THE FIRST FUNDAMENTAL THEOREM OF COINVARIANT THEORY FOR THE QUANTUM GENERAL LINEAR GROUP

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Abstract. We prove First Fundamental Theorems of Coinvariant Theory for the standard coactions of the quantum groups $\mathcal{O}_q(GL_t(K))$ and $\mathcal{O}_q(SL_t(K))$ on the quantized algebra $\mathcal{O}_q(M_{m,t}(K)) \otimes \mathcal{O}_q(M_{t,n}(K))$. (Here K is an arbitrary field and q an arbitrary nonzero scalar.) In both cases, the set of coinvariants is a subalgebra of $\mathcal{O}_q(M_{m,t}(K)) \otimes \mathcal{O}_q(M_{t,n}(K))$, which we identify.

Introduction

One of the highlights of classical invariant theory is the determination of the algebra of invariant functions for the standard action of the general linear group on the variety of pairs of matrices over a field K. More precisely, the standard action of $GL_t = GL_t(K)$ on the variety $V := M_{m,t}(K) \times M_{t,n}(K)$ induces an action of GL_t on $\mathcal{O}(V)$, and the classical theorem determines the algebra of invariants, $\mathcal{O}(V)^{GL_t}$. We recall the details below, since, if we assume that K is algebraically closed, the method of proof we follow has an easy geometric translation.

The main theorem of this paper, Theorem 4.5, gives a quantum analog of the above theorem. Since the quantum group $\mathcal{O}_q(GL_t)$ is not a group but a Hopf algebra, we first place the classical situation into a Hopf algebra context. This is standard: the action of GL_t on V induces a coaction of $\mathcal{O}(GL_t)$ on $\mathcal{O}(V)$, under which $\mathcal{O}(V)$ becomes an $\mathcal{O}(GL_t)$-comodule algebra, and $\mathcal{O}(V)^{GL_t}$ equals the algebra of $\mathcal{O}(GL_t)$-coinvariants in $\mathcal{O}(V)$. It is this situation which has a natural quantization: the coordinate ring $\mathcal{O}(V)$ becomes the algebra $\mathcal{O}_q(V) := \mathcal{O}_q(M_{m,t}) \otimes \mathcal{O}_q(M_{t,n})$, and the coaction of $\mathcal{O}(GL_t)$ on $\mathcal{O}(V)$ becomes a coaction of $\mathcal{O}_q(GL_t)$ on $\mathcal{O}_q(V)$. We prove the First Fundamental Theorem of Coinvariant Theory for this coaction, that is, we identify the set of coinvariants, $\mathcal{O}_q(V)^{\mathrm{co}\,\mathcal{O}_q(GL_t)}$.

There is a natural comultiplication map $\theta_q^* : \mathcal{O}_q(M_{m,n}) \to \mathcal{O}_q(M_{m,t}) \otimes \mathcal{O}_q(M_{t,n})$, which is the quantum analog of matrix multiplication $M_{m,t} \times M_{t,n} \to M_{m,n}$. We prove that the

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set of coinvariants is equal to the image of θ_q^*. In an earlier paper, [**2**], the first two authors have shown that the kernel of θ_q^* is the ideal generated by the $(t+1)\times(t+1)$ quantum minors of $\mathcal{O}_q(M_{m,n})$: this is the Second Fundamental Theorem of Coinvariant Theory for this comodule action. Taken together, these two results give a complete description of $\mathcal{O}_q(V)^{\mathrm{co}\,\mathcal{O}_q(GL_t)}$. Further, we investigate the coaction of $\mathcal{O}_q(SL_t)$ on $\mathcal{O}_q(V)$ induced by that of $\mathcal{O}_q(GL_t)$ and identify the coinvariants of this coaction.

The basic structure of our proof follows the outline of one of the possible proofs in the classical invariant theoretical setting. However, there are significant problems that arise due to the noncommutative setting. The most striking one is that, unlike in the commutative case, the $\mathcal{O}_q(GL_t)$-comodule $\mathcal{O}_q(V)$ is not a comodule algebra. For this reason, it is not even obvious at the outset that the set of coinvariants forms a subalgebra. More generally, the quantum analogs of several maps that we need are not algebra morphisms, and so their properties cannot be analyzed simply by checking how they behave on sets of algebra generators.

Nevertheless, it is useful to start by reviewing the classical situation, to provide a skeleton for our approach.

The classical situation. We fix an algebraically closed field K and positive integers m, n, t. For integers $u, v > 0$, we write $M_{u,v} = M_{u,v}(K)$ for the set of $u \times v$ matrices with entries in K. We will be mainly interested in the general linear group $GL_t = GL_t(K)$ and its standard action on the algebraic variety $V = V_{m,t,n} := M_{m,t} \times M_{t,n}$. This action is given by:

$$\begin{aligned} GL_t \times V &\xrightarrow{\quad\gamma\quad} V \\ (g,(A,B)) &\longmapsto (Ag^{-1}, gB). \end{aligned}$$

Thus GL_t acts on $\mathcal{O}(V) \cong \mathcal{O}(M_{m,t}) \otimes \mathcal{O}(M_{t,n})$. Classical invariant theory is interested in computing the subalgebra $\mathcal{O}(V)^{GL_t}$ of invariants for this action. The description of this algebra goes as follows. Consider the morphism of varieties

$$\begin{aligned} M_{m,t} \times M_{t,n} &\xrightarrow{\quad\theta\quad} M_{m,n} \\ (A,B) &\longmapsto AB \end{aligned}$$

and its associated comorphism $\theta^* : \mathcal{O}(M_{m,n}) \to \mathcal{O}(M_{m,t}) \otimes \mathcal{O}(M_{t,n})$. Let X_{ij} (for $1 \leq i \leq m$ and $1 \leq j \leq n$) stand for the usual coordinate functions on the variety $M_{m,n}$, and let $\mathcal{I}_{t+1}$ denote the ideal of $\mathcal{O}(M_{m,n})$ generated by all the $(t+1)\times(t+1)$ minors of the generic matrix (X_{ij}) over $\mathcal{O}(M_{m,n})$. (This ideal is zero if $t \geq \min\{m,n\}$.)

Theorem 0.1. *The ring of invariants $\mathcal{O}(V)^{GL_t}$ equals* $\operatorname{Im}\theta^*$. $\square$

Theorem 0.2. *The kernel of θ^* is $\mathcal{I}_{t+1}$.* $\square$

Theorems 0.1 and 0.2 are respectively known as the *First Fundamental Theorem of Invariant Theory* and the *Second Fundamental Theorem of Invariant Theory* (for GL_t). They give a complete description of $\mathcal{O}(V)^{GL_t}$.

We denote by $M_{m,n}^{\leq t}$ the subvariety of $M_{m,n}$ of $m \times n$ matrices with rank at most t. This variety is just the image of the morphism θ, and so we can factor θ in the form

$$M_{m,t} \times M_{t,n} \xrightarrow{\mu} M_{m,n}^{\leq t} \xrightarrow{\subseteq} M_{m,n}.$$

Since the restriction map $r : \mathcal{O}(M_{m,n}) \to \mathcal{O}(M^{\leq t}_{m,n})$ is surjective, the comorphism μ^* is injective and has the same image as θ^*. Thus, Theorem 0.1 can be rephrased in the form $\mathcal{O}(V)^{GL_t} = \operatorname{Im}\mu^*$. Further, Theorem 0.2 shows that $\ker(r) = \ker(\theta^*) = \mathcal{I}_{t+1}$, and therefore $\mathcal{O}(M^{\leq t}_{m,n}) = \mathcal{O}(M_{m,n})/\mathcal{I}_{t+1}$.

The proof of De Concini and Procesi for Theorem 0.1. We briefly describe the proof of Theorem 0.1 given by De Concini and Procesi in [**1**]; more precisely, we follow the exposition of that proof given in [**8**].

The general case can be easily reduced to that where $t < \min\{m, n\}$. So, we restrict attention to that particular case. Let us fix some notation. We will denote by $M^{\leq t,\circ}_{m,n}$ the open subset of $M^{\leq t}_{m,n}$ consisting of matrices with rank at most t whose upper leftmost $t \times t$ minor is invertible. In a similar way, $M^{\circ}_{m,t}$ denotes the set of $m \times t$ matrices whose uppermost $t \times t$ minor is invertible, and $M^{\circ}_{t,n}$ denotes the set of $t \times n$ matrices whose leftmost $t \times t$ minor is invertible. Finally, we set $V^\circ = M^{\circ}_{m,t} \times M^{\circ}_{t,n}$. Clearly the action γ of GL_t on V restricts to an action γ° on V° and we can ask for the invariants of that restricted action. It turns out that they are easy to compute.

Let us adopt the convention that if (A, B) is an element of $V = M_{m,t} \times M_{t,n}$ we write A and B in the forms

$$A = \begin{bmatrix} A_0 \\ A_1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} B_0 & B_1 \end{bmatrix}$$

with $A_0, B_0 \in M_t$ while $A_1 \in M_{m-t,t}$ and $B_1 \in M_{t,n-t}$.

With these notations, we define a morphism of varieties:

$$\begin{aligned} V^\circ &\xrightarrow{\ i\ } M^{\leq t,\circ}_{m,n} \times GL_t \\ (A, B) &\longmapsto (AB, B_0). \end{aligned}$$

It turns out that i is actually an isomorphism of varieties (this is an elementary fact, and the inverse morphism can be explicitly written down) and that the action of GL_t on $M^{\leq t,\circ}_{m,n} \times GL_t$ induced by γ° is just the natural one:

$$\begin{aligned} GL_t \times (M^{\leq t,\circ}_{m,n} \times GL_t) &\xrightarrow{\ \xi\ } M^{\leq t,\circ}_{m,n} \times GL_t \\ (g, C, h) &\longmapsto (C, gh). \end{aligned}$$

The invariants for ξ are easy to compute, using for instance the coinvariants of the associated $\mathcal{O}(GL_t)$-comodule structure on $\mathcal{O}(M^{\leq t,\circ}_{m,n} \times GL_t)$ (see the quantum case below). One then proves that $\mathcal{O}(M^{\leq t,\circ}_{m,n} \times GL_t)^{GL_t} = \mathcal{O}(M^{\leq t,\circ}_{m,n}) \subseteq \mathcal{O}(M^{\leq t,\circ}_{m,n} \times GL_t)$.

To recover from this the invariants for γ° it is enough to use the comorphism i^* of i. Let us consider the morphism

$$\mu^\circ : M^{\circ}_{m,t} \times M^{\circ}_{t,n} = V^\circ \longrightarrow M^{\leq t,\circ}_{m,n}$$

given by multiplication of matrices, and denote by $(\mu^\circ)^*$ its comorphism:

$$(\mu^\circ)^* : \mathcal{O}(M^{\leq t,\circ}_{m,n}) \longrightarrow \mathcal{O}(V^\circ) = \mathcal{O}(M^{\circ}_{m,t}) \otimes \mathcal{O}(M^{\circ}_{t,n}).$$

Then we can describe i^* as the composition

$$i^* : \mathcal{O}(M_{m,n}^{\leq t,\circ}) \otimes \mathcal{O}(GL_t) \xrightarrow{(\mu^\circ)^* \otimes \mathrm{id}} \mathcal{O}(V^\circ) \otimes \mathcal{O}(GL_t) \xrightarrow{\mathrm{id} \otimes \subseteq} \mathcal{O}(V^\circ) \otimes \mathcal{O}(M_{t,n}^\circ)$$
$$\xrightarrow{=} \mathcal{O}(M_{m,t}^\circ) \otimes \mathcal{O}(M_{t,n}^\circ) \otimes \mathcal{O}(M_{t,n}^\circ) \xrightarrow{\mathrm{id} \otimes \underline{\mathrm{m}}} \mathcal{O}(V^\circ),$$

where $\underline{\mathrm{m}}$ denotes multiplication in the algebra $\mathcal{O}(M_{t,n}^\circ)$. From the above we get that the ring of invariants for the action γ° is just $i^*(\mathcal{O}(M_{m,n}^{\leq t,\circ}) \otimes 1) = (\mu^\circ)^*(\mathcal{O}(M_{m,n}^{\leq t,\circ}))$.

