From totally nonnegative matrices to quantum matrices and back, via Poisson geometry

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The nonnegative world

- A matrix is **totally positive** if each of its minors is positive.

- A matrix is **totally nonnegative** if each of its minors is non-negative.

History

- Fekete (1910s)
- Gantmacher and Krein, Schoenberg (1930s): small oscillations, eigenvalues
- Karlin and McGregor (1950s): statistics, birth and death processes
- Lindström (1970s): planar networks
- Gessel and Viennot (1985): binomial determinants, Young tableaux
- Gasca and Peña (1992): optimal checking
- Lusztig (1990s): reductive groups, canonical bases
- Fomin and Zelevinsky (1999/2000): survey articles (eg Math Intelligencer)
- Postnikov (2007): the totally nonnegative grassmannian
- Oh (2008): Positroids and Schubert matroids

Examples

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 \end{pmatrix} \qquad \begin{pmatrix} 5 & 6 & 3 & 0 \\ 4 & 7 & 4 & 0 \\ 1 & 4 & 4 & 2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

¿ How much work is involved in checking if a matrix is totally positive/totally nonnegative?

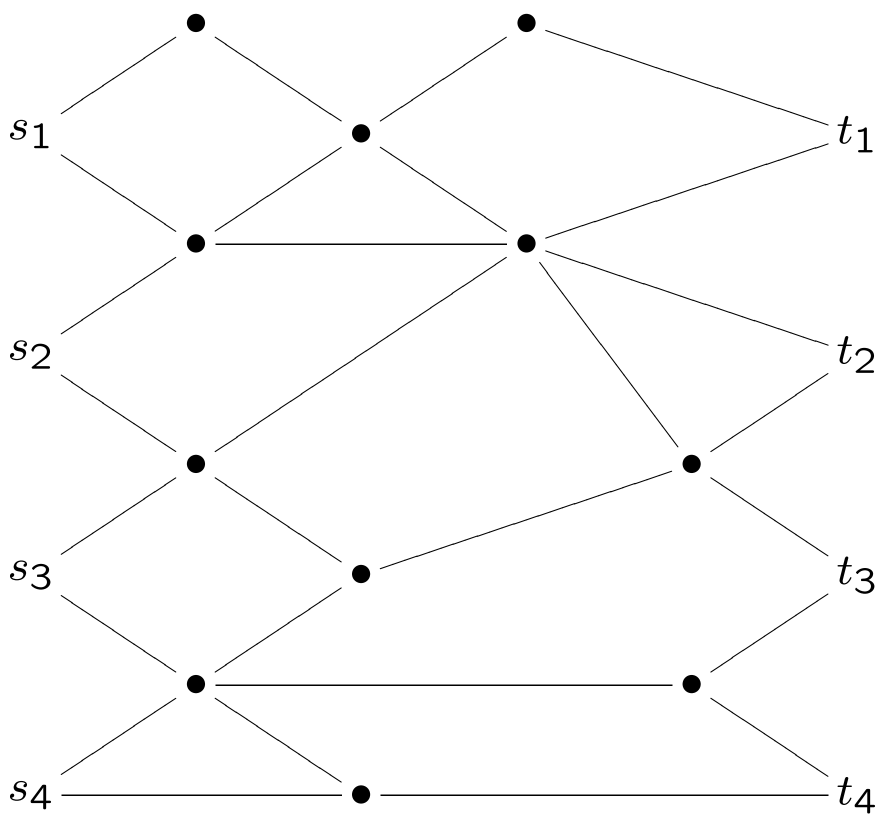
Eg. $n = 4$:

$$\#\text{minors} \quad = \quad \sum_{k=1}^{n} \binom{n}{k}^2 \quad = \quad \binom{2n}{n} - 1 \quad \approx \quad \frac{4^n}{\sqrt{\pi n}}$$

by using Stirling's approximation

$$n! \approx \sqrt{2\pi n}\frac{n^n}{e^n}$$

Planar networks Consider an directed graph with no directed cycles, n sources and n sinks.



$M = \left(m_{ij}\right)$ where m_{ij} is the number of paths from source s_i to sink t_j.

$$\begin{pmatrix} 5 & 6 & 3 & 0 \\ 4 & 7 & 4 & 0 \\ 1 & 4 & 4 & 2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

Edges directed left to right.

