an immediate corollary we obtain the following.

Theorem 5.4 *Let ξ be a $K(q)$-point of M_q such that the centralizer of $\eta(\xi)$ in $M(N,K)$ has dimension N. Then the kernel of $\beta^{\xi}_{K(q)}$ coincides with the right ideal $\sum_{i=1}^N(\tau_i - \tau_i(\xi))\mathcal{O}_{K(q)}(M_q)$.*

We determine the comodule structure of $\beta^{\xi}_{K(q)}$ with the aid of corepresentation theory. The material we shall summarize below can be found in [6, 11.5] (the base field is $\mathbb{C}$ in that book; however, the irreducible corepresentations of $\mathcal{O}(GL_q)$ are defined over the field of rational numbers, so the results mentioned below obviously hold over K and $K(q)$). In the sequel (A,q) will stand for either of $(K(q),q)$ or $(K,1)$. Recall that $\mathcal{O}_A(GL_q)$ is cosemisimple. The irreducible corepresentations are indexed by the set P of dominant integral weights for $GL(N)$. Denote by $T_q(\lambda)$ the irreducible $\mathcal{O}_A(GL_q)$-corepresentation associated with $\lambda \in P$, write $\chi(T_q(\lambda))$ for its character. Recall that by the *character* $\chi(T)$ of a finite dimensional $\mathcal{O}_A(GL_q)$-corepresentation $T: V \to V \otimes \mathcal{O}_A(GL_q)$ we mean the image under π_D of the sum of the diagonal matrix coefficients. In other words, the character $\chi(T)$ is the same as the Hilbert series $\chi(V)$ with respect to the $\mathbb{Z}^N$-grading of V determined by the $\mathcal{O}_A(D)$-corepresentation $(\mathrm{id}\otimes\pi_D)\circ T$. Any $\mathcal{O}_A(GL_q)$-corepresentation decomposes as a direct sum of copies of the irreducible corepresentations $T_q(\lambda)$. Given a finite dimensional corepresentation T, the multiplicity of $T_q(\lambda)$ as a summand of T is the same as the (uniquely determined) coefficient of $\chi(T_q(\lambda))$ in $\chi(T)$, expressed as a linear combination of the characters $\chi(T_q(\mu))$, $\mu \in P$. We shall need also the fact that the character of the irreducible $\mathcal{O}_{K(q)}(GL_q)$-corepresentation $T_q(\lambda)$ coincides with the character of the $\mathcal{O}_K(GL)$-corepresentation $T_1(\lambda)$ (which coincides with the character of the irreducible rational representation of $GL(N,\mathbb{C})$ associated with λ). Consequently, if we are given an $\mathcal{O}_{K(q)}(GL_q)$-corepresentation $T: V \to V \otimes \mathcal{O}_{K(q)}(GL_q)$ and an $\mathcal{O}_K(GL)$-corepresentation $U: W \to W \otimes \mathcal{O}_K(GL)$ such that $\chi(V) = \chi(W)$, then the multiplicity of $T_q(\lambda)$ as a summand of T is the same as the multiplicity of $T_1(\lambda)$ as a summand of U, for all $\lambda \in P$.

To simplify the notation we write $T(\lambda) := T_1(\lambda)$. The $\mathcal{O}_K(GL)$-corepresentation on $\mathrm{im}(\beta^{\eta(\xi)}_K)$ decomposes as

$$\mathrm{im}(\beta^{\eta(\xi)}_K) \cong \bigoplus_{\lambda\in P} m(\lambda)T(\lambda),$$

where $m(\lambda)$ denotes the dimension of the zero weight space in the dual corepresentation $T^*(\lambda)$, by [7, Theorem 0.4 and formula (0.1.6)]. Although the result is stated in [7] in the language of representations, it can clearly be converted to the language of corepresentations. Indeed, observe that the character of the $\mathcal{O}_K(GL)$-corepresentation on $\mathrm{im}(\beta^{\eta(\xi)}_K)$ coincides with the usual formal character of the natural right action of $GL(N,K)$ on this space, and $\mathrm{im}(\beta^{\eta(\xi)}_K)$ is the coordinate ring of the closure of the adjoint orbit of $\eta(\xi)$. So $m(\lambda)$ is the dimension of the subspace of the underlying vector space of $T^*(\lambda)$ consisting of the vectors v with $(\mathrm{id}\otimes\pi_D)\circ T^*(\lambda)(v) = v \otimes 1$. In particular, $m(\lambda)$ is finite for all $\lambda \in P$, and does not depend on $\eta(\xi)$ (still assuming that the centralizer of $\eta(\xi)$ has dimension