The ring of invariants for γ° can thus be described as the localisation of $\mathrm{Im}\,\mu^*$ with respect to the multiplicative set generated by $d_Y \otimes d_Z$ where d_Y is the uppermost $t \times t$ minor of the generic matrix (Y_{ij}) of generators of $\mathcal{O}(M_{m,t})$, and d_Z is the leftmost $t \times t$ minor of the generic matrix (Z_{ij}) of generators of $\mathcal{O}(M_{t,n})$.

The second step of the proof is to show that one can "remove denominators" to deduce invariants for the action γ from invariants for the localised action γ°. Indeed, if ϕ is an invariant function in $\mathcal{O}(V)$ (i.e., for the action γ) then of course it is an invariant function in the localised ring $\mathcal{O}(V^\circ) = \mathcal{O}(V)_{d_Y \otimes d_Z}$ (for the action γ°). Hence, there is a non-negative integer s such that $\phi(d_Y \otimes d_Z)^s \in \mathrm{Im}\,\mu^*$. Thus, in order to establish that $\mathcal{O}(V)^{GL_t} = \mathrm{Im}\,\mu^*$, it is enough to prove that for $\psi \in \mathcal{O}(V)$, if there is a non-negative integer s such that $\psi(d_Y \otimes d_Z)^s \in \mathrm{Im}\,\mu^*$, then $\psi \in \mathrm{Im}\,\mu^*$. This last result is proved using the theory of standard bases.

In the quantum situation, the lower left quantum minors play a special rôle, since they are normal elements. It is for this reason that, unlike in the commutative case, we will invert leftmost *lower* $t \times t$ minors instead of leftmost upper ones.

Throughout the paper, we work over an arbitrary base field K and make an arbitrary choice of a nonzero element $q \in K$. We will have to deal with the following four quantized coordinate rings: $\mathcal{O}_q(M_{m,n})$, $\mathcal{O}_q(M_{m,t})$, $\mathcal{O}_q(M_{t,n})$ and $\mathcal{O}_q(GL_t)$. In order to avoid confusion, we will denote their respective canonical generators by X_{ij}, Y_{ij}, Z_{ij} and T_{ij}. The definitions of these algebras will be as in [**6**, **2**]; for instance, $X_{ij}X_{ik} = qX_{ik}X_{ij}$ when $j < k$. Thus, each time we use results from [**7**] we must replace q by q^{-1}. Finally, a convention concerning notation: each time that we have to deal with the multiplication map in an algebra, we will denote it by $\underline{\mathrm{m}}$. The context will make clear which algebra is concerned.

1. The general setup

Below, we will have to deal with the following situation combining a right and a left comodule algebra. Let $(H, \underline{\mathrm{m}}, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra, (A, ρ^*) a right comodule algebra over H, and (B, λ^*) a left comodule algebra over H. Here,

$$\rho^* : A \longrightarrow A \otimes H \qquad \text{and} \qquad \lambda^* : B \longrightarrow H \otimes B$$

are the comodule structure maps; that A and B are *comodule algebras* means that ρ^* and λ^* are also algebra morphisms. It is well known that A can be turned into a left H-comodule using the structure map $A \xrightarrow{\rho^*} A \otimes H \xrightarrow{\mathrm{id} \otimes S} A \otimes H \xrightarrow{\tau_{12}} H \otimes A$, where τ_{12} is the flip.

Thus, $A \otimes B$ can be equipped with the structure of a left H-comodule via the following structure map:

$$\gamma^* : A \otimes B \xrightarrow{\rho^* \otimes \lambda^*} A \otimes H \otimes H \otimes B \xrightarrow{\mathrm{id} \otimes S \otimes \mathrm{id} \otimes \mathrm{id}} A \otimes H \otimes H \otimes B$$
$$\xrightarrow{\tau_{(132)}} H \otimes H \otimes A \otimes B \xrightarrow{\underline{\mathrm{m}} \otimes \mathrm{id} \otimes \mathrm{id}} H \otimes A \otimes B,$$

where $\tau_{(132)}$ is the isomorphism which permutes the factors according to the cycle (132), that is, $\tau_{(132)}(a \otimes h \otimes h' \otimes b) = h \otimes h' \otimes a \otimes b$. Using the standard comodule notations $\rho^*(a) = \sum_{(a)} a_0 \otimes a_1$ and $\lambda^*(b) = \sum_{(b)} b_{-1} \otimes b_0$ for $a \in A$ and $b \in B$ (cf. [**5**, p. 11]), one thus has

$$\gamma^*(a \otimes b) = \sum_{(a),(b)} S(a_1)b_{-1} \otimes a_0 \otimes b_0.$$

Of course, $(A \otimes B, \gamma^*)$ is not a comodule algebra any longer. Nevertheless, this comodule continues to have nice properties that are now described. Let us recall that if (M, ν^*) is a left H-comodule, then the set of *H-coinvariants of* (M, ν^*) (or *ν^*-coinvariants* for short) is the sub-comodule of M defined by $M^{\mathrm{co}\, H} := \{x \in M \mid \nu^*(x) = 1 \otimes x\}$. It is immediate that if (M, ν^*) is a comodule algebra then $M^{\mathrm{co}\, H}$ is a subalgebra of M. In the more general situation described above this property is not automatic but still true.

Proposition 1.1. *In the above notation:*

(a) *If $v \in A \otimes B$ is such that $\gamma^*(v) = z \otimes v$ for some central element $z \in H$, and if $w \in A \otimes B$ is any element, then $\gamma^*(vw) = \gamma^*(v)\gamma^*(w)$. In particular, this holds when v is a γ^*-coinvariant.*

(b) *If $v, w \in A \otimes B$ are γ^*-coinvariants, then vw is again a γ^*-coinvariant.*

(c) *The set $(A \otimes B)^{\mathrm{co}\, H}$ is a subalgebra of $A \otimes B$.*

Proof. Without loss of generality, we may assume that $w = a' \otimes b'$ is a pure tensor. Moreover, let us write $v = \sum_{i=1}^r a_i \otimes b_i$. Since both ρ^* and λ^* are algebra morphisms, we have

$$(\rho^* \otimes \lambda^*)(vw) = \sum_{i=1}^r \sum_{(a_i),(b_i)} \sum_{(a'),(b')} a_{i,0}a'_0 \otimes a_{i,1}a'_1 \otimes b_{i,-1}b'_{-1} \otimes b_{i,0}b'_0,$$

where $a_{i,0} = (a_i)_0$ etc. Hence,

$$\begin{aligned}
(\gamma^*)(vw) &= \sum_{i=1}^r \sum_{(a_i),(b_i)} \sum_{(a'),(b')} S(a_{i,1}a'_1)b_{i,-1}b'_{-1} \otimes a_{i,0}a'_0 \otimes b_{i,0}b'_0 \\
&= \sum_{i=1}^r \sum_{(a_i),(b_i)} \sum_{(a'),(b')} S(a'_1)S(a_{i,1})b_{i,-1}b'_{-1} \otimes a_{i,0}a'_0 \otimes b_{i,0}b'_0 \\
&= \sum_{(a'),(b')} \big(S(a'_1) \otimes 1 \otimes 1\big)\left(\sum_{i=1}^r \sum_{(a_i),(b_i)} S(a_{i,1})b_{i,-1} \otimes a_{i,0} \otimes b_{i,0}\right)\big(b'_{-1} \otimes a'_0 \otimes b'_0\big) \\
&= \sum_{(a'),(b')} \big(S(a'_1) \otimes 1 \otimes 1\big)\gamma^*(v)\big(b'_{-1} \otimes a'_0 \otimes b'_0\big).
\end{aligned}$$

But, since $\gamma^*(v) = z \otimes v$ and z is central, we have

$$(\gamma^*)(vw) = \sum_{(a'),(b')} \big(S(a'_1) \otimes 1 \otimes 1\big)(z \otimes v)\big(b'_{-1} \otimes a'_0 \otimes b'_0\big)$$
$$= (z \otimes v) \sum_{(a'),(b')} S(a'_1)b'_{-1} \otimes a'_0 \otimes b'_0 = \gamma^*(v)\gamma^*(w).$$

This proves statement (a). Clearly, (b) as well as (c) follow at once from (a). $\square$

We now record a special case of a coaction for which computing the coinvariants is easy. Again, $(H, \underline{\mathrm{m}}, \eta, \Delta, \varepsilon, S)$ is a Hopf algebra, and M denotes a vector space. Clearly, the map

$$\begin{aligned} M &\longrightarrow H \otimes M \\ a &\longmapsto 1 \otimes a \end{aligned}$$

defines a (trivial) left coaction of H on M. Moreover, the comultiplication Δ of H makes H itself into a left H-comodule. The tensor product of these two coactions gives a left coaction of H on $M \otimes H$:

$$\begin{aligned} M \otimes H &\xrightarrow{\xi^*} H \otimes M \otimes H \\ a \otimes h &\longmapsto \textstyle\sum_{(h)} h_1 \otimes a \otimes h_2. \end{aligned}$$

Lemma 1.2. *With the above notation,* $(M \otimes H)^{\operatorname{co} H} = M \otimes 1$.

Proof. Consider $x \in (M \otimes H)^{\operatorname{co} H}$, and write $x = \sum_{i=1}^{s} a_i \otimes h_i$ where the a_i are linearly independent. Now

$$\sum_{i=1}^{s} \sum_{(h_i)} h_{i,1} \otimes a_i \otimes h_{i,2} = \xi^*(x) = 1 \otimes x = \sum_{i=1}^{s} 1 \otimes a_i \otimes h_i.$$

Since the a_i are linearly independent, we obtain

$$\Delta(h_i) = \sum_{(h_i)} h_{i,1} \otimes h_{i,2} = 1 \otimes h_i$$

for each i, whence $h_i = (\mathrm{id} \otimes \varepsilon)\Delta(h_i) = 1 \cdot \varepsilon(h_i)$. This shows that the h_i are scalars, and so $x \in M \otimes 1$. Thus, $(M \otimes H)^{\operatorname{co} H} \subseteq M \otimes 1$. The reverse inclusion is obvious. $\square$

2. Quantization of the standard action of GL_t on V

As in the introduction, we fix our base field K and positive integers m, n, t. By analogy with the commutative situation, we put

$$\mathcal{O}_q(V) = \mathcal{O}_q(V_{m,t,n}) := \mathcal{O}_q(M_{m,t}) \otimes \mathcal{O}_q(M_{t,n}).$$

It is easily checked that $\mathcal{O}_q(V)$ is an iterated skew polynomial extension of the base field K, and so $\mathcal{O}_q(V)$ is a noetherian domain. A quantum analogue of the action γ is obtained as a coaction of $\mathcal{O}_q(GL_t)$ on $\mathcal{O}_q(V)$ that we now describe.

It is easy to check that one can define morphisms of algebras satisfying the following rules:

$$\begin{array}{ccc} \mathcal{O}_q(M_{m,t}) & \xrightarrow{\rho_q^*} & \mathcal{O}_q(M_{m,t}) \otimes \mathcal{O}_q(GL_t) \\ Y_{ij} & \longmapsto & \sum_{k=1}^t Y_{ik} \otimes T_{kj} \end{array} \qquad \begin{array}{ccc} \mathcal{O}_q(M_{t,n}) & \xrightarrow{\lambda_q^*} & \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(M_{t,n}) \\ Z_{ij} & \longmapsto & \sum_{k=1}^t T_{ik} \otimes Z_{kj}. \end{array}$$

Moreover, ρ_q^* endows $\mathcal{O}_q(M_{m,t})$ with a right $\mathcal{O}_q(GL_t)$-comodule algebra structure, while λ_q^* endows $\mathcal{O}_q(M_{t,n})$ with a left $\mathcal{O}_q(GL_t)$-comodule algebra structure. It follows that $\tau_{12} \circ (\mathrm{id} \otimes S) \circ \rho_q^*$ gives $\mathcal{O}_q(M_{m,t})$ the structure of a left $\mathcal{O}_q(GL_t)$-comodule, where S denotes the antipode in $\mathcal{O}_q(GL_t)$. The tensor product of these two left coactions allows us to define a left comodule structure on $\mathcal{O}_q(V)$ that we denote by γ_q^*. So, as in Section 1,

$$\gamma_q^* : \mathcal{O}_q(V) \longrightarrow \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(V)$$

is given by the rule

$$\gamma_q^*(a \otimes b) = \sum_{(a),(b)} S(a_1) b_{-1} \otimes a_0 \otimes b_0$$

for $a \in \mathcal{O}_q(M_{m,t})$ and $b \in \mathcal{O}_q(M_{t,n})$.

