Noncommutative unique factorisation rings

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Commutative case

- R is UFD : each nonzero, nonunit uniquely expressible $c = u p_1^{n_1} \cdots p_m^{n_m}$ some irred els $p_1, \ldots, p_n$

- R UFD iff each height 1 prime principal

- Thm R UFD $\Rightarrow$ $R[x]$ UFD
- eg $k[x_1, \ldots, x_n]$ UFD

- R UFD $\Rightarrow$ R integrally closed
- (Aus-Buchs) 1959

  R regular, local $\Rightarrow$ R UFD
- Nagata's Lemma

  R noeth. int. domain, $p_1, \ldots, p_m$ prime elts

  $T = R[p_1^{-1}, \ldots, p_m^{-1}]$

  T UFD $\Rightarrow$ R UFD.

<u>Noncommutative case</u>

- Chatters-Jordan (1986)
- $I \triangleleft R$ is <u>principal</u> : $\exists\ x$ normal s.t.

  $I = xR = Rx$

  R noeth. domain is UFD : every height 1 prime is principal (& completely prime)

- <u>Thm</u> R UFD $\Rightarrow$ $R[x]$ UFD
- <u>Thm</u> R UFD $\Rightarrow$ R maximal order

$\dot{c}$ Skew poly extension $R[x]$ has

- elts $\sum r_i x^i$

& if degree arguments to work need

$$xr = cx + d \quad , c, d \in R$$
$$= \sigma(r)x + \delta(r)$$

- need σ automorphism of R
- δ a σ-derivation ; ie

$$\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$$

- Notation $\quad R[x; \sigma, \delta]$
- Special cases

$\sigma =$ identity $\quad R[x; \delta]$

$\delta = 0 \quad R[x; \sigma]$

Examples

- **Weyl algebra**

  $A = k[x, y] \qquad xy - yx = 1$

  $A = k[y][x \,;\, \frac{d}{dy}]$

- **Env. alg. $\mathfrak{g}$ 2-dim solvable Lie alg**

  $\mathfrak{g} = kx + ky \qquad [x\ y] = x$

  $U(\mathfrak{g}) = k[x, y] \quad xy \ \ yx = x$

  or $\quad xy = (y+1)x$

  ie $\quad xy = \sigma(y)x \qquad \sigma : y \to y+1$

  $U(\mathfrak{g}) = k[y][x \,;\, \sigma]$

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- <u>Quantum Weyl Algebra</u>

$A = k[x\,y] \quad xy - qyx = 1$

$0 \neq q \in k.$

$A = k[y][x;\sigma,\delta]$

$\sigma : y \to qy$

¿ δ

$xy^2 - y^2x$

$= (xy - qyx)y + qy(xy - qyx)$

$= \quad y \quad + \quad qy$

$= (1+q)y \qquad$ (& <u>not</u> $2y$)

Ch-Jo Thms

- $R[x;\sigma]$

Thm R noeth UFR & each non-zero σ-prime ideal contains nonzero principal σ-ideal. Then $R[x;\sigma]$ is UFR.

Cor R noeth UFR, $\sigma^n = 1 \Rightarrow R[x;\sigma]$ UFR

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- $R[x;\delta]$

Thm R noeth UFR & each non-zero δ-prime ideal contains nonzero principal δ-ideal. Then $R[x;\delta]$ is UFR.

Eg $\mathfrak{g}_{\mathbb{C}}$ solvable $\Rightarrow$ $U(\mathfrak{g})$ UFR

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- $R[x;\sigma\ \delta]$

Hopeless!