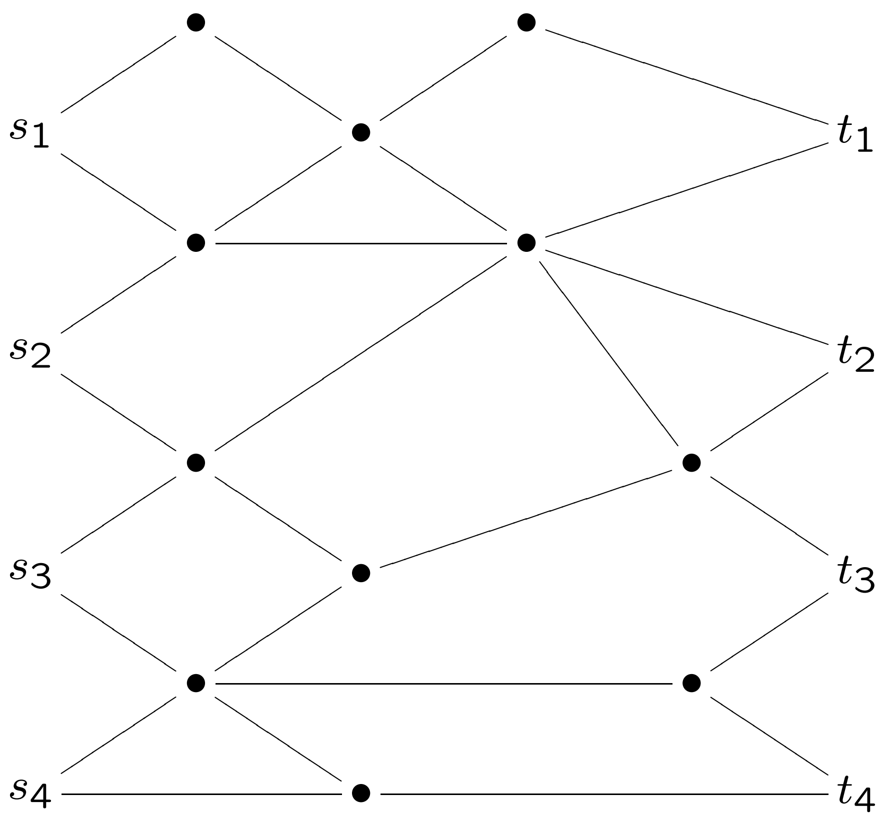
(Skandera: Introductory notes on total positivity)

Notation The minor formed by using rows from a set I and columns from a set J is denoted by $[I \mid J]$.

Theorem (Lindström)
The path matrix of any planar network is totally nonnegative. In fact, the minor $[I \mid J]$ is equal to the number of families of non-intersecting paths from sources indexed by I and sinks indexed by J.

If we allow weights on paths then even more is true.

Theorem
Every totally nonnegative matrix is the weighted path matrix of some planar network.



Edges directed left to right.

$M = \left(m_{ij}\right)$ where m_{ij} is the number of paths from source s_i to sink t_j.

$$\begin{pmatrix} 5 & 6 & 3 & 0 \\ 4 & 7 & 4 & 0 \\ 1 & 4 & 4 & 2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

Let $\mathcal{M}_{m,p}^{\mathsf{tnn}}$ be the set of totally nonnegative $m \times p$ real matrices.

Let Z be a subset of minors. The **cell** S_Z^o is the set of matrices in $\mathcal{M}_{m,p}^{\mathsf{tnn}}$ for which the minors in Z are zero (and those not in Z are nonzero).

Some cells may be empty. The space $\mathcal{M}_{m,p}^{\mathsf{tnn}}$ is partitioned by the nonempty cells.

Example In $\mathcal{M}_2^{\text{tnn}}$ the cell $S^\circ_{\{[2,2]\}}$ is empty.

For, suppose that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is tnn and $d = 0$.

Then $a, b, c \geq 0$ and also $ad - bc \geq 0$.

Thus, $-bc \geq 0$ and hence $bc = 0$ so that $b = 0$ or $c = 0$.

Exercise There are 14 nonempty cells in $\mathcal{M}_2^{\text{tnn}}$.

Postnikov (arXiv:math/0609764) defines **Le-diagrams**: an $m \times p$ array with entries either 0 or 1 is said to be a **Le-diagram** if it satisfies the following rule: if there is a 0 in a given square then either each square to the left is also filled with 0 or each square above is also filled with 0.

An example and a non-example of a Le-diagram on a 5×5 array

1	1	0	1	0
0	0	0	1	0
1	1	1	1	0
0	0	0	1	0
1	1	1	1	0

1	1	0	1	0
0	0	1	0	1
1	1	1	0	1
0	0	1	1	1
1	1	1	1	1

• **Postnikov (arXiv:math/0609764)** There is a bijection between Le-diagrams on an $m \times p$ array and nonempty cells S_Z° in $\mathcal{M}_{m,p}^{\mathsf{tnn}}$.