The main objective of this paper is to calculate $\mathcal{O}_q(V)^{\mathrm{co}\, \mathcal{O}_q(GL_t)}$, the set of coinvariants of γ_q^*. As we see by Proposition 1.1, this set is a subalgebra of $\mathcal{O}_q(V)$.

Quantum minors. Recall that quantum minors in a quantum matrix algebra $\mathcal{O}_q(M_{u,v})$ correspond to quantum determinants in subalgebras of $\mathcal{O}_q(M_{u,v})$. More precisely, if $1 \leq r_1 < r_2 < \cdots < r_l \leq u$ and $1 \leq c_1 < c_2 < \cdots < c_l \leq v$, then the subalgebra of $\mathcal{O}_q(M_{u,v})$ generated by $\{X_{r_i c_j} \mid 1 \leq i, j \leq l\}$ is naturally isomorphic to $\mathcal{O}_q(M_l)$, and the element of this subalgebra corresponding to the quantum determinant in $\mathcal{O}_q(M_l)$ is called the quantum minor of $\mathcal{O}_q(M_{u,v})$ corresponding to the rows $r_1, \dots, r_l$ and columns $c_1, \dots, c_l$ (see [**6**, §1.2; **7**, §4.3]). Since we require a number of the formulas and results developed in [**2**], we shall use the notation of that paper for quantum minors. Thus, the $l \times l$ quantum minor described above will be denoted by the symbol

$$[r_l r_{l-1} \cdots r_1 \mid c_1 c_2 \cdots c_l],$$

or by $[R|C]$ where R and C denote the sets $\{r_1, \dots, r_l\}$ and $\{c_1, \dots, c_l\}$, respectively.

Remark 2.1. (i) The ideal of $\mathcal{O}_q(M_{m,n})$ generated by all the $(t+1) \times (t+1)$ quantum minors will be denoted $\mathcal{I}_{t+1}$, or $\mathcal{I}_{t+1}^{m,n}$ when m and n require emphasis. (In case $t \geq \min\{m, n\}$, there are no $(t+1) \times (t+1)$ quantum minors in $\mathcal{O}_q(M_{m,n})$, and $\mathcal{I}_{t+1} = 0$.) Moreover, we put

$$\mathcal{O}_q(M_{m,n}^{\leq t}) := \mathcal{O}_q(M_{m,n}) / \mathcal{I}_{t+1}^{m,n}$$

and $x_{ij} := X_{ij} + \mathcal{I}_{t+1} \in \mathcal{O}_q(M_{m,n}^{\leq t})$.

(ii) For any $m' \geq m$ and $n' \geq n$, we will identify $\mathcal{O}_q(M_{m,n})$ with the subalgebra of $\mathcal{O}_q(M_{m',n'})$ generated by those X_{ij} with $i \leq m$ and $j \leq n$. Now there is a K-algebra retraction

$$\pi_{m,n} = \pi_{m,n}^{m',n'} : \mathcal{O}_q(M_{m',n'}) \longrightarrow \mathcal{O}_q(M_{m,n})$$

such that $\pi_{m,n}(X_{ij}) = X_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, and $\pi_{m,n}(X_{ij}) = 0$ for $i > m$ or $j > n$. It is worth mentioning here that the inclusion map $\mathcal{O}_q(M_{m,n}) \to \mathcal{O}_q(M_{m',n'})$ (corresponding to the identification above) provides a section for $\pi_{m,n}$; this fact will be used without further comment. Note that $\pi_{m,n}[I|J] = [I|J]$ when $I \subseteq \{1,\dots,m\}$ and $J \subseteq \{1,\dots,n\}$, while $\pi_{m,n}[I|J] = 0$ otherwise. It follows that $\pi_{m,n}(\mathcal{I}_{t+1}^{m',n'}) = \mathcal{I}_{t+1}^{m,n}$, and consequently $\mathcal{I}_{t+1}^{m,n} = \mathcal{I}_{t+1}^{m',n'} \cap \mathcal{O}_q(M_{m,n})$.

(iii) As is easily checked, there is a morphism of algebras

$$\begin{aligned} \mathcal{O}_q(M_{m,n}) &\xrightarrow{\theta_q^*} \mathcal{O}_q(V) \\ X_{ij} &\longmapsto \textstyle\sum_{k=1}^t Y_{ik} \otimes Z_{kj}. \end{aligned}$$

For instance, this can be checked by using the commutative diagram

$$\begin{tikzcd} \mathcal{O}_q(M_l) \arrow[r, "\Delta"] & \mathcal{O}_q(M_l) \otimes \mathcal{O}_q(M_l) \arrow[d, "\pi_{m,t}\otimes\pi_{t,n}"] \\ \mathcal{O}_q(M_{m,n}) \arrow[u, "\subseteq"] \arrow[r, "\theta_q^*"] & \mathcal{O}_q(V) \end{tikzcd}$$

where $l \geq \max\{m,n\}$ and Δ denotes the comultiplication of $\mathcal{O}_q(M_l)$.

(iv) The comultiplication rule for quantum minors in a square quantum matrix algebra, say $\mathcal{O}_q(M_l)$, states that

$$\Delta[I|J] = \sum_{\substack{L \subseteq \{1,\dots,l\} \\ |L|=|I|}} [I|L] \otimes [L|J]$$

for all quantum minors $[I|J] \in \mathcal{O}_q(M_l)$ (e.g., [**6**, (1.9)]).

(v) From points (ii), (iii) and (iv), it is clear that $\mathcal{I}_{t+1}$ is contained in the kernel of θ_q^*. Hence, there is an induced morphism of algebras

$$\mu_q^* : \mathcal{O}_q(M_{m,n}^{\leq t}) \longrightarrow \mathcal{O}_q(V)$$

such that $\mu_q^*(x_{ij}) = \sum_{k=1}^t Y_{ik} \otimes Z_{kj}$. The work of the first and second authors in [**2**] leads to the following result:

Theorem 2.2. *In the above notation, the morphism μ_q^* is an injection. In particular, $\mathcal{O}_q(M_{m,n}^{\leq t})$ is a domain.*

Proof. We must show that $\ker(\theta_q^*) = \mathcal{I}_{t+1}$. In view of Remark 2.1(ii), there is no loss of generality in assuming that $m = n > t$. That case is proved in [**2**, Proposition 2.4]. □

Clearly, Theorem 2.2 gives a quantum analogue of the Second Fundamental Theorem of Invariant Theory; see Theorem 0.2. The rest of this work is devoted to proving a quantum analogue of the First Fundamental Theorem of Invariant Theory; see Theorem 0.1. A first easy proposition is in order.

Proposition 2.3. *The set of coinvariants $\mathcal{O}_q(V)^{\operatorname{co}\mathcal{O}_q(GL_t)}$ is a subalgebra of $\mathcal{O}_q(V)$ containing* $\operatorname{Im}\mu_q^*$.

Proof. We are in the setup defined at the begining of Section 1, so we may apply Proposition 1.1(c) to see that $\mathcal{O}_q(V)^{\operatorname{co}\mathcal{O}_q(GL_t)}$ is a subalgebra of $\mathcal{O}_q(V)$. For the rest of the statement, since the map μ_q^* is a morphism of algebras, it is enough to prove that $\mu_q^*(x_{ij})$ is a γ_q^*-coinvariant, for $1 \le i \le m$ and $1 \le j \le n$. However,

$$\begin{aligned}\gamma_q^*(\mu_q^*(x_{ij})) &= \sum_{k=1}^{t} \gamma_q^*(Y_{ik} \otimes Z_{kj}) = \sum_{r,s,k=1}^{t} S(T_{rk})T_{ks} \otimes Y_{ir} \otimes Z_{sj} \\ &= \sum_{r,s=1}^{t} \varepsilon(T_{rs}) \otimes Y_{ir} \otimes Z_{sj} = \sum_{r=1}^{t} 1 \otimes Y_{ir} \otimes Z_{rj} = 1 \otimes \mu_q^*(x_{ij}). \qquad \square\end{aligned}$$

Remark 2.4. By using the retraction maps $\pi_{\bullet,\bullet}$ discussed in Remark 2.1(ii), most results proved for quantized coordinate rings of square matrices immediately carry over to the rectangular case, as in the proof of Theorem 2.2. For the purposes of the present paper, rather than carry over a large number of results in this way, it is more efficient to check that it suffices to prove our main theorem in the case $m = n$. This also allows us to assume that $t < n$, as follows.

Choose an integer $l \ge \max\{m, n\}$ (later, it will be helpful to take $l > \max\{m, n\}$), and observe that we have a commutative diagram as follows:

$$\begin{tikzcd}
\mathcal{O}_q(M_{m,n}^{\le t}) \arrow[rr, equal] \arrow[dd, "\mu_q^*"'] \arrow[dr, "\subseteq"'] & & \mathcal{O}_q(M_{m,n}^{\le t}) \arrow[dd, "\mu_q^*"] \\
& \mathcal{O}_q(M_l^{\le t}) \arrow[ur, "\overline{\pi}_{m,n}"'] \arrow[dd, "\mu_q^*", near start] & \\
\mathcal{O}_q(V_{m,t,n}) \arrow[rr, equal] \arrow[dd, "\gamma_q^*"'] \arrow[dr, "\subseteq"'] & & \mathcal{O}_q(V_{m,t,n}) \arrow[dd, "\gamma_q^*"] \\
& \mathcal{O}_q(V_{l,t,l}) \arrow[ur, "\pi_{m,t}\otimes\pi_{t,n}"'] \arrow[dd, "\gamma_q^*", near start] & \\
\mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(V_{m,t,n}) \arrow[rr, equal] \arrow[dr, "\operatorname{id}\otimes\subseteq"'] & & \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(V_{m,t,n}) \\
& \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(V_{l,t,l}) \arrow[ur, "\operatorname{id}\otimes\pi_{m,t}\otimes\pi_{t,n}"'] &
\end{tikzcd}$$

Here '$\subseteq$' is shorthand for various embeddings induced by identifications (cf. Remark 2.1(ii)), and $\overline{\pi}_{m,n}$ denotes the map induced by the retraction $\pi_{m,n} : \mathcal{O}_q(M_l) \to \mathcal{O}_q(M_{m,n})$ (recall from Remark 2.1(ii) that $\pi_{m,n}(\mathcal{I}_{t+1}^{l,l}) = \mathcal{I}_{t+1}^{m,n}$). It is clear from the above diagram that

$$\mathcal{O}_q(V_{l,t,l})^{\operatorname{co}\mathcal{O}_q(GL_t)} = \mu_q^*\big(\mathcal{O}_q(M_l^{\le t})\big) \quad \Longrightarrow \quad \mathcal{O}_q(V_{m,t,n})^{\operatorname{co}\mathcal{O}_q(GL_t)} = \mu_q^*\big(\mathcal{O}_q(M_{m,n}^{\le t})\big).$$

Hence, to prove our main theorem it suffices to consider the case ($m = n > t$).

3. A localisation of γ_q^*

In view of Remark 2.4, we assume, until further notice, that $m = n > t$. Certain notations contract slightly in this case; e.g., $\mathcal{O}_q(M_{m,n}^{\leq t})$ becomes $\mathcal{O}_q(M_n^{\leq t})$.

