The traces of quantum powers commute

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Abstract

The traces of the quantum powers of a generic quantum matrix pairwise commute. This was conjectured by Kaoru Ikeda, in connection with certain Hamiltonian systems. The proof involves Newton's formulae for quantum matrices, relating traces of quantum powers with sums of principal minors.

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The quasi-classical limit of the quantum group $GL_q(n)$ is the n^2-variable commutative polynomial algebra generated by y_{ij}, $1 \leq i, j \leq n$, together with the Poisson bracket given by

$$\{y_{ij}, y_{kl}\} = (\theta(i,k) + \theta(j,l))y_{il}y_{kj},$$

where $\theta(i,j) = 1$, if $i < j$; 0, if $i = j$; -1, if $i > j$; see [1]. This Poisson bracket was used in [2] to construct a Hamiltonian system, which can be reduced to the classical Toda lattice. The approach of [2] is based on the fact that

$$\{\mathrm{Tr}(Y^k), \mathrm{Tr}(Y^m)\} = 0 \quad \text{for} \quad k, m = 1, 2, \ldots, \tag{1}$$

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where Y is the $n \times n$ matrix with y_{ij} as the (i,j) entry; thus $\mathrm{Tr}(Y^k)$, $k = 1, 2, \ldots$ can be considered as an involutive set of integrals. It is therefore natural to look for a quantum analogue of (1). This problem is addressed in [3], [4], [5]. Recall that the coordinate ring $\mathcal{O}(M_q)$ of $n \times n$ quantum matrices is the $\mathbb{C}$-algebra generated by x_{ij}, $1 \le i, j \le n$, subject to the relations $x_{ij}x_{kl} - x_{kl}x_{ij} = (q^{\theta(j,l)} - q^{-\theta(i,k)})x_{il}x_{kj}$, $1 \le i, j, k, l \le n$, where q is a fixed non-zero element of $\mathbb{C}$ (in some papers $q = e^{-h/2}$ is considered to be an element of the ring of formal power series $\mathbb{C}[[h]]$, and $\mathcal{O}(M_q)$ is the $\mathbb{C}[[h]]$-algebra generated by the x_{ij}; this convention has to be used in the definition of the 'quasi-classical limit'). The so-called q-multiplication of matrices was introduced in [6]. If $A = (a_{ij})$, $B = (b_{ij})$ are $n \times n$ matrices with entries from $\mathcal{O}(M_q)$, then let $A \star B$ be the matrix whose (i,j)-entry is

$$(A \star B)_{ij} = \sum_{k=1}^{n} q^{\theta(j,k)} a_{ik} b_{kj}.$$

Note that $\star$ is not an associative multiplication. Write X for the $n \times n$ matrix with x_{ij} as the (i,j) entry. Define ${}_qX^k$ for $k = 0, 1, 2, \ldots$ as follows. Set ${}_qX^0 = I$, the identity matrix, ${}_qX^1 = X$, ${}_qX^2 = X \star X$, ${}_qX^3 = X \star (X \star X)$, and recursively, ${}_qX^k = X \star ({}_qX^{k-1})$. Denote by $\mathrm{Tr}({}_qX^k)$ the usual trace (the sum of the diagonal entries) of ${}_qX^k$. When q is specialized to 1, the element $\mathrm{Tr}({}_qX^k)$ goes to $\mathrm{Tr}(Y^k)$ from (1). Ikeda [4] has conjectured that the following quantum analogue of (1) holds:

$$[\mathrm{Tr}({}_qX^k), \mathrm{Tr}({}_qX^m)] = 0 \quad \text{for} \quad k, m = 1, 2, \ldots, \tag{2}$$

where $[a, b] = ab - ba$. The special cases of (2) when $k = 1$, $m = 2$, and later, when $k = 1$, m arbitrary were verified in [4], [5]. In the present paper we prove (2) in general. Note that this explains the reason behind (1).

We need to recall a related set of elements in $\mathcal{O}(M_q)$. The *quantum determinant* is

$$\sigma_n := \sum_{\pi \in S_n} (-q)^{l(\pi)} x_{1,\pi(1)} \ldots x_{n,\pi(n)},$$

where $l(\pi)$ denotes the number of pairs $i < j$ with $\pi(i) > \pi(j)$ for a permutation π. Take k-element subsets K, L of $\{1, \ldots, n\}$. Then the x_{ij} with $i \in K$, $j \in L$ generate a subalgebra of $\mathcal{O}(M_q)$, isomorphic to the coordinate ring of $k \times k$ quantum matrices. So we can form the quantum determinant of this subalgebra. The resulting element is denoted by $[K|L]$, and is called a $k \times k$ *quantum minor* in $\mathcal{O}(M_q)$. The sum $\sum_K [K|K]$ of principal $k \times k$ quantum minors is denoted by σ_k for $k = 1, \ldots, n$. In particular, $\sigma_1 = \mathrm{Tr}(X)$. For the sake of notational convenience, we define $\sigma_{n+1}, \sigma_{n+2}, \ldots$ to be zero.