2×2 **Le-diagrams**

$$\begin{array}{|c|c|}\hline 1&1\\\hline 1&1\\\hline\end{array}\qquad\begin{array}{|c|c|}\hline 0&1\\\hline 1&1\\\hline\end{array}\qquad\begin{array}{|c|c|}\hline 1&0\\\hline 1&1\\\hline\end{array}\qquad\begin{array}{|c|c|}\hline 1&1\\\hline 0&1\\\hline\end{array}$$

$$\begin{array}{|c|c|}\hline 1&1\\\hline 1&0\\\hline\end{array}\qquad\begin{array}{|c|c|}\hline 0&0\\\hline 1&1\\\hline\end{array}\qquad\begin{array}{|c|c|}\hline 0&1\\\hline 0&1\\\hline\end{array}\qquad\begin{array}{|c|c|}\hline 0&1\\\hline 1&0\\\hline\end{array}$$

$$\begin{array}{|c|c|}\hline 1&0\\\hline 0&1\\\hline\end{array}\qquad\begin{array}{|c|c|}\hline 1&0\\\hline 1&0\\\hline\end{array}\qquad\begin{array}{|c|c|}\hline 1&1\\\hline 0&0\\\hline\end{array}\qquad\begin{array}{|c|c|}\hline 0&0\\\hline 0&1\\\hline\end{array}$$

$$\begin{array}{|c|c|}\hline 0&0\\\hline 1&0\\\hline\end{array}\qquad\begin{array}{|c|c|}\hline 0&1\\\hline 0&0\\\hline\end{array}\qquad\begin{array}{|c|c|}\hline 1&0\\\hline 0&0\\\hline\end{array}\qquad\begin{array}{|c|c|}\hline 0&0\\\hline 0&0\\\hline\end{array}$$

Postnikov's Algorithm starts with a Le-Diagram and produces a planar network from which one generates a totally nonnegative matrix which defines a nonempty cell.

Example

	0	
0	0	

The quantum world

Quantum matrices

$\mathcal{O}_q(\mathcal{M}_2)$, the *quantised coordinate ring of* 2×2 *matrices*

$$\mathcal{O}_q(\mathcal{M}_2) := k \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with relations

$$ab = qba \qquad ac = qca \qquad bc = cb$$

$$bd = qdb \qquad cd = qdc \qquad ad - da = (q - q^{-1})bc.$$

The *quantum determinant* is $D_q := ad - qbc$

Exercise Check that the quantum determinant is central.

Overall problem Describe Spec($\mathcal{O}_q(\mathcal{M}_2)$), q generic ($q^m \neq 1$)

Set $\mathcal{H} := (k^*)^4$.

There is an action of $\mathcal{H}$ on $\mathcal{O}_q(\mathcal{M}_2)$ given by

$$(\alpha, \beta; \gamma, \delta) \circ \begin{bmatrix} a & b \\ c & d \end{bmatrix} := \begin{bmatrix} \alpha\gamma a & \alpha\delta b \\ \beta\gamma c & \beta\delta d \end{bmatrix};$$

that is, by row and column multiplications.

Subproblem Identify all of the prime ideals of $\mathcal{O}_q(\mathcal{M}_2)$ that are $\mathcal{H}$-invariant.

• **Overall problem**: describe $\text{Spec}(\mathcal{O}_q(\mathcal{M}_2))$, when q is not a root of unity.

Theorem (Goodearl-Letzter) Let $P \in \text{Spec}(\mathcal{O}_q(\mathcal{M}_2))$. Then $\mathcal{O}_q(\mathcal{M}_2)/P$ is an integral domain; that is, all primes are completely prime.

Theorem (Goodearl-Letzter)

$$|\mathcal{H} - \text{Spec}(\mathcal{O}_q(\mathcal{M}_2))| \leq 2^4 = 16 < \infty$$

• **Sub-problem**: describe $\mathcal{H} - \text{Spec}(\mathcal{O}_q(\mathcal{M}_2))$

Example Let P be a prime ideal of $\mathcal{O}_q(\mathcal{M}_2)$ that contains d. Then