Remark 3.1. (i) We label the following $t \times t$ quantum minors:

$$\begin{aligned}
d_X &:= [n, \dots, n-t+1 \mid 1, \dots, t] \in \mathcal{O}_q(M_n)\\
d_Y &:= [n, \dots, n-t+1 \mid 1, \dots, t] \in \mathcal{O}_q(M_{n,t})\\
d_Z &:= [t, \dots, 1 \mid 1, \dots, t] \in \mathcal{O}_q(M_{t,n})\\
d_T &:= [t, \dots, 1 \mid 1, \dots, t] \in \mathcal{O}_q(GL_t).
\end{aligned}$$

Further, we set $d_x = d_X + \mathcal{I}_{t+1} \in \mathcal{O}_q(M_n^{\leq t})$. It is known that $d_X \notin \mathcal{I}_{t+1}$ (for instance, it is clear from Remark 3.2(i) below that $\theta_q^*(d_X) \neq 0$), and so $d_x \neq 0$.

(ii) It is well known that in any $\mathcal{O}_q(M_{u,v})$, the "lower left" quantum minors – i.e., those corresponding to the lowest l rows and the leftmost l columns, for any l – are normal elements (e.g., see [**2**, Corollary 5.2]). In particular, the quantum minors d_X, d_Y, d_Z are normal in $\mathcal{O}_q(M_n)$, $\mathcal{O}_q(M_{n,t})$, $\mathcal{O}_q(M_{t,n})$, respectively. Since d_T is the quantum determinant in $\mathcal{O}_q(GL_t)$, it is central. It follows from the normality of d_X that d_x is normal in $\mathcal{O}_q(M_n^{\leq t})$.

(iii) Since d_Y is a normal element in $\mathcal{O}_q(M_{n,t})$, we can consider the localisation of $\mathcal{O}_q(M_{n,t})$ with respect to the multiplicative set generated by d_Y. By analogy with the commutative case, we denote this localisation, $\mathcal{O}_q(M_{n,t})[d_Y^{-1}]$, by $\mathcal{O}_q(M_{n,t}^\circ)$. Similarly, the localisations $\mathcal{O}_q(M_{t,n})[d_Z^{-1}]$ and $\mathcal{O}_q(M_n^{\leq t})[d_x^{-1}]$ will be denoted $\mathcal{O}_q(M_{t,n}^\circ)$ and $\mathcal{O}_q(M_n^{\leq t,\circ})$, respectively. At this stage it is worth mentioning that

$$\mathcal{O}_q(M_n^{\leq t,\circ}) \cong \big(\mathcal{O}_q(M_n)[d_X^{-1}]\big)\big/\big(\mathcal{I}_{t+1}[d_X^{-1}]\big);$$

this is an easy consequence of the universal properties of quotients and localisations. We will use this isomorphism without further comment.

Remark 3.2. (i) It follows from Remark 2.1(ii,iv) that in the notation of Section 2, one has

$$\rho_q^*(d_Y) = d_Y \otimes d_T \qquad \text{and} \qquad \lambda_q^*(d_Z) = d_T \otimes d_Z.$$

Further, $\theta_q^*(d_X) = d_Y \otimes d_Z$, and so

$$\mu_q^*(d_x) = d_Y \otimes d_Z.$$

(ii) In view of Remark 3.1(iii) and point (i) above, ρ_q^* and λ_q^* extend uniquely to morphisms of algebras

$$\begin{aligned}
\rho_q^{*\circ} &: \mathcal{O}_q(M_{n,t}^\circ) \longrightarrow \mathcal{O}_q(M_{n,t}^\circ) \otimes \mathcal{O}_q(GL_t)\\
\lambda_q^{*\circ} &: \mathcal{O}_q(M_{t,n}^\circ) \longrightarrow \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(M_{t,n}^\circ).
\end{aligned}$$

(iii) Clearly, $\lambda_q^{*\circ}$ defines a left $\mathcal{O}_q(GL_t)$-comodule algebra structure on $\mathcal{O}_q(M_{t,n}^\circ)$, while $\rho_q^{*\circ}$ defines a right $\mathcal{O}_q(GL_t)$-comodule algebra structure on $\mathcal{O}_q(M_{n,t}^\circ)$. Again, the composition $\tau_{12} \circ (\mathrm{id} \otimes S) \circ \rho_q^{*\circ}$ defines a left $\mathcal{O}_q(GL_t)$-comodule structure on $\mathcal{O}_q(M_{n,t}^\circ)$. Finally, if we set

$$\mathcal{O}_q(V^\circ) := \mathcal{O}_q(M_{n,t}^\circ) \otimes \mathcal{O}_q(M_{t,n}^\circ)$$

and tensor the above two left coactions, we obtain a map

$$\gamma_q^{*\circ} : \mathcal{O}_q(V^\circ) \longrightarrow \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(V^\circ),$$

endowing $\mathcal{O}_q(V^\circ)$ with the structure of a left $\mathcal{O}_q(GL_t)$-comodule. The map $\gamma_q^{*\circ}$ follows the same formula as γ_q^*; namely,

$$\gamma_q^{*\circ}(a \otimes b) = \sum_{(a),(b)} S(a_1)b_{-1} \otimes a_0 \otimes b_0$$

for $a \in \mathcal{O}_q(M_{n,t}^\circ)$ and $b \in \mathcal{O}_q(M_{t,n}^\circ)$. Obviously, the restriction of $\gamma_q^{*\circ}$ to $\mathcal{O}_q(V)$ is just γ_q^*.

(iv) Recall from Theorem 2.2 that the morphism $\mu_q^* : \mathcal{O}_q(M_n^{\leq t}) \longrightarrow \mathcal{O}_q(V)$ is injective. Because of Remark 3.1(iii) and point (i) above, we can extend μ_q^* to the corresponding localised algebras and obtain another injective morphism of algebras:

$$\mu_q^{*\circ} : \mathcal{O}_q(M_n^{\leq t,\circ}) \longrightarrow \mathcal{O}_q(V^\circ).$$

We shall need a technical lemma. We have already mentioned that γ_q^* is not an algebra morphism, and so neither is $\gamma_q^{*\circ}$. Nevertheless, we have the following statement, the first part of which is a special case of Proposition 1.1.

Lemma 3.3. *The set of coinvariants $\mathcal{O}_q(V^\circ)^{\mathrm{co}\,\mathcal{O}_q(GL_t)}$ is a subalgebra of $\mathcal{O}_q(V^\circ)$, containing $\mathrm{Im}\,\mu_q^{*\circ}$.*

Proof. As noted, Proposition 1.1(c) implies that $\mathcal{O}_q(V^\circ)^{\mathrm{co}\,\mathcal{O}_q(GL_t)}$ is a subalgebra of $\mathcal{O}_q(V^\circ)$.

Let us observe that for $r \in \mathbb{N}$, the element $(d_Y \otimes d_Z)^{-r}$ is a coinvariant for $\gamma_q^{*\circ}$. Indeed, since both $\rho_q^{*\circ}$ and $\lambda_q^{*\circ}$ are morphism of algebras, we have $\rho_q^{*\circ}(d_Y^{-r}) = d_Y^{-r} \otimes d_T^{-r}$ and $\lambda_q^{*\circ}(d_Z^{-r}) = d_T^{-r} \otimes d_Z^{-r}$. Recall that $S(d_T) = d_T^{-1}$ [**6**, p. 40; **7**, proof of Theorem 5.3.2]. Hence,

$$\begin{aligned}\gamma_q^{*\circ}((d_Y \otimes d_Z)^{-r}) &= \gamma_q^{*\circ}(d_Y^{-r} \otimes d_Z^{-r}) = S(d_T^{-r})d_T^{-r} \otimes d_Y^{-r} \otimes d_Z^{-r} \\ &= d_T^r d_T^{-r} \otimes d_Y^{-r} \otimes d_Z^{-r} = 1 \otimes d_Y^{-r} \otimes d_Z^{-r}.\end{aligned}$$

This proves the claim. Now since $\mu_q^{*\circ}$ is a morphism of algebras, any element $c \in \mathrm{Im}\,\mu_q^{*\circ}$ has the form $c = a(d_Y \otimes d_Z)^{-r}$ for some $a \in \mathrm{Im}\,\mu_q^*$ and $r \in \mathbb{N}$. So, from the observation above, the fact that $\mathrm{Im}\,\mu_q^* \subseteq \mathcal{O}_q(V)^{\mathrm{co}\,\mathcal{O}_q(GL_t)}$ (see Proposition 2.3), and the fact that $\mathcal{O}_q(V^\circ)^{\mathrm{co}\,\mathcal{O}_q(GL_t)}$ is a subalgebra of $\mathcal{O}_q(V^\circ)$, we conclude that $c \in \mathcal{O}_q(V^\circ)^{\mathrm{co}\,\mathcal{O}_q(GL_t)}$. □

Coaction of $\mathcal{O}_q(GL_t)$ on $\mathcal{O}_q(M_n^{\leq t,\circ}) \otimes \mathcal{O}_q(GL_t)$. One key point in the commutative case is that the action of GL_t on V° turns out to identify with the natural action of GL_t on $M_n^{\leq t,\circ} \times GL_t$ by left translation on the second factor. Here, we define and study a quantum analogue for the latter action.

We can apply the situation described at the end of Section 1 with $M = \mathcal{O}_q(M_n^{\leq t,\circ})$ and $H = \mathcal{O}_q(GL_t)$. We thus obtain a left $\mathcal{O}_q(GL_t)$-comodule structure on $\mathcal{O}_q(M_n^{\leq t,\circ}) \otimes \mathcal{O}_q(GL_t)$ via:

$$\begin{aligned}\mathcal{O}_q(M_n^{\leq t,\circ}) \otimes \mathcal{O}_q(GL_t) &\xrightarrow{\xi_q^*} \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(M_n^{\leq t,\circ}) \otimes \mathcal{O}_q(GL_t) \\ x \otimes h &\longmapsto \textstyle\sum_{(h)} h_1 \otimes x \otimes h_2.\end{aligned}$$

According to what we proved in Lemma 1.2, we have the following statement:

Proposition 3.4. *The set of ξ_q^*-coinvariants is a subalgebra of $\mathcal{O}_q(M_n^{\leq t,\circ}) \otimes \mathcal{O}_q(GL_t)$. More precisely,*

$$\big(\mathcal{O}_q(M_n^{\leq t,\circ}) \otimes \mathcal{O}_q(GL_t)\big)^{\operatorname{co}\mathcal{O}_q(GL_t)} = \mathcal{O}_q(M_n^{\leq t,\circ}) \otimes 1. \qquad \square$$

Link between the coactions $\gamma_q^{*\circ}$ and ξ_q^*. In light of the commutative case, one expects that $\mathcal{O}_q(M_n^{\leq t,\circ}) \otimes \mathcal{O}_q(GL_t)$ and $\mathcal{O}_q(V^\circ)$ are isomorphic as $\mathcal{O}_q(GL_t)$-comodules. This is, in fact, true, although we do not need the full statement here, only that $\mathcal{O}_q(V^\circ)$ is a comodule retraction of $\mathcal{O}_q(M_n^{\leq t,\circ}) \otimes \mathcal{O}_q(GL_t)$. The morphism that appears as the most natural candidate for an isomorphism between these two comodules is a quantum analog of i^* that we describe below.