Binomial Theorem

Quantum plane $k[x,y]$ $\quad xy = qyx$

- $(x+y)^2 = x^2 + xy + yx + y^2$

  $= x^2 + (1+q)yx + y^2$

- $(x+y)^3 = x^3 + (1+q)xyx + xy^2$

  $yx^2 + (1+q)yxy + y^3$

  $= x^3 + (1+q+q^2)yx^2 + (1+q+q^2)y^2x + y^3$

- Define $[3]_q := 1 + q + q^2$

  $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$

$[n]_q! := [n]_q \cdot [n-1]_q!$

$$\binom{n}{r}_q := \frac{[n]_q!}{[r]_q!\,[n-r]_q!}$$

generic q

$q^m \neq 1$

$$\boxed{(x+y)^n = \sum_{r=0}^{n} \binom{n}{r}_q y^r x^{n-r}}$$

Many interesting quantum algebras are of the form

$$k[x_1][x_2; \sigma_2, \delta_2] \cdots [x_n; \sigma_n, \delta_n]$$

& $\exists q$ with $\sigma\delta = q\delta\sigma$

$R = \mathcal{O}_q(M_2) = k\begin{bmatrix} a & b \\ & d \end{bmatrix}$ $0 \neq q \in k$, generic

$$ab = qba \qquad ac = qca$$
$$bd = qdb \qquad cd = qdc$$
$$bc = cb \qquad ad\ da = (q - q^{-1})bc$$

$R = k[a, b, c][d] = A[d \cdot \sigma, \delta]$

~~Cauchon - deleting derivations~~

- $H = (k^*)^4$ acts by multiplying thro' rows & cols
- d H-eigenerector
- σ action is via H
- δ is locally nilpotent

Cauchon - deleting derivations

- $R = \mathcal{O}_q(M_2)$ $\quad A = k[a, b, c]$

  $R = A[d; \sigma, \delta]$ $\quad D := ad - qbc$ is central.

- Inside $Q = \mathrm{Fract}(R)$, set

  $a = a - qbcd^{-1}$ $\quad (= (ad - qbc)d^{-1} = Dd^{-1})$

- $a'd - da' = Dd^{-1}d - dDd^{-1} = 0$

- Set $A' = k[a', b, c]$

  $R' = A'[d, \alpha]$

Fact $S = \{d^n\}$ is an Ore set in $R = \mathcal{O}_q(M_n)$ & so can form

$\hat{R} - RS^{-1}$ a subring of Q.

Note $A \cong A$

$$\begin{tikzcd}[column sep=small]
& RS^{-1} = \hat{R} = R'S^{-1} \arrow[dl, dash] \arrow[dr, dash] & \\
A[d;\sigma,\delta] = R & & R' = A'[d;\alpha]
\end{tikzcd}$$

- Pass info about prime ideals of R not containing d to & from R via the common localisation $\hat{R}$

- For primes P containing d, go to $P/\langle d\rangle$ & find inverse image in R' of a surjective homo

$$\begin{aligned}
R &\twoheadrightarrow R/\langle d\rangle \\
d &\longrightarrow 0 \\
a &\longrightarrow \bar{a} = a + \langle d\rangle
\end{aligned}$$

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General deleting derivations homo

$$\theta(a) = \sum_{n=0}^{\infty} \frac{(1-q)^{-n}}{[n]_q!} \delta^n \circ \sigma^{-n}(a) X^{-n}$$

Thm (LLR)

Let $R[x;\sigma,\delta]$ be a Cauchon Extension

Then R H-UFD $\Rightarrow R[x;\sigma,\delta]$ H-UFD.

- So, in $O_q(M_2)$ the height 1 H-primes are principal. They are

  $\langle D\rangle \quad \langle b\rangle \quad \langle c\rangle$

- Set $T = O_q(M_2)[D^{-1}\ b^{-1}\ c^{-1}]$

  Goodearl-Letzter theory gives

  centre of $T = k[D^{\pm}, (bc^{-1})^{\pm}]$

  & ideals of T are centrally generated.

  ~~Then noncomm~~

  and so T is UFD

Then, a noncommutative Nagata Lemma gives $\mathcal{O}_q(M_2)$ UFD.

Thm (LLR)

If $R = k[x_1][x_2,\sigma_2,\delta_2]\cdots[x_n;\sigma_n,\delta_n]$ is an iterated Cauchon extension then R is a UFD.

Examples

$\mathcal{O}_q(M_n)$, $U_q(\mathfrak{n}^+)$, $U_q(\mathfrak{b}^+)$

$G_q(m,n)$ by noncommutative dehomog^{n}

$G_q(m,n)[u^{-1}] \cong \mathcal{O}_q(m,n-m)[y,y^{-1};\phi]$

$\mathcal{O}_q(G)$ G ss up to a localisation.