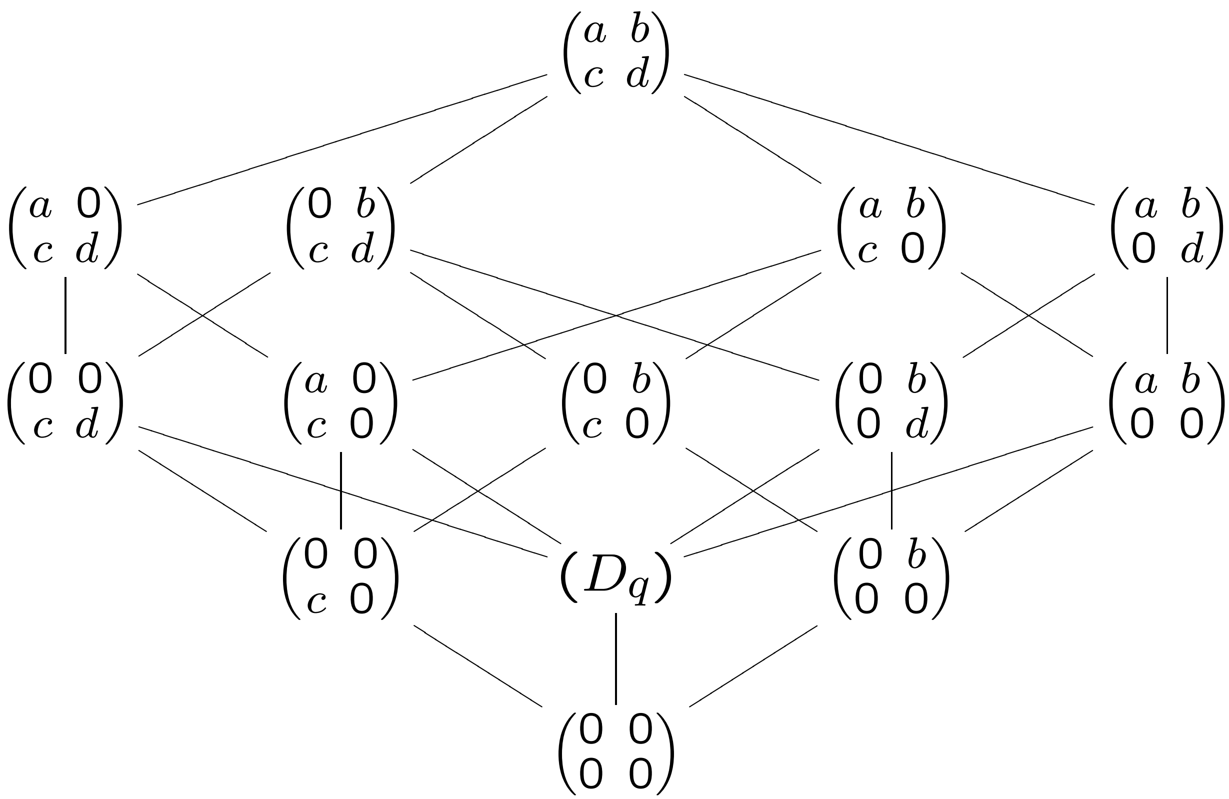
$$(q - q^{-1})bc = ad - da \in P$$

As $0 \neq (q - q^{-1}) \in \mathbb{C}$ and P is completely prime, we deduce that either $b \in P$ or $c \in P$.

Thus, there is no prime ideal in $\mathcal{O}_q(\mathcal{M}_2)$ such that d is the only quantum minor that is in P.

You should notice the analogy with the corresponding result in the space of 2×2 totally nonnegative matrices: the cell corresponding to d being the only vanishing minor is empty.

Claim The following 14 $\mathcal{H}$-invariant ideals are all prime and these are the only $\mathcal{H}$-prime ideals in $\mathcal{O}_q(\mathcal{M}_2)$.

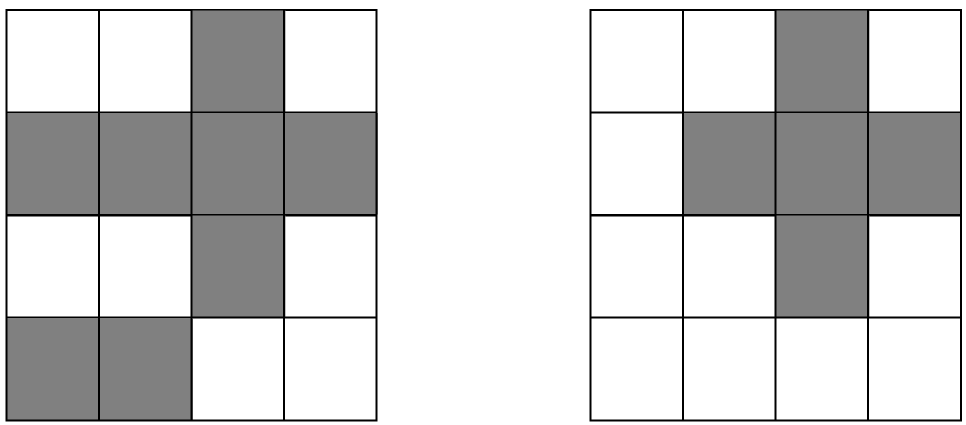


To interpret this picture, note that, for example, $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ denotes the ideal generated by a, b and c.

In quantum $m \times p$ matrices there is an action of $\mathcal{H} = (k^*)^{m+p}$ and the problem is to describe the finitely many $\mathcal{H}$-prime ideals.