Remark 3.5. (i) The following injective homomorphisms of algebras will be of constant use in the sequel. One should pay attention to the fact that, due to the choice of the minors we have inverted, these are not the usual maps one might expect to use. These embeddings are all into the lower left corner of the target algebra, in order to make use of the normality of the elements d_Y etc. We denote all these morphisms by "inj" in order to avoid introducing too much notation; the context will make clear which one we are using.

$$\begin{tikzcd}
 & T_{ij} \arrow[r, mapsto] & Y_{n-t+i,j} & \\
T_{ij} \arrow[d, mapsto] & \mathcal{O}_q(GL_t) \arrow[r, "\mathrm{inj}"] \arrow[d, "\mathrm{inj}"'] & \mathcal{O}_q(M_{n,t}^\circ) \arrow[d, "\mathrm{inj}"] & Y_{ij} \arrow[d, mapsto] \\
Z_{ij} & \mathcal{O}_q(M_{t,n}^\circ) \arrow[r, "\mathrm{inj}"] & \mathcal{O}_q(M_n^{\leq t,\circ}) & x_{ij} \\
 & Z_{ij} \arrow[r, mapsto] & x_{n-t+i,j} &
\end{tikzcd}$$

The injectivity of these maps is easy to prove, since in each case we can exhibit a left inverse. We show how to deal with the case of the left hand one; similar arguments hold for the others. The retraction $\pi_{t,t}$ (see Remark 2.1(ii)) induces a morphism of algebras $\mathcal{O}_q(M_{t,n}^\circ) \longrightarrow \mathcal{O}_q(GL_t)$ that maps Z_{ij} to T_{ij} if $j \leq t$ and to 0 if $j > t$. Clearly, this map provides a left inverse for inj.

(ii) Identify $\mathcal{O}_q(M_{n,t}^\circ)$ (respectively, $\mathcal{O}_q(M_{t,n}^\circ)$) with its image in $\mathcal{O}_q(M_n^{\leq t,\circ})$. Then, the restriction of $\mu_q^{*\circ}$ to $\mathcal{O}_q(M_{n,t}^\circ)$ (respectively, $\mathcal{O}_q(M_{t,n}^\circ)$) coincides with $\rho_q^{*\circ}$ (respectively, $\lambda_q^{*\circ}$), as one can easily check on generators (this is enough since the maps involved are morphism of algebras). More precisely, we are claiming that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{O}_q(M_{n,t}^\circ) \arrow[r, "\mathrm{inj}"] \arrow[d, "\rho_q^{*\circ}"'] & \mathcal{O}_q(M_n^{\leq t,\circ}) \arrow[d, "\mu_q^{*\circ}"] & \mathcal{O}_q(M_{t,n}^\circ) \arrow[l, "\mathrm{inj}"'] \arrow[d, "\lambda_q^{*\circ}"] \\
\mathcal{O}_q(M_{n,t}^\circ) \otimes \mathcal{O}_q(GL_t) \arrow[r, "\mathrm{id}\otimes\mathrm{inj}"] & \mathcal{O}_q(M_{n,t}^\circ) \otimes \mathcal{O}_q(M_{t,n}^\circ) & \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(M_{t,n}^\circ) \arrow[l, "\mathrm{inj}\otimes\mathrm{id}"']
\end{tikzcd}$$

For example, to see that the left hand square commutes, it suffices to check that $\mu_q^{*\circ} \circ \mathrm{inj}$ and $(\mathrm{id} \otimes \mathrm{inj}) \circ \rho_q^{*\circ}$ agree on the Y_{ij} and on d_Y. This is clear – both maps send Y_{ij} to

$\sum_{k=1}^{t} Y_{ik} \otimes Z_{kj}$ and d_Y to $d_Y \otimes d_Z$. The commutativity of the right hand square is proved analogously.

(iii) Finally, the restriction of $\lambda_q^{*\circ}$ (respectively, $\mu_q^{*\circ}$) to $\mathcal{O}_q(GL_t)$ coincides with Δ, i.e., the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{O}_q(GL_t) \arrow[r, "\mathrm{inj}"] \arrow[d, "\Delta"'] & \mathcal{O}_q(M_{t,n}^\circ) \arrow[r, "\mathrm{inj}"] \arrow[d, "\lambda_q^{*\circ}"] & \mathcal{O}_q(M_n^{\leq t,\circ}) \arrow[d, "\mu_q^{*\circ}"] \\
\mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(GL_t) \arrow[r, "\mathrm{id}\otimes\mathrm{inj}"] & \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(M_{t,n}^\circ) \arrow[r, "\mathrm{inj}\otimes\mathrm{id}"] & \mathcal{O}_q(M_{n,t}^\circ) \otimes \mathcal{O}_q(M_{t,n}^\circ)
\end{tikzcd}$$

In the following, we will identify the algebras $\mathcal{O}_q(GL_t)$, $\mathcal{O}_q(M_{n,t}^\circ)$, $\mathcal{O}_q(M_{t,n}^\circ)$ with their images under inj and we will use the commutativity of the above diagrams without explicit mention.

Quantization of i^*. Due to our identification of $\mathcal{O}_q(GL_t)$ with a subalgebra of $\mathcal{O}_q(M_{t,n}^\circ)$, we can use the multiplication map $\underline{\mathrm{m}}$ on $\mathcal{O}_q(M_{t,n}^\circ)$ to endow $\mathcal{O}_q(M_{t,n}^\circ)$ with the structure of a right $\mathcal{O}_q(GL_t)$-module. We now consider the following quantum analogue of i^*:

$$\begin{aligned} i_q^* : \mathcal{O}_q(M_n^{\leq t,\circ}) \otimes \mathcal{O}_q(GL_t) &\xrightarrow{\mu_q^{*\circ}\otimes\mathrm{id}} \mathcal{O}_q(M_{n,t}^\circ) \otimes \mathcal{O}_q(M_{t,n}^\circ) \otimes \mathcal{O}_q(GL_t) \\ &\xrightarrow{\mathrm{id}\otimes\underline{\mathrm{m}}} \mathcal{O}_q(M_{n,t}^\circ) \otimes \mathcal{O}_q(M_{t,n}^\circ) = \mathcal{O}_q(V^\circ). \end{aligned}$$

It can be proved that i_q^* is an isomorphism of left $\mathcal{O}_q(GL_t)$-comodules, but we shall not need the full result – for our purposes, it suffices to construct a right inverse for i_q^*.

Recall that the antipode S in $\mathcal{O}_q(GL_t)$ is bijective (e.g., [**7**, Theorem 5.4.2]).

Consider the map $j_q^* : \mathcal{O}_q(V^\circ) \to \mathcal{O}_q(M_n^{\leq t,\circ}) \otimes \mathcal{O}_q(GL_t)$ defined as the composition of the following maps:

$$\begin{aligned} \mathcal{O}_q(V^\circ) = \mathcal{O}_q(M_{n,t}^\circ) \otimes \mathcal{O}_q(M_{t,n}^\circ) &\xrightarrow{\rho_q^{*\circ}\otimes\lambda_q^{*\circ}} \mathcal{O}_q(M_{n,t}^\circ) \otimes \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(M_{t,n}^\circ) \\ &\xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\Delta\otimes\mathrm{id}} \mathcal{O}_q(M_{n,t}^\circ) \otimes \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(M_{t,n}^\circ) \\ &\xrightarrow{\mathrm{id}\otimes S\otimes\mathrm{id}\otimes S^{-1}\otimes\mathrm{id}} \mathcal{O}_q(M_{n,t}^\circ) \otimes \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(M_{t,n}^\circ) \\ &\xrightarrow{\tau_{(135)(24)}} \mathcal{O}_q(M_{t,n}^\circ) \otimes \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(M_{n,t}^\circ) \otimes \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(GL_t) \\ &\xrightarrow{\mathrm{inj}\otimes\mathrm{inj}\otimes\mathrm{inj}\otimes\underline{\mathrm{m}}} \mathcal{O}_q(M_n^{\leq t,\circ}) \otimes \mathcal{O}_q(M_n^{\leq t,\circ}) \otimes \mathcal{O}_q(M_n^{\leq t,\circ}) \otimes \mathcal{O}_q(GL_t) \\ &\xrightarrow{\underline{\mathrm{m}}(\underline{\mathrm{m}}\otimes\mathrm{id})\otimes\mathrm{id}} \mathcal{O}_q(M_n^{\leq t,\circ}) \otimes \mathcal{O}_q(GL_t). \end{aligned}$$

Here $\tau_{(135)(24)}$ denotes the isomorphism which shuffles the factors via the permutation $(135)(24)$, that is,

$$\tau_{(135)(24)}(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) = x_5 \otimes x_4 \otimes x_1 \otimes x_2 \otimes x_3.$$

For a pure tensor $a \otimes b \in \mathcal{O}_q(V^\circ)$, we can give the following formula for $j_q^*(a \otimes b)$ using the notation $\rho_q^*(a) = \sum_{(a)} a_0 \otimes a_1$ and $(\Delta \otimes \mathrm{id})\lambda_q^*(b) = \sum_{(b)} b_{-2} \otimes b_{-1} \otimes b_0$. Namely,

$$j_q^*(a \otimes b) = \sum_{(a),(b)} b_0 S^{-1}(b_{-1}) a_0 \otimes S(a_1) b_{-2}.$$

Lemma 3.6. *The map j_q^* is a morphism of left $\mathcal{O}_q(GL_t)$-comodules.*

Proof. We must prove that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{O}_q(V^\circ) \arrow[rr, "j_q^*"] \arrow[d, "\gamma_q^{*\circ}"'] & & \mathcal{O}_q(M_n^{\leq t,\circ}) \otimes \mathcal{O}_q(GL_t) \arrow[d, "\xi_q^*"] \\
\mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(V^\circ) \arrow[rr, "\mathrm{id}\otimes j_q^*"] & & \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(M_n^{\leq t,\circ}) \otimes \mathcal{O}_q(GL_t)
\end{tikzcd}$$

If $a \in \mathcal{O}_q(M_{n,t}^\circ)$ and $b \in \mathcal{O}_q(M_{t,n}^\circ)$, then (with the usual conventions for expanding components)

$$\begin{aligned}
(\mathrm{id}\otimes j_q^*)\gamma_q^{*\circ}(a\otimes b) &= \sum_{(a),(b)} S(a_1)b_{-1}\otimes j_q^*(a_0\otimes b_0)\\
&= \sum_{(a),(b)} S(a_2)b_{-3}\otimes b_0S^{-1}(b_{-1})a_0\otimes S(a_1)b_{-2}.
\end{aligned}$$

Since $\Delta(S(c)d) = \sum_{(c),(d)} S(c_2)d_1 \otimes S(c_1)d_2$ for $c, d \in \mathcal{O}_q(GL_t)$, we also have

$$\begin{aligned}
\xi_q^* j_q^*(a\otimes b) &= \sum_{(a),(b)} \xi_q^*\big(b_0S^{-1}(b_{-1})a_0\otimes S(a_1)b_{-2}\big)\\
&= \sum_{(a),(b)} S(a_2)b_{-3}\otimes b_0S^{-1}(b_{-1})a_0\otimes S(a_1)b_{-2}.
\end{aligned}$$

Therefore $(\mathrm{id}\otimes j_q^*)\gamma_q^{*\circ}(a\otimes b) = \xi_q^* j_q^*(a\otimes b)$, as desired. $\square$

In the sequel, we shall need the identity

$$\sum_{(c)} c_2S^{-1}(c_1) = \sum_{(c)} S^{-1}(c_2)c_1 = \varepsilon(c)\cdot 1$$

for $c \in \mathcal{O}_q(GL_t)$. This follows from the defining property of the antipode S by applying the antiautomorphism S^{-1}. Moreover, in view of the identity $\Delta\circ S = \tau_{12}\circ(S\otimes S)\circ\Delta$ (e.g., [**5**, Proposition 1.5.10]), we also have

$$(S^{-1}\otimes S^{-1})\circ\tau_{12}\circ\Delta = \Delta\circ S^{-1}.$$

Lemma 3.7. *The composition $i_q^*\circ j_q^*$ is the identity map on $\mathcal{O}_q(V^\circ)$.*