The elements $\sigma_1, \ldots, \sigma_n$ pairwise commute by [7]. Denote by R the subalgebra of $\mathcal{O}(M_q)$ generated by $\sigma_1, \ldots, \sigma_n$. So R is an n-variable commutative polynomial algebra. The commutativity of R was obtained in [7] as a by-product of its Hopf algebraic interpretation: the σ_k are the basic coinvariants for a coaction (a q-analogue of the adjoint action) of $GL_q(n)$ on $\mathcal{O}(M_q)$; in an alternative formulation, R is the subset of cocommutative elements in the coalgebra $\mathcal{O}(M_q)$.

To simplify notation, write $t_k = \mathrm{Tr}({}_qX^k)$, $k = 1, 2, \ldots$. We shall show that the t_k pairwise commute by proving that they are all contained in R. (Actually, $t_1, \ldots, t_n$ turns out to be another generating set of the algebra R.) More precisely, we shall show that the sequences $\sigma_1, \sigma_2, \ldots$ and $t_1, t_2, \ldots$ are related by the same Newton's formulae as in the classical case $q = 1$, see Lemma 3. This will be derived from the proof of a Cayley-Hamilton theorem due to Zhang [6]. We present this proof in detail, because (P 1.2) on page 103 of [6] is wrong, consequently, the formulae we need are not contained in [6].

Definition 1 *We define Z_k, for $k = 0, 1, 2, \ldots$, recursively, via*

$$\begin{aligned} Z_0 &:= I \\ Z_k &:= X \star Z_{k-1} + (-1)^k I \sigma_k, \qquad \text{for } k \geq 1. \end{aligned}$$

In order to prove the next lemma, we need to use a quantum Laplace expansion in the following form for a quantum minor $[K|L]$ with $i, r \in K$. This relation can be easily obtained from Proposition 8 in 9.2.2 of [8]. Set $\delta_{i,r} = 1$ if $i = r$, and 0, otherwise.

$$\delta_{i,r}[K|L] = \sum_{s \in L} (-q)^{l(s,L)-l(r,K)} x_{is}[K \backslash r | L \backslash s], \tag{3}$$

where $l(u, J)$ stands for the number of elements $j \in J$ with $u > j$, for an element u and a subset J of $\{1, \ldots, n\}$, and $K \backslash r$ is the difference of the sets K and $\{r\}$. Later we shall abbreviate $l(u, \{j\})$ as $l(u, j)$, so $l(u, j)$ is 1, if $u > j$, and 0, otherwise.

Lemma 2 *The (i, j) entry of Z_k is*

$$(-1)^k \sum_{\substack{|J|=k+1 \\ i,j \in J}} q^{\theta(i,j)} (-q)^{l(i,J)-l(j,J)} [J \backslash j | J \backslash i]$$

for $k = 0, 1, \ldots, n-1$, and $Z_n = Z_{n+1} = Z_{n+2} = \ldots$ is the zero matrix.

Proof. The proof is by induction on k, the starting case where $k = 0$ is easily checked. Take $0 \leq k \leq n-2$ and assume that the formula holds for k. First, we compute the (i, j) entry of $X \star Z_k$:

$$
\begin{aligned}
(X \star Z_k)_{i,j} &= \sum_{s=1}^{n} x_{is} q^{\theta(j,s)} (Z_k)_{s,j} \\
&= \sum_{s=1}^{n} x_{is} q^{\theta(j,s)} (-1)^k \left\{ \sum_{\substack{|J|=k+1 \\ s,j\in J}} q^{\theta(s,j)} (-q)^{l(s,J)-l(j,J)} [J\backslash j | J\backslash s] \right\} \\
&= \sum_{\substack{|J|=k+1 \\ j\in J}} (-1)^k \left\{ \sum_{s\in J} (-q)^{l(s,J)-l(j,J)} x_{is} [J\backslash j | J\backslash s] \right\} \\
&= \sum_{\substack{|J|=k+1 \\ j\in J}} (-1)^k f(i,j,J)
\end{aligned}
$$

where we have set

$$
f(i,j,J) = \sum_{s\in J} (-q)^{l(s,J)-l(j,J)} x_{is} [J\backslash j | J\backslash s].
$$

If $i \in J$, then $f(i,j,J) = \delta_{i,j}[J|J]$ by (3); and so,

$$
(X \star Z_k)_{i,j} = \sum_{\substack{|J|=k+1 \\ i,j\in J}} (-1)^k \delta_{i,j} [J|J] + \sum_{\substack{|J|=k+1 \\ j\in J, i\notin J}} (-1)^k f(i,j,J).
$$

Hence,

$$
\begin{aligned}
(Z_{k+1})_{j,j} &= \sum_{\substack{|J|=k+1 \\ j\in J}} (-1)^k [J|J] + (-1)^{k+1} \sum_{|J|=k+1} [J|J] \\
&= \sum_{\substack{|J|=k+1 \\ j\notin J}} (-1)^{k+1} [J|J] \\
&= \sum_{\substack{|J|=k+2 \\ j\in J}} (-1)^{k+1} [J\backslash j | J\backslash j]
\end{aligned}
$$

which agrees with the formula in the statement of the lemma.