Theorem (Cauchon) The $\mathcal{H}$-prime ideals in quantum $m \times p$ matrices are in bijection with Cauchon diagrams:

Cauchon Diagrams



The rule for a Cauchon diagram is that if a square is black then either each square to the left of it is black, or each square above it is black.

The Poisson world

Poisson algebra: definition

A Poisson algebra is a commutative finitely generated $\mathbb{C}$-algebra A with a "Poisson bracket" $\{-,-\} : A \times A \to A$ such that

1. $(A, \{-,-\})$ is a Lie algebra;
2. for all $a \in A$, the linear map $\{a,-\} : A \to A$ is a derivation, that is:

$$\{a, bc\} = b\{a,c\} + \{a,b\}c \qquad \forall a,b,c \in A.$$

Example. $\mathbb{C}[X,Y]$ is a Poisson algebra with Poisson bracket given by:

$$\{P,Q\} := \frac{\partial P}{\partial X} \cdot \frac{\partial Q}{\partial Y} - \frac{\partial P}{\partial Y} \cdot \frac{\partial Q}{\partial X}.$$

The semiclassical limit of $\mathcal{O}_q(\mathcal{M}_2)$ is the commutative algebra of polynomials $\mathbb{C}[a, b, c, d]$ with

$$\boxed{\begin{array}{ll} \{a, b\} = ab, & \{c, d\} = cd \\ \{a, c\} = ac, & \{b, d\} = bd \\ \{b, c\} = 0, & \{a, d\} = 2bc. \end{array}},$$

Symplectic leaves

Let A be the algebra of complex-valued C^∞ functions on a smooth affine variety V.

- *Hamiltonian derivations*: $H_a := \{a, -\}$ with $a \in A$.

- A *Hamiltonian path in V* is a smooth path $c : [0, 1] \to V$ such that there exists $H \in C^\infty(V)$ with

$$\frac{d}{dt}(f \circ c)(t) = \{H, f\} \circ c(t)$$

for all $0 < t < 1$.

- It is easy to check that the relation "connected by a piecewise Hamiltonian path" is an equivalence relation.

- The equivalence classes of this relation are called the *symplectic leaves* of V; they form a partition of V.

Again, there is an action of a torus $\mathcal{H}$ on the space of matrices as Poisson automorphisms and one can look at *torus orbits of symplectic leaves*.

Exercise There are 14 torus orbits of symplectic leaves in the space of 2×2 matrices over $\mathbb{C}$ equipped with the Poisson bracket coming from the semiclassical limit of $\mathcal{O}_q(\mathcal{M}_2)$.

The torus orbits of symplectic leaves have been described by Brown, Goodearl and Yakimov.

Set

$$\mathcal{S} = \{w \in S_{m+p} \mid -p \leq w(i) - i \leq m \text{ for all } i = 1, 2, \ldots, m+p\}.$$

Theorem (Brown, Goodearl and Yakimov)