Proof. Let $a\otimes b\in\mathcal{O}_q(V^\circ)$. Using Remark 3.5(ii,iii) and the fact that $\mu_q^{*\circ}$ is a morphism of algebras, we have

$$\begin{aligned}
(\mu_q^{*\circ}\otimes\mathrm{id})j_q^*(a\otimes b) &= \sum_{(a),(b)} \big(\mu_q^{*\circ}(b_0)\big(\mu_q^{*\circ}S^{-1}(b_{-1})\big)\mu_q^{*\circ}(a_0)\big)\otimes S(a_1)b_{-2}\\
&= \sum_{(a),(b)} \big(\lambda_q^{*\circ}(b_0)\big(\Delta S^{-1}(b_{-1})\big)\rho_q^{*\circ}(a_0)\big)\otimes S(a_1)b_{-2}\\
&= \sum_{(a),(b)} \big((b_{-1}\otimes b_0)\big(\Delta S^{-1}(b_{-2})\big)(a_0\otimes a_1)\big)\otimes S(a_2)b_{-3}.
\end{aligned}$$

Consequently, with the help of the identities above, we see that

$$\begin{aligned}(\mu_q^{*\circ}\otimes \mathrm{id})j_q^*(a\otimes b) &= \sum_{(a),(b)} b_{-1}S^{-1}(b_{-2})a_0\otimes b_0S^{-1}(b_{-3})a_1\otimes S(a_2)b_{-4}\\ &= \sum_{(a),(b)} \varepsilon(b_{-1})a_0\otimes b_0S^{-1}(b_{-2})a_1\otimes S(a_2)b_{-3}\\ &= \sum_{(a),(b)} a_0\otimes b_0S^{-1}\big(b_{-2}\varepsilon(b_{-1})\big)a_1\otimes S(a_2)b_{-3}\\ &= \sum_{(a),(b)} a_0\otimes b_0S^{-1}(b_{-1})a_1\otimes S(a_2)b_{-2}.\end{aligned}$$

Thus,

$$\begin{aligned}i_q^*j_q^*(a\otimes b) &= \sum_{(a),(b)} a_0\otimes b_0S^{-1}(b_{-1})a_1S(a_2)b_{-2} = \sum_{(a),(b)} a_0\otimes b_0S^{-1}(b_{-1})\varepsilon(a_1)b_{-2}\\ &= \sum_{(a),(b)} a_0\varepsilon(a_1)\otimes b_0S^{-1}(b_{-1})b_{-2} = \sum_{(b)} a\otimes b_0\varepsilon(b_{-1}) = a\otimes b. \qquad \square\end{aligned}$$

Proposition 3.8. $\mathcal{O}_q(V^\circ)^{\mathrm{co}\,\mathcal{O}_q(GL_t)} = \mathrm{Im}\,\mu_q^{*\circ}$.

Proof. We have the inclusion "$\supseteq$" by Lemma 3.3. To prove the reverse inclusion, consider an arbitrary $\gamma_q^{*\circ}$-coinvariant $c\in\mathcal{O}_q(V^\circ)$. Since j_q^* is a comodule morphism (Lemma 3.6), $j_q^*(c)$ is a ξ_q^*-coinvariant in $\mathcal{O}_q(M_n^{\leq t,\circ})\otimes\mathcal{O}_q(GL_t)$. Hence, $j_q^*(c)\in\mathcal{O}_q(M_n^{\leq t,\circ})\otimes 1$ by Proposition 3.4. Since $i_q^*j_q^*(c)=c$ by Lemma 3.7, and since clearly $i_q^*(\mathcal{O}_q(M_n^{\leq t,\circ})\otimes 1) = \mathrm{Im}\,\mu_q^{*\circ}$, we conclude that $c\in\mathrm{Im}\,\mu_q^{*\circ}$. This verifies the inclusion "$\subseteq$". $\square$

4. The First Fundamental Theorem for $\mathcal{O}_q(GL_t)$-coinvariants

In this section, we complete the proof of our main theorem – that the set of γ_q^*-coinvariants in $\mathcal{O}_q(V)$ equals the image of μ_q^*. We first discuss how to obtain γ_q^*-coinvariants from $\gamma_q^{*\circ}$-coinvariants by "removing denominators".

Remark 4.1. (i) As we already mentioned in Remark 3.2(iii), the restriction of $\gamma_q^{*\circ}$ to $\mathcal{O}_q(V)$ is just γ_q^*. So, if $w\in\mathcal{O}_q(V)$ is a γ_q^*-coinvariant, it is also a $\gamma_q^{*\circ}$-coinvariant. Thus, according to Proposition 3.8, $w\in\mathrm{Im}\,\mu_q^{*\circ}$. This means that there exists $s\in\mathbb{N}$ such that $w(d_Y\otimes d_Z)^s\in\mathrm{Im}\,\mu_q^*$. So, to prove that the set of γ_q^*-coinvariants equals $\mathrm{Im}\,\mu_q^*$, it is enough to show the following property (cf. Proposition 2.3):

$$(\dagger)\qquad \forall w\in\mathcal{O}_q(V),\ \text{ if }\ w(d_Y\otimes d_Z)\in\mathrm{Im}\,\mu_q^*,\ \text{ then }\ w\in\mathrm{Im}\,\mu_q^*.$$

(ii) Recall from Remark 3.2(i) that the image under μ_q^* of the (normal) element $d_x\in\mathcal{O}_q(M_n^{\leq t})$ is the (normal) element $d_Y\otimes d_Z\in\mathcal{O}_q(V)$. We claim that to prove ($\dagger$), it is enough (in fact, equivalent) to verify the following property:

$$(\ddagger)\qquad (\mu_q^*)^{-1}\big(\langle d_Y\otimes d_Z\rangle\big) = \langle d_x\rangle.$$

Thus, assume that (‡) holds, and let $w \in \mathcal{O}_q(V)$ such that $w(d_Y \otimes d_Z) \in \operatorname{Im} \mu_q^*$. Then $w(d_Y \otimes d_Z) = \mu_q^*(y)$ for some $y \in \mathcal{O}_q(M_n^{\leq t})$, and (‡) implies that $y = zd_x$ for some $z \in \mathcal{O}_q(M_n^{\leq t})$. Since μ_q^* is a morphism of algebras, $w(d_Y \otimes d_Z) = \mu_q^*(y) = \mu_q^*(z)(d_Y \otimes d_Z)$, and so $w = \mu_q^*(z)$ because $\mathcal{O}_q(V)$ is a domain. This shows that (‡) implies (†). The converse follows easily from the injectivity of μ_q^* (see Theorem 2.2).

Remark 4.2. (i) The computations in this section and in Section 6 below rely on the *preferred bases* for the algebras $\mathcal{O}_q(M_n)$, $\mathcal{O}_q(M_{n,t})$, and $\mathcal{O}_q(M_{t,n})$ developed in [**2**, Section 1], and we follow the notation of that paper. See, in particular, [**2**, Corollary 1.11] for the rectangular case. We recall the notation $[T|T']$ for the product of quantum minors corresponding to an allowable bitableaux (T, T'), and we recall also that it is sometimes convenient to label rows of (T, T') in the form (I, J) where I and J are sets of row and column indices, respectively.

(ii) On tensoring the preferred basis elements of $\mathcal{O}_q(M_{n,t})$ with those of $\mathcal{O}_q(M_{t,n})$, we obtain a preferred basis for $\mathcal{O}_q(V)$ consisting of all pure tensors $[S|S'] \otimes [T|T']$ where (S, S') and (T, T') are preferred bitableaux with appropriate entries. In particular, S and T each have at most t columns.

(iii) Because of the quantum Laplace expansions [**6**, Proposition 1.1 and Corollary; **7**, Corollary 4.4.4], the ideal $\mathcal{I}_{t+1}$ in $\mathcal{O}_q(M_n)$ contains all the $r \times r$ quantum minors for $t+1 \leq r \leq n$. Hence, $[T|T'] \in \mathcal{I}_{t+1}$ whenever (T, T') is a preferred bitableau and T has at least $t+1$ columns.

The statement and proof of [**2**, Proposition 2.4] show that $\mathcal{I}_{t+1}$ does not contain any nonzero linear combinations of distinct products $[S|S']$ where the (S, S') are preferred bitableaux and S has at most t columns. Consequently, $\mathcal{I}_{t+1}$ has a K-basis consisting of the products $[T|T']$ such that (T, T') is a preferred bitableau and T has at least $t+1$ columns.

(iv) In view of point (iii), the cosets $[T|T'] + \mathcal{I}_{t+1}$ such that (T, T') is a preferred bitableau and T has at most t columns form a K-basis for $\mathcal{O}_q(M_n^{\leq t})$. To simplify the notation, we shall write cosets $a + \mathcal{I}_{t+1}$ in the form $\overline{a}$. Thus, $\overline{[T|T']} = 0$ whenever T has more than t columns.

It is convenient to label two special index sets:

$$\widetilde{I} = \{n-t+1, \dots, n\} \qquad \text{and} \qquad \widetilde{J} = \{1, \dots, t\}.$$

With this notation, observe that d_X and d_Y can both be labeled $[\widetilde{I}|\widetilde{J}]$, while d_Z and d_T can both be labeled $[\widetilde{J}|\widetilde{J}]$.

Lemma 4.3. (a) *A K-basis for $\langle d_Y \rangle$ consists of all $[T|T']$ in $\mathcal{O}_q(M_{n,t})$ where (T, T') is a preferred bitableau with first row $(\widetilde{I}, \widetilde{J})$.*

(b) *A K-basis for $\langle d_Z \rangle$ consists of all $[T|T']$ in $\mathcal{O}_q(M_{t,n})$ where (T, T') is a preferred bitableau with first row $(\widetilde{J}, \widetilde{J})$.*

(c) *A K-basis for $\langle d_Y \otimes d_Z \rangle$ consists of all pure tensors $[S|S'] \otimes [T|T']$ in $\mathcal{O}_q(V)$ where (S, S') and (T, T') are preferred bitableaux with first rows $(\widetilde{I}, \widetilde{J})$ and $(\widetilde{J}, \widetilde{J})$, respectively.*

(d) *A K-basis for $\langle d_x \rangle$ consists of all $\overline{[T|T']}$ in $\mathcal{O}_q(M_n^{\leq t})$ where (T, T') is a preferred bitableau with first row $(\widetilde{I}, \widetilde{J})$.*

Proof. (a) Obviously any such $[T|T']$ lies in $\langle d_Y\rangle$, because $d_Y = [\widetilde{I}|\widetilde{J}]$. Recall that d_Y is a normal element of $\mathcal{O}_q(M_{n,t})$, so that $\langle d_Y\rangle = d_Y\mathcal{O}_q(M_{n,t})$. Since $(\widetilde{I},\widetilde{J})$ is the minimum index pair labeling quantum minors in $\mathcal{O}_q(M_{n,t})$, the preferred basis of $\mathcal{O}_q(M_{n,t})$ is closed under left multiplication by d_Y. Hence, $\langle d_Y\rangle$ is spanned by products $d_Y[S|S']$ as (S,S') runs through all preferred bitableaux, and each such $d_Y[S|S'] = [T|T']$ for some preferred bitableau (T,T') with first row $(\widetilde{I},\widetilde{J})$.

(b) This is proved in the same manner as part (a).

(c) Note that because d_Y and d_Z are normal elements of $\mathcal{O}_q(M_{n,t})$ and $\mathcal{O}_q(M_{t,n})$, respectively,

$$\langle d_Y\otimes d_Z\rangle = (d_Y\otimes d_Z)\mathcal{O}_q(V) = d_Y\mathcal{O}_q(M_{n,t})\otimes d_Z\mathcal{O}_q(M_{t,n}) = \langle d_Y\rangle\otimes\langle d_Z\rangle.$$

Hence, part (c) follows directly from parts (a) and (b).