N). The multiplicity of $T_q(\lambda)$ in the $\mathcal{O}_{K(q)}(GL_q)$-corepresentation on $\mathcal{O}_{K(q)}(GL_q)$ is the dimension of $T_q(\lambda)$ (see [6, 11.5.4 Theorem 51 and 11.1.4 Proposition 8 (ii)]), hence the subcorepresentation on $\mathrm{im}(\beta^{\xi}_{K(q)})$ decomposes as

$$\mathrm{im}(\beta^{\xi}_{K(q)}) \cong \bigoplus_{\lambda\in P} m^{\xi}_q(\lambda) T_q(\lambda)$$

with $m^{\xi}_q(\lambda)$ finite for all $\lambda \in P$.

Proposition 5.5 *Let ξ be a $K(q)$-point of M_q such that the centralizer of $\eta(\xi)$ in $M(N, K)$ has dimension N. Then for all $\lambda \in P$ we have $m^{\xi}_q(\lambda) = m(\lambda)$.*

Proof. Denote by $m^d(\lambda)$ the multiplicity of $T(\lambda)$ as a summand in Y^d_1, and denote by $m^{\xi,d}_q(\lambda)$ the multiplicity of $T_q(\lambda)$ as a summand in Y^d. Obviously

$$m(\lambda) = \sup\{m^d(\lambda) \mid d \in \mathbb{N}_0\} \tag{17}$$

and

$$m^{\xi}_q(\lambda) = \sup\{m^{\xi,d}_q(\lambda) \mid d \in \mathbb{N}_0\}, \tag{18}$$

because $\mathrm{im}(\beta^{\xi}_K) = \bigcup_{d\in\mathbb{N}_0} Y^d_1$ and $\mathrm{im}(\beta^{\xi}_{K(q)}) = \bigcup_{d\in\mathbb{N}_0} Y^d$. On the other hand, $\chi(Y^d) = \chi(Y^d_1)$ by Proposition 5.3, implying

$$m^{\xi,d}_q(\lambda) = m^d(\lambda) \text{ for all } d \in \mathbb{N}_0,\ \lambda \in P. \tag{19}$$

So our statement follows from equations (17), (18), (19). $\square$

If $\eta(\xi)$ is diagonal with pairwise different eigenvalues, then the stabilizer of $\eta(\xi)$ with respect to the classical adjoint action is the diagonal subgroup of $GL(N, K)$, and the orbit of $\eta(\xi)$ is closed in $M(N, K)$. It follows then that the image of $\beta^{\eta(\xi)}_K$ is $\mathcal{O}_K(D \setminus GL)$. In particular, the multiplicity of $T(\lambda)$ as a direct summand of the $\mathcal{O}_K(GL)$-corepresentation on $\mathcal{O}_K(D \setminus GL)$ is $m(\lambda)$. We show that the same holds in the q-deformed case.

Lemma 5.6 *The multiplicity of $T_q(\lambda)$ as a direct summand of the $\mathcal{O}_{K(q)}(GL_q)$-corepresentation on $\mathcal{O}_{K(q)}(D \setminus GL_q)$ is $m(\lambda)$ for all $\lambda \in P$.*

Proof. In the sequel (A, q) will stand for either of $(K(q), q)$ or $(K, 1)$. First we describe a spanning set of $\mathcal{O}_A(D \setminus GL_q)$. For an $N \times N$ matrix $a = (a_{ij})$ with non-negative integer entries and a non-negative integer d set

$$x^a_d := \det{}_q^{-d} x_{11}^{a_{11}} x_{12}^{a_{12}} \cdots x_{NN}^{a_{NN}}$$

(the variables x_{ij} are ordered lexicographically). The elements of this form span $\mathcal{O}_A(GL_q)$, and

$$(\pi_D \otimes \mathrm{id}) \circ \Delta(x^a_d) = t^a_d \otimes x^a_d,$$

where

$$t_d^a := \prod_{i=1}^{N} t_i^{-d+\sum_{j=1}^{N} a_{ij}}.$$

It follows that $\mathcal{O}_A(D \setminus GL_q)$ is spanned by the x_d^a with $\sum_{j=1}^{N} a_{ij} = d$ for $i = 1, \dots, N$. Set

$$L_A^d := \mathrm{Span}_A\{x_d^a \mid \sum_{j=1}^{N} a_{ij} = d,\ i = 1, \dots, N\}.$$