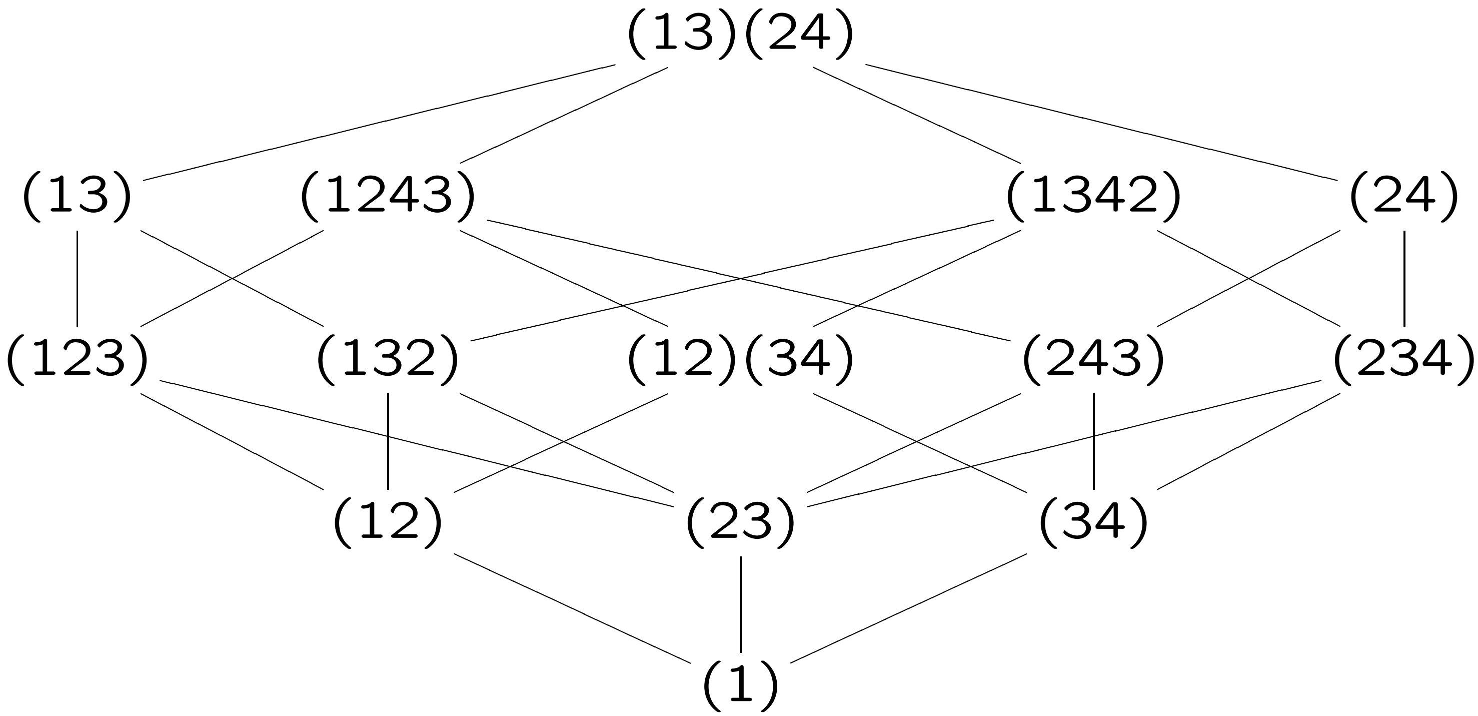
• There is an explicit $1:1$ correspondence between $\mathcal{S}$ and the torus orbits of symplectic leaves in $\mathcal{M}_{m,p}(\mathbb{C})$.

• Each $\mathcal{H}$-orbit of symplectic leaves is defined by some rank conditions; that is, by the vanishing and nonvanishing of certain minors.

In the 2×2 case, this subposet of the Bruhat poset of S_4 is

$$\mathcal{S} = \{w \in S_4 \mid -2 \leq w(i) - i \leq 2 \text{ for all } i = 1, 2, 3, 4\}.$$

and is shown below.



Inspection of this poset reveals that it is isomorphic to the poset of the $\mathcal{H}$-prime ideals of $\mathcal{O}_q(\mathcal{M}_2)$ displayed earlier; and so to a similar poset of the Cauchon diagrams corresponding to the $\mathcal{H}$-prime ideals.

The Grand Unifying Theory

Totally nonnegative cells

Totally nonnegative cells are defined by the vanishing of families of minors. Some of the TNN cells are empty.

We denote by S^0_Z the TNN cell associated to the family of minors Z.

A family of minors is *admissible* if the corresponding TNN cell is nonempty.

Question: what are the admissible families of minors?

Matrix Poisson varieties

$\mathcal{H}$-orbits of symplectic leaves are algebraic, and are defined by rank conditions. In other words, they are defined by the vanishing and non-vanishing of some families of minors.

Question: which families of minors?

Generators of $\mathcal{H}$-primes in quantum matrices.

Theorem (Launois): Assume that q is transcendental.
Then $\mathcal{H}$-primes of $\mathcal{O}_q(\mathcal{M}(m,p))$ are generated by quantum minors.

Question: which families of quantum minors?

An algorithm to rule them all

Deleting derivations algorithm:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} a - bd^{-1}c & b \\ c & d \end{pmatrix}$$

Restoration algorithm:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} a + bd^{-1}c & b \\ c & d \end{pmatrix}$$

An example

Set $M = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Apply the restoration algorithm:

$$M^{(2,2)} = M^{(2,1)} = M^{(1,3)} = M^{(1,2)} = M^{(1,1)} = M,$$

$$M^{(3,1)} = M^{(2,3)} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad M^{(3,2)} = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and

$$M^{(3,3)} = \begin{pmatrix} 3 & 2 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Exercise. Is this matrix TNN?

TNN Matrices and restoration algorithm

Theorem (Goodearl-Launois-Lenagan 2009).

• If the entries of M are nonnegative and its zeros form a Cauchon diagram, then $M^{(m,p)}$ is TNN.

• Let M be a matrix with real entries. We can apply the deleting derivation algorithm to M. Let N denote the resulting matrix.

Then M is TNN iff the matrix N is nonnegative and its zeros form a Cauchon diagram.