(d) By Remark 4.2(iv), a K-basis for $\mathcal{O}_q(M_n^{\le t})$ consists of all $\overline{[T|T']}$ such that (T,T') is a preferred bitableau and T has at most t columns. Among index pairs (I,J) with $|I|\le t$, the minimum element is $(\widetilde{I},\widetilde{J})$. Since $d_x = \overline{[\widetilde{I}|\widetilde{J}]}$, we therefore obtain part (d) in exactly the same manner as part (a). □

In the following proof, we shall need the natural multigradings on quantum matrix algebras (cf. [**2**, §1.5]). For instance, $\mathcal{O}_q(M_{n,t})$ is graded by $\mathbb{Z}^n\times\mathbb{Z}^t$ with each generator Y_{ij} having degree (ϵ_i,ϵ_j), where $\epsilon_1,\epsilon_2,\dots$ denote the standard basis elements in $\mathbb{Z}^n$ and $\mathbb{Z}^t$. In the proof, we use the label 'homogeneous' to refer to homogeneous elements with respect to the above gradings.

Proposition 4.4. $(\mu_q^*)^{-1}\big(\langle d_Y\otimes d_Z\rangle\big) = \langle d_x\rangle$.

Proof. The inclusion '$\supseteq$' is clear since $\mu_q^*(d_x) = d_Y\otimes d_Z$. If this inclusion is proper, choose an element

$$x = \sum_{i=1}^{r}\alpha_i\overline{[T_i|T_i']}\in(\mu_q^*)^{-1}\big(\langle d_Y\otimes d_Z\rangle\big)\setminus\langle d_x\rangle$$

where the α_i are nonzero scalars, the (T_i,T_i') are distinct preferred bitableaux, and the T_i have at most t columns. We may assume that none of the $\overline{[T_i|T_i']}$ lie in $\langle d_x\rangle$. Thus, by Lemma 4.3(d), none of the (T_i,T_i') has first row $(\widetilde{I},\widetilde{J})$. Define n-tuples $\overline{\rho}(T_i)$ as in [**2**, §2.2], and let $\overline{\rho}$ be the minimum of the $\overline{\rho}(T_i)$ under reverse lexicographic order.

After re-indexing, we may assume that there is some r' such that $\overline{\rho}(T_i) = \overline{\rho}$ for $i\le r'$ and $\overline{\rho}(T_i) >_{\mathrm{rlex}} \overline{\rho}$ for $i>r'$. Applying [**2**, Lemma 2.3] to each $\mu_q^*\overline{[T_i|T_i']}$ $(=\theta_q^*[T_i|T_i'])$ and collecting terms, we see (using the notation of [**2**, §2.2]) that

$$\mu_q^*(x) = \sum_{i=1}^{r'}\alpha_i[T_i|\mu(T_i)]\otimes[\mu'(T_i)|T_i'] + \sum_j X_j\otimes Y_j$$

where the X_j and Y_j are homogeneous with $\overline{c}(X_j) = \overline{r}(Y_j) >_{\mathrm{rlex}} \overline{\rho}$. We then observe (as in the proof of [**2**, Proposition 2.4]) that all of the X_j belong to different homogeneous components than the $[T_i|\mu(T_i)]$ for $i \le r'$. Since $\mu_q^*(x) \in \langle d_Y \otimes d_Z\rangle = \langle d_Y\rangle \otimes \langle d_Z\rangle$ and the ideal $\langle d_Y\rangle$ is homogeneous, it follows that

$$\sum_{i=1}^{r'} \alpha_i [T_i|\mu(T_i)] \otimes [\mu'(T_i)|T_i'] \in \langle d_Y \otimes d_Z\rangle.$$

For $1 \le i < j \le r'$, either $T_i \ne T_j$ or $T_i' \ne T_j'$, whence $(T_i, \mu(T_i)) \ne (T_j, \mu(T_j))$ or $(\mu'(T_i), T_i') \ne (\mu'(T_j), T_j')$. Recall from [**2**, §2.2] that the $(T_i, \mu(T_i))$ and the $(\mu'(T_i), T_i')$ are preferred bitableaux. In view of Lemma 4.3(c), it follows that T_i has first row $\widetilde{I}$ and T_i' has first row $\widetilde{J}$ for $1 \le i \le r'$. However, this contradicts our choices above, and therefore the proposition is proved. □

Theorem 4.5. *Let m, n, t be arbitrary positive integers. The set of γ_q^*-coinvariants in $\mathcal{O}_q(V) = \mathcal{O}_q(M_{m,t}) \otimes \mathcal{O}_q(M_{t,n})$ is $\operatorname{Im}\mu_q^*$, that is,*

$$\big(\mathcal{O}_q(M_{m,t}) \otimes \mathcal{O}_q(M_{t,n})\big)^{\operatorname{co}\mathcal{O}_q(GL_t)} = \mu_q^*\big(\mathcal{O}_q(M_{m,n}^{\le t})\big).$$

Proof. In view of Remark 2.4, there is no loss of generality in assuming that $m = n > t$. The theorem is then immediate using Remark 4.1 and Proposition 4.4. □

5. $\mathcal{O}_q(SL_t)$-COINVARIANTS

For arbitrary positive integers m, n, t, there is a natural coaction Γ_q^* of $\mathcal{O}_q(SL_t)$ on $\mathcal{O}_q(V)$ induced from the $\mathcal{O}_q(GL_t)$-coaction γ_q^* that we have been studying. Denote by π the natural quotient map $\mathcal{O}_q(GL_t) \to \mathcal{O}_q(SL_t)$. The coaction $\Gamma_q^* : \mathcal{O}_q(V) \to \mathcal{O}_q(SL_t) \otimes \mathcal{O}_q(V)$ is given by $\Gamma_q^* := (\pi \otimes \mathrm{id}) \circ \gamma_q^*$. In this section and the next, we compute the Γ_q^*-coinvariants.

In fact, Γ_q^* can be constructed from the left coaction of $\mathcal{O}_q(SL_t)$ on $\mathcal{O}_q(M_{t,n})$ given by

$$(\pi \otimes \mathrm{id}) \circ \lambda_q^* : \mathcal{O}_q(M_{t,n}) \to \mathcal{O}_q(SL_t) \otimes \mathcal{O}_q(M_{t,n})$$

and the right coaction of $\mathcal{O}_q(SL_t)$ on $\mathcal{O}_q(M_{m,t})$ given by

$$(\mathrm{id} \otimes \pi) \circ \rho_q^* : \mathcal{O}_q(M_{m,t}) \to \mathcal{O}_q(M_{m,t}) \otimes \mathcal{O}_q(SL_t)$$

by using the construction described in Section 1.

Proposition 5.1. *The set of coinvariants $\mathcal{O}_q(V)^{\operatorname{co}\mathcal{O}_q(SL_t)}$ is a subalgebra of $\mathcal{O}_q(V)$ containing $\operatorname{Im}\mu_q^*$.*

Proof. That $\mathcal{O}_q(V)^{\operatorname{co}\mathcal{O}_q(SL_t)}$ is a subalgebra of $\mathcal{O}_q(V)$ follows from Proposition 1.1(c). It contains $\operatorname{Im}\mu_q^*$ because the elements of $\operatorname{Im}\mu_q^*$ are γ_q^*-coinvariants and hence also Γ_q^*-coinvariants. □

In order to describe the Γ_q^*-coinvariants, we may assume that $m = n$ and that $t < n$, as in Remark 2.4. We start by observing that there is a natural $\mathbb{Z}$-grading on quantum matrices which will simplify the problem. The algebra $\mathcal{O}_q(M_{n,t})$ can be graded by total degree in the variables Y_{ij} and we put $\mathcal{O}_q(M_{n,t}) = \bigoplus_{i \ge 0} \mathcal{O}_q(M_{n,t})_i$, where $\mathcal{O}_q(M_{n,t})_i$ is the subspace spanned by monomials of total degree i in $\mathcal{O}_q(M_{n,t})$. In the same way, and with obvious notation, $\mathcal{O}_q(M_{t,n}) = \bigoplus_{j\geq 0} \mathcal{O}_q(M_{t,n})_j$. It follows that

$$\mathcal{O}_q(V) = \bigoplus_{i,j=0}^{\infty} \mathcal{O}_q(V)_{i,j},$$

where $\mathcal{O}_q(V)_{i,j} = \mathcal{O}_q(M_{n,t})_i \otimes \mathcal{O}_q(M_{t,n})_j$.

Recall that $\rho_q^* : \mathcal{O}_q(M_{n,t}) \to \mathcal{O}_q(M_{n,t}) \otimes \mathcal{O}_q(GL_t)$ is the algebra morphism defined by $\rho_q^*(Y_{ij}) = \sum_{k=1}^t Y_{ik} \otimes T_{kj}$. Thus, it is clear that $\rho_q^*(\mathcal{O}_q(M_{n,t})_i) \subseteq \mathcal{O}_q(M_{n,t})_i \otimes \mathcal{O}_q(GL_t)$. Similarly, one has $\lambda_q^*(\mathcal{O}_q(M_{t,n})_j) \subseteq \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(M_{t,n})_j$. It follows that

$$\gamma_q^*(\mathcal{O}_q(V)_{i,j}) \subseteq \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(V)_{i,j}$$
$$\Gamma_q^*(\mathcal{O}_q(V)_{i,j}) \subseteq \mathcal{O}_q(SL_t) \otimes \mathcal{O}_q(V)_{i,j}$$

for all i, j.

Given $w \in \mathcal{O}_q(V)$, we will write $w = \sum_{i,j} w_{ij}$ with $w_{ij} \in \mathcal{O}_q(V)_{i,j}$.

Lemma 5.2. *Let $w = \sum_{i,j} w_{ij} \in \mathcal{O}_q(V)$. Then*

(a) *w is a γ_q^*-coinvariant if and only if each w_{ij} is a γ_q^*-coinvariant.*

(b) *w is a Γ_q^*-coinvariant if and only if each w_{ij} is a Γ_q^*-coinvariant.*

Proof. If each w_{ij} is a γ_q^*-coinvariant then, obviously, w is a γ_q^*-coinvariant. Conversely, suppose that w is a γ_q^*-coinvariant. One has $\sum_{i,j} 1 \otimes w_{ij} = 1 \otimes w = \gamma_q^*(w) = \sum_{i,j} \gamma_q^*(w_{ij})$, and from the discussion above, we see that $\gamma_q^*(w_{ij}) \in \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(V)_{i,j}$. Thus, since $\mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(V) = \bigoplus_{i,j} \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(V)_{i,j}$, one has $\gamma_q^*(w_{ij}) = 1 \otimes w_{ij}$ for all i, j. This finishes the proof of (a); the proof of (b) is similar. $\square$

The previous result shows that, in order to describe the Γ_q^*-coinvariants, it is enough to describe the coinvariants which are in each $\mathcal{O}_q(V)_{i,j}$. It is obvious from the definition of Γ_q^* that any γ_q^*-coinvariant will be a Γ_q^*-coinvariant. Our first aim is to show that any homogeneous Γ_q^*-coinvariant w is a γ_q^*-*semi-coinvariant*, meaning that $\gamma_q^*(w) = d_T^s \otimes w$ for some integer s; this is achieved in Theorem 5.5. We begin by constructing some maps which will help us to prove this fact.

The above grading on $\mathcal{O}_q(V)$ relates to the following comodule structure. Consider the Hopf algebra $\mathcal{O}_q(K^\times) = K[T^{\pm 1}]$ and the algebra morphism $\pi' : \mathcal{O}_q(GL_t) \twoheadrightarrow \mathcal{O}_q(K^\times)$ such that $T_{ii} \mapsto T$ for all i, while $T_{ij} \mapsto 0$ for $i \neq j$. It is easily checked that π' is, in fact, a morphism of Hopf algebras. The composite map

$$\mathcal{O}_q(V) \xrightarrow{\gamma_q^*} \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(V) \xrightarrow{\pi' \otimes \mathrm{id}} \mathcal{O}_q(K^\times) \otimes \mathcal{O}_q(V)$$

thus gives $\mathcal{O}_q(V)$ the structure of a left comodule over $\mathcal{O}_q(K^\times)$. The following proposition shows that, in fact, $\mathcal{O}_q(V)$ is an $\mathcal{O}_q(K^\times)$-comodule algebra.