Observe that the given spanning set of L_A^d is actually a basis, because after multiplication by $\det_q^d$, it becomes a set of linearly independent monomials of degree dN from the standard basis of the domain $\mathcal{O}_A(M_q)$. Clearly

$$\mathcal{O}_A(D \setminus GL_q) = \bigcup_{d \in \mathbb{N}_0} L_A^d,$$

and L_A^d is a right $\mathcal{O}_A(GL_q)$-subcomodule of $\mathcal{O}_A(D \setminus GL_q)$ for all d.

Note that

$$(\mathrm{id} \otimes \pi_D) \circ \Delta(x_d^a) = x_d^a \otimes t_{a,d} \quad \text{with} \quad t_{a,d} := \prod_{j=1}^{N} t_j^{-d+\sum_{i=1}^{N} a_{ij}}.$$

It follows that

$$\chi(L_A^d) = \sum_{a \in \omega_d} t_{a,d},$$

where

$$\omega_d := \{a = (a_{ij}) \in M(N, \mathbb{N}_0) \mid \sum_{j=1}^{N} a_{ij} = d \text{ for } i = 1, \dots, N\}.$$

Observe that the above computation of $\chi(L_A^d)$ is the same when $A = K(q)$ and when $A = K$. It follows that for all $d \in \mathbb{N}_0$ the multiplicity of $T_q(\lambda)$ in $L_{K(q)}^d$ is the same as the multiplicity of $T(\lambda)$ in L_K^d. Consequently, the multiplicity of $T_q(\lambda)$ in $\mathcal{O}_{K(q)}(D \setminus GL_q)$ is the same as the multiplicity of $T(\lambda)$ in $\mathcal{O}_K(D \setminus GL) = \mathrm{im}(\beta_K^{\eta(\xi)})$. $\square$

Theorem 5.7 *Let ξ be a diagonal $K[q, q^{-1}]$-point of M_q such that $\eta(\xi)$ has pairwise different eigenvalues. Then the image of $\beta_{K(q)}^{\xi}$ coincides with $\mathcal{O}_{K(q)}(D \setminus GL_q)$.*

Proof. We have $\mathrm{im}(\beta_{K(q)}^{\xi}) \subseteq \mathcal{O}_{K(q)}(D \setminus GL_q)$ by the same argument as in the proof of Proposition 2.6. The multiplicity of $T_q(\lambda)$ is $m(\lambda)$ both in $\mathcal{O}_{K(q)}(D \setminus GL_q)$ and in $\mathrm{im}(\beta_{K(q)}^{\xi})$ by Proposition 5.5 and Lemma 5.6. This implies that $\mathrm{im}(\beta_{K(q)}^{\xi})$ is the whole of $\mathcal{O}_{K(q)}(D \setminus GL_q)$. $\square$

Finally we return to the coordinate ring of quantum matrices over $\mathbb{C}$. The results of this section have the following corollary in the framework of Section 2.

Corollary 5.8 *Let ξ be a $\mathbb{C}$-point of M_q such that the centralizer of ξ in $M(N,K)$ has dimension N. Assume that $q \in \mathbb{C}^*$ is transcendental over the subfield K of $\mathbb{C}$ generated by the entries of ξ. Then the kernel of β^ξ coincides with the right ideal of $\mathcal{O}(M_q)$ generated by $\tau_i - \tau_i(\xi)$, $i = 1, \dots, N$. If in addition ξ is diagonal, then the image of β^ξ is $\mathcal{O}(D \setminus GL_q)$.*

Proof. Since q is transcendental over K, we may apply Theorems 5.4 and 5.7 for the kernel and the image of $\beta^\xi_{K(q)}$. The $\mathbb{C}$-linear map β^ξ is obtained from the $K(q)$-linear map $\beta^\xi_{K(q)}$ by extending scalars to $\mathbb{C}$. Therefore a generating set of $\ker(\beta^\xi_{K(q)})$ as a $K(q)$-vector space spans $\ker(\beta^\xi)$ over $\mathbb{C}$, and a generating set of $\mathrm{im}(\beta^\xi)$ as a $K(q)$-vector space spans $\mathrm{im}(\beta^\xi)$ over $\mathbb{C}$. $\square$

It is straightforward to modify the statements of this section to obtain similar results for the coaction α.

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