Exercise. Use the deleting derivation algorithm to test whether the following matrices are TNN:

$$M_1 = \begin{pmatrix} 11 & 7 & 4 & 1 \\ 7 & 5 & 3 & 1 \\ 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} 7 & 5 & 4 & 1 \\ 6 & 5 & 3 & 1 \\ 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Main Result

Theorem. (GLL) Let $\mathcal{F}$ be a family of minors in the coordinate ring of $\mathcal{M}_{m,p}(\mathbb{C})$, and let $\mathcal{F}_q$ be the corresponding family of quantum minors in $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$. Then the following are equivalent:

1. The totally nonnegative cell associated to $\mathcal{F}$ is nonempty.

2. $\mathcal{F}$ is the set of minors that vanish on the closure of a torus-orbit of symplectic leaves in $\mathcal{M}_{m,p}(\mathbb{C})$.

3. $\mathcal{F}_q$ is the set of quantum minors that belong to torus-invariant prime in $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$.

Consequences of the Main Result

The TNN cells are the traces of the $\mathcal{H}$-orbits of symplectic leaves on $\mathcal{M}_{m,p}^{\text{tnn}}$.

The sets of minors that vanish on the closure of a torus-orbit of symplectic leaves in $\mathcal{M}_{m,p}(\mathbb{C})$ can be explicitely described thanks to results of Fulton and Brown-Goodearl-Yakimov. So, as a consequence of the previous result, **the sets of minors that define nonempty totally nonnegative cells are explicitely described.**

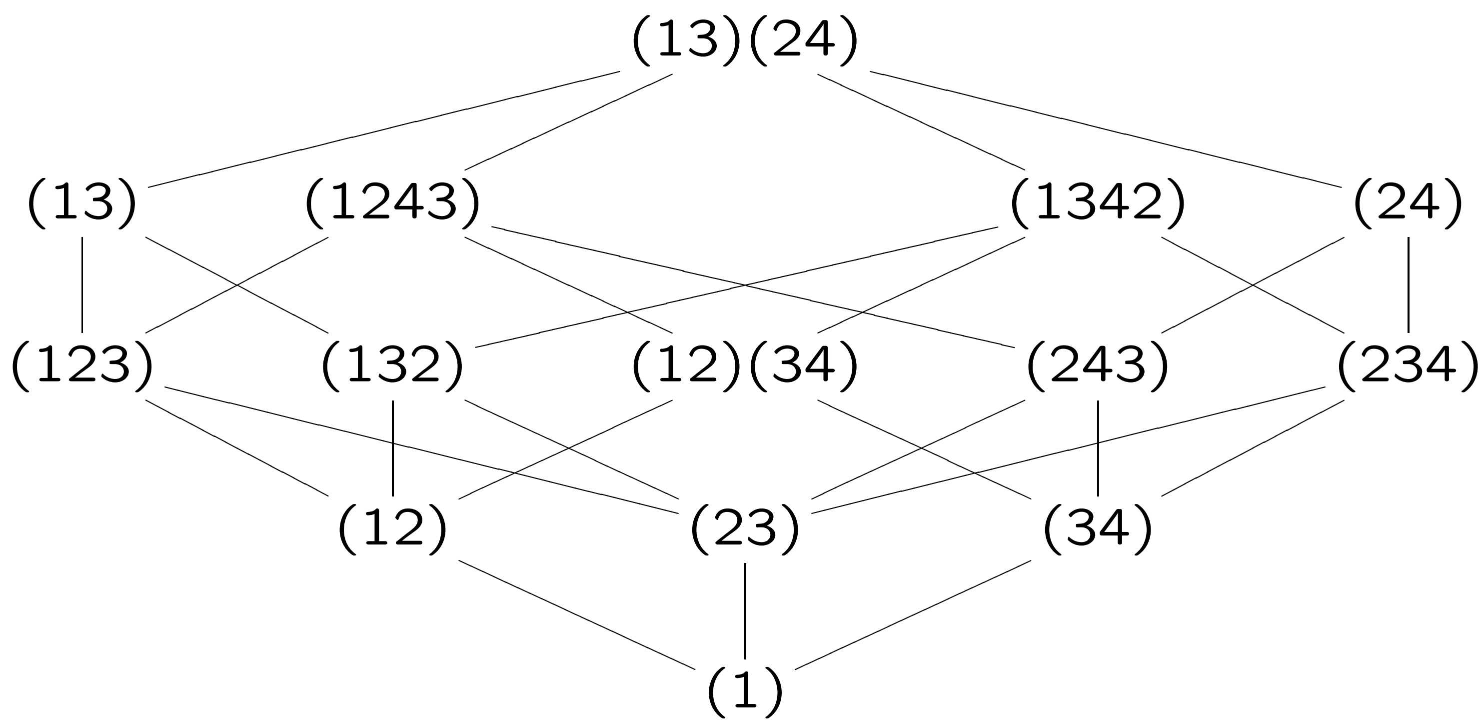
On the other hand, when the deformation parameter q is transcendental over the rationals, then the torus-invariant primes in $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$ are generated by quantum minors, and so we deduce from the above result **explicit generating sets of quantum minors for the torus-invariant prime ideals of** $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$.