If $i \neq j$, then

$$
\begin{aligned}
(Z_{k+1})_{i,j} &= (X \star Z_k)_{i,j} \\
&= \sum_{\substack{|J|=k+1 \\ j\in J, i\notin J}} (-1)^k f(i,j,J)
\end{aligned}
$$

For $|J| = k+1$ with $j \in J$ and $i \notin J$, set $K := J \sqcup i$, the disjoint union of J and $\{i\}$. By (3),

$$\begin{aligned}
[K\backslash j|K\backslash i] &= \sum_{s\in K\backslash i} (-q)^{l(s,K\backslash i)-l(i,K\backslash j)} x_{is}[K\backslash\{i,j\}|K\backslash\{i,s\}] \\
&= \sum_{s\in J} (-q)^{l(s,J)-l(i,J\sqcup i\backslash j)} x_{is}[J\backslash j|J\backslash s] \\
&= (-q)^{l(j,J)-l(i,J\sqcup i\backslash j)} f(i,j,J).
\end{aligned}$$

Hence,

$$f(i,j,J) = (-q)^{l(i,J\sqcup i\backslash j)-l(j,J)}[K\backslash j|[K\backslash i],$$

where $K = J \sqcup i$ when $j \in J$ and $i \notin J$. Thus, for $i \neq j$, we see that

$$\begin{aligned}
(Z_{k+1})_{i,j} &= \sum_{\substack{|K|=k+2\\ i,j\in K}} (-1)^k (-q)^{l(i,K\backslash j)-l(j,K\backslash i)}[K\backslash j|K\backslash i] \\
&= (-q)^{l(j,i)-l(i,j)} \left\{ \sum_{\substack{|J|=k+2\\ i,j\in J}} (-1)^k (-q)^{l(i,J)-l(j,J)}[J\backslash j|J\backslash i] \right\},
\end{aligned}$$

which is the formula in the lemma for Z_{k+1}, since $(-q)^{l(j,i)-l(i,j)} = -q^{\theta(i,j)}$, provided that $i \neq j$. So we have proved the lemma for $k = 0, 1, \ldots, n-1$. Applying the case $k = n-1$ and (3) once more, we obtain that $X \star Z_{n-1} = (-1)^{n-1} I\sigma_n$; and so $Z_n = 0$. Consequently, we have $0 = Z_n = Z_{n+1} = Z_{n+2} = \ldots$. $\square$

From the definition of Z_k, it follows that

$$Z_k = {}_qX^k - {}_qX^{k-1}\sigma_1 + {}_qX^{k-2}\sigma_2 + \cdots + (-1)^k I\sigma_k, \tag{4}$$

where ${}_qX^{k-j}\sigma_j$ is the matrix obtained after multiplying each entry of ${}_qX^{k-j}$ by σ_j from the right. Thus the assertion $Z_n = 0$ in Lemma 2 is the (second) Cayley-Hamilton theorem in [6].

By taking the trace of the equality (4) we obtain

$$\mathrm{Tr}(Z_k) = t_k - t_{k-1}\sigma_1 + t_{k-2}\sigma_2 + \cdots + (-1)^{k-1}t_1\sigma_{k-1} + (-1)^k n\sigma_k.$$

Now, for $k = 0, 1, \ldots, n-1$ we have by Lemma 2 that

$$(Z_k)_{i,i} = (-1)^k \sum_{\substack{|K|=k\\ i\notin K}} [K|K]$$

and so $\mathrm{Tr}(Z_k) = (-1)^k(n-k)\sigma_k$. For $k \geq n$ we have $\mathrm{Tr}(Z_k) = \mathrm{Tr}(0) = 0$. Comparing the two expressions for $\mathrm{Tr}(Z_k)$ we see that Newton's formulae hold:

Lemma 3 *For $k = 1, 2, \ldots$ we have*

$$t_k - t_{k-1}\sigma_1 + t_{k-2}\sigma_2 + \cdots + (-1)^{k-1} t_1 \sigma_{k-1} + (-1)^k k\sigma_k = 0.$$

(Recall that $0 = \sigma_{n+1} = \sigma_{n+2} = \ldots$.)

A result of [7] together with Lemma 3 imply the following:

Theorem 4 *We have $[t_k, t_m] = 0$ for $k, m = 1, 2, \ldots$. The elements $t_{n+1}, t_{n+2}, \ldots$ can be expressed as polynomials of $t_1, \ldots, t_n$.*

Proof. It follows from Lemma 3 by induction on k that t_k is contained in $R = \mathbb{C}[\sigma_1, \ldots, \sigma_n]$ for all k. Since R is commutative by [7], the t_k pairwise commute. Moreover, $R = \mathbb{C}[t_1, \ldots, t_n]$, since $\sigma_1, \ldots, \sigma_n$ can be expressed as a polynomial of $t_1, \ldots, t_n$. This implies the second statement. (The fact that $t_{n+1}, t_{n+2}, \ldots$ are polynomials of $t_1, \ldots, t_{n-1}, \sigma_n$ was shown in [4] by an elaborate argument.) $\square$

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