Restricted permutations

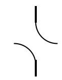
$w \in S_{m+p}$ with

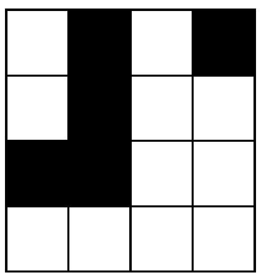
$$-p \leq w(i) - i \leq m \text{ for all } i = 1, 2, \ldots, m + p.$$

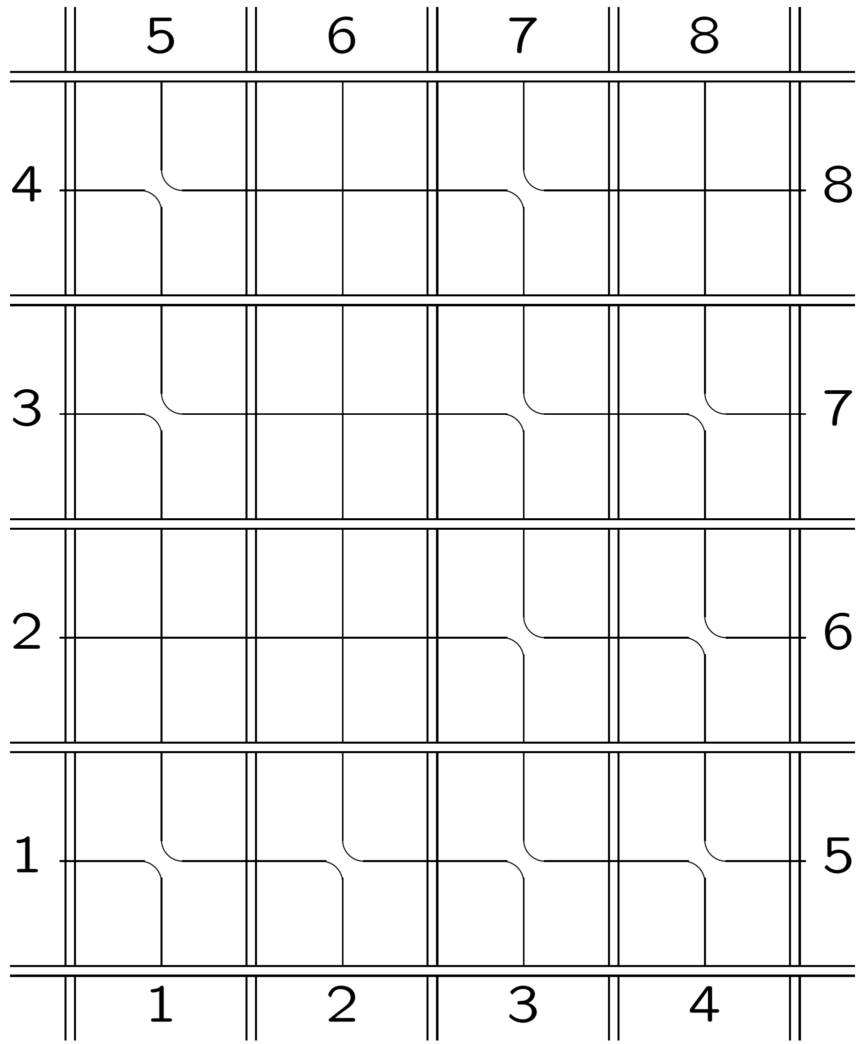
When $m = p = 2$, there are 14 of them.



Restricted permutations versus Cauchon diagrams

Replace ■ by $+$ and □ by 

$C =$ 



Related articles

• K Casteels, *A Graph Theoretic Method for Determining Generating Sets of Prime Ideals in* $\mathcal{O}_q(M_{m,n}(\mathbb{C}))$, http://arxiv.org/abs/0907.1617

• A Knutson, T Lam, and D E Speyer: *Positroid varieties I: juggling and geometry*, htpp://arxiv.org/abs/0903.3694.

• S Launois and T H Lenagan, *From totally nonnegative matrices to quantum matrices and back, via Poisson geometry*, http://arxiv.org/abs/0911.2990

• S Oh, *Positroids and Schubert matroids*, http://arxiv.org/abs/0803.1018

• K Talaska, *Combinatorial formulas for Le-coordinates in a totally nonnegative Grassmannian*, http://arxiv.org/abs/0812.0640

• M Yakimov, *Invariant prime ideals in quantizations of nilpotent Lie algebras*, http://arxiv.org/abs/0905.0852,