Proposition 5.3. (a) $(\pi' \otimes \mathrm{id}) \circ \gamma_q^*$ *is an algebra morphism.*

(b) $(\pi' \otimes \mathrm{id})\gamma_q^*(Y_{ij} \otimes 1) = T^{-1} \otimes Y_{ij} \otimes 1$ *and* $(\pi' \otimes \mathrm{id})\gamma_q^*(1 \otimes Z_{ij}) = T \otimes 1 \otimes Z_{ij}$ *for all* i, j.

(c) $(\pi' \otimes \mathrm{id})\gamma_q^*(w) = T^{j-i} \otimes w$ *for all* $w \in \mathcal{O}_q(V)_{i,j}$.

Proof. (a) Recall that, for $a \in \mathcal{O}_q(M_{n,t})$ and $b \in \mathcal{O}_q(M_{t,n})$, if we put $\rho_q^*(a) = \sum_{(a)} a_0 \otimes a_1$ and $\lambda_q^*(b) = \sum_{(b)} b_{-1} \otimes b_0$, then $\gamma_q^*(a \otimes b) = \sum_{(a),(b)} S(a_1)b_{-1} \otimes a_0 \otimes b_0$. If, also, $a' \otimes b' \in \mathcal{O}_q(V)$ then, since ρ_q^* and λ_q^* are algebra morphisms, one has

$$\gamma_q^*\big((a \otimes b)(a' \otimes b')\big) = \sum_{(a'),(b')} S(a_1')S(a_1)b_{-1}b'_{-1} \otimes a_0a_0' \otimes b_0b_0'.$$

Using the fact that $\mathcal{O}_q(K^\times)$ is a commutative algebra we then get

$$(\pi' \otimes \mathrm{id})\gamma_q^*\big((a \otimes b)(a' \otimes b')\big) = \big((\pi' \otimes \mathrm{id})\gamma_q^*\big)(a \otimes b)\big((\pi' \otimes \mathrm{id})\gamma_q^*\big)(a' \otimes b').$$

(b) If A_{jk} denotes the $(t-1)\times(t-1)$ quantum minor of $\mathcal{O}_q(GL_t)$ obtained by deleting row j and column k, then $S(T_{kj}) = (-q)^{k-j}A_{jk}d_T^{-1}$, by [**7**, Theorem 5.3.2], and so

$$\gamma_q^*(Y_{ij} \otimes 1) = \sum_{k=1}^{t} S(T_{kj}) \otimes Y_{ik} \otimes 1 = \sum_{k=1}^{t} (-q)^{k-j}A_{jk}d_T^{-1} \otimes Y_{ik} \otimes 1$$

and $\gamma_q^*(1 \otimes Z_{ij}) = \sum_{k=1}^{t} T_{ik} \otimes 1 \otimes Z_{kj}$. It follows that $(\pi' \otimes \mathrm{id})\gamma_q^*(Y_{ij} \otimes 1) = T^{-1} \otimes Y_{ij} \otimes 1$ and that $(\pi' \otimes \mathrm{id})\gamma_q^*(1 \otimes Z_{ij}) = T \otimes 1 \otimes Z_{ij}$.

(c) This is clear from parts (a) and (b). $\square$

We shall need the algebra morphism

$$\phi_q^* : \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(K^\times) \to \mathcal{O}_q(GL_t)[\overline{\chi}] := \mathcal{O}_q(GL_t)[\chi]/\langle \chi^t - d_T^{-1} \rangle,$$

where χ is an indeterminate and $\overline{\chi} = \chi + \langle \chi^t - d_T^{-1} \rangle$, such that each $T_{ij} \otimes 1 \mapsto T_{ij}\overline{\chi}$ and $1 \otimes T \mapsto \overline{\chi}^{-1}$. It is easy to check that $\phi_q^*(d_T \otimes 1) = d_T\overline{\chi}^t = 1$, and so there is an induced algebra morphism

$$\varphi_q^* : \mathcal{O}_q(SL_t) \otimes \mathcal{O}_q(K^\times) \to \mathcal{O}_q(GL_t)[\overline{\chi}]$$

such that $\varphi_q^* \circ (\pi \otimes \mathrm{id}) = \phi_q^*$.

Finally, let $\alpha_q : \mathcal{O}_q(GL_t) \to \mathcal{O}_q(GL_t)[\overline{\chi}]$ denote the canonical injection.

Lemma 5.4. (a) $\phi_q^* \circ (\mathrm{id} \otimes \pi') \circ \Delta = \alpha_q$.

(b) $(\varphi_q^* \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \pi' \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \gamma_q^*) \circ \Gamma_q^* = (\alpha_q \otimes \mathrm{id}) \circ \gamma_q^*$.

Proof. (a) It is enough to check this on the generators T_{ij}, since each of the maps involved is an algebra morphism. One has

$$\phi_q^*(\mathrm{id} \otimes \pi')\Delta(T_{ij}) = \phi_q^*(\mathrm{id} \otimes \pi')\Big(\sum_{k=1}^{t} T_{ik} \otimes T_{kj}\Big) = \phi_q^*(T_{ij} \otimes T) = T_{ij}.$$

(b) Since γ_q^* is the structure map of a left $\mathcal{O}_q(GL_t)$-comodule, we have $(\Delta \otimes \mathrm{id}) \circ \gamma_q^* = (\mathrm{id} \otimes \gamma_q^*) \circ \gamma_q^*$. In view of part (a), it follows that

$$\begin{aligned}(\alpha_q \otimes \mathrm{id}) \circ \gamma_q^* &= (\phi_q^* \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \pi' \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}) \circ \gamma_q^* \\ &= (\phi_q^* \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \pi' \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \gamma_q^*) \circ \gamma_q^*.\end{aligned}$$

We also have the following commutative diagram:

$$\begin{tikzcd}
& \mathcal{O}_q(V) \arrow[dl, "\gamma_q^*"'] \arrow[dr, "\Gamma_q^*"] & \\
\mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(V) \arrow[rr, "\pi\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\gamma_q^*"'] & & \mathcal{O}_q(SL_t) \otimes \mathcal{O}_q(V) \arrow[d, "\mathrm{id}\otimes\gamma_q^*"] \\
\mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(V) \arrow[rr, "\pi\otimes\mathrm{id}\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\pi'\otimes\mathrm{id}"'] & & \mathcal{O}_q(SL_t) \otimes \mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(V) \arrow[d, "\mathrm{id}\otimes\pi'\otimes\mathrm{id}"] \\
\mathcal{O}_q(GL_t) \otimes \mathcal{O}_q(K^\times) \otimes \mathcal{O}_q(V) \arrow[rr, "\pi\otimes\mathrm{id}\otimes\mathrm{id}"] \arrow[dr, "\phi_q^*\otimes\mathrm{id}"'] & & \mathcal{O}_q(SL_t) \otimes \mathcal{O}_q(K^\times) \otimes \mathcal{O}_q(V) \arrow[dl, "\varphi_q^*\otimes\mathrm{id}"] \\
& \mathcal{O}_q(GL_t)[\overline{\chi}] \otimes \mathcal{O}_q(V) &
\end{tikzcd}$$

Thus $(\phi_q^* \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \pi' \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \gamma_q^*) \circ \gamma_q^* = (\varphi_q^* \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \pi' \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \gamma_q^*) \circ \Gamma_q^*$, which completes the proof of part (b). $\square$

Theorem 5.5. *Let $w \in \mathcal{O}_q(V)_{i,j}$ be a Γ_q^*-coinvariant.*

(a) *The difference $i - j$ is divisible by t, and $\gamma_q^*(w) = d_T^{-s} \otimes w$ where $s = (i-j)/t$.*

(b) *If $i \geq j$, then $(1 \otimes d_Z^s)w$ is a γ_q^*-coinvariant.*

(c) *If $i \leq j$, then $(d_Y^{-s} \otimes 1)w$ is a γ_q^*-coinvariant.*

Proof. (a) By Proposition 5.3(c), we have $(\pi' \otimes \mathrm{id})\gamma_q^*(w) = T^{j-i} \otimes w$. From this and the assumption that $\Gamma_q^*(w) = 1 \otimes w$, it follows that

$$\begin{aligned}
(\varphi_q^* \otimes \mathrm{id})(\mathrm{id} \otimes \pi' \otimes \mathrm{id})(\mathrm{id} \otimes \gamma_q^*)\Gamma_q^*(w) &= (\varphi_q^* \otimes \mathrm{id})(\mathrm{id} \otimes \pi' \otimes \mathrm{id})(\mathrm{id} \otimes \gamma_q^*)(1 \otimes w) \\
&= (\varphi_q^* \otimes \mathrm{id})(1 \otimes T^{j-i} \otimes w) = \overline{\chi}^{i-j} \otimes w.
\end{aligned}$$

Thus, by Lemma 5.4(b), $(\alpha_q \otimes \mathrm{id})\gamma_q^*(w) = \overline{\chi}^{i-j} \otimes w$. In particular,

$$\overline{\chi}^{i-j} \otimes w \in \mathrm{Im}(\alpha_q \otimes \mathrm{id}) = \mathcal{O}_q(GL_t) \cdot 1 \otimes \mathcal{O}_q(V).$$

Since $\mathcal{O}_q(GL_t)[\overline{\chi}]$ is a free $\mathcal{O}(GL_t)$-module with basis $\{1, \overline{\chi}, \ldots, \overline{\chi}^{t-1}\}$, we have $t \mid i - j$. Set $s = (i-j)/t$. Now

$$(\alpha_q \otimes \mathrm{id})\gamma_q^*(w) = \overline{\chi}^{ts} \otimes w = (\alpha_q \otimes \mathrm{id})(d_T^{-s} \otimes w),$$

and therefore $\gamma_q^*(w) = d_T^{-s} \otimes w$, as required.

(b) Since λ_q^* is an algebra morphism, we see using Remark 3.2(i) that $\lambda_q^*(d_Z^s) = d_T^s \otimes d_Z^s$, and hence $\gamma_q^*(1 \otimes d_Z^s) = d_T^s \otimes 1 \otimes d_Z^s$. As d_T^s is central in $\mathcal{O}_q(GL_t)$, we may apply Proposition 1.1(a) to conclude that

$$\gamma_q^*\big((1 \otimes d_Z^s)w\big) = \gamma_q^*(1 \otimes d_Z^s)\gamma_q^*(w) = (d_T^s \otimes 1 \otimes d_Z^s)(d_T^{-s} \otimes w) = 1 \otimes \big((1 \otimes d_Z^s)w\big).$$

(c) By Remark 3.2(i), $\rho_q^*(d_Y^{-s}) = d_Y^{-s} \otimes d_T^{-s}$. Since $S(d_T) = d_T^{-1}$, we thus obtain

$$\gamma_q^*(d_Y^{-s} \otimes 1) = d_T^s \otimes d_Y^{-s} \otimes 1,$$

and therefore part (c) follows from a second application of Proposition 1.1(a). $\square$

6. The First Fundamental Theorem for $\mathcal{O}_q(SL_t)$-coinvariants

In this final section, we determine the Γ_q^*-coinvariants in $\mathcal{O}_q(V)$. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ denote the respective subalgebras of $\mathcal{O}_q(M_{m,t})$ and $\mathcal{O}_q(M_{t,n})$ generated by all the $t\times t$ quantum minors. We shall prove that the set of Γ_q^*-coinvariants in $\mathcal{O}_q(V)$ is the subalgebra generated by $\mathcal{A}_1\otimes\mathcal{A}_2$ together with $\operatorname{Im}\mu_q^*$. In fact, this subalgebra turns out to be equal to the product $(\mathcal{A}_1\otimes\mathcal{A}_2)\cdot\operatorname{Im}\mu_q^*$. On the road to this goal, Theorem 5.5 puts us in a position roughly similar to that of Remark 4.1(ii), and to finish we need some computations analogous to Proposition 4.4.

Until further notice, we continue to assume that $m=n>t$.

Remark 6.1. (i) As in Section 4, we set $\widetilde{I}=\{n-t+1,\dots,n\}$ and $\widetilde{J}=\{1,\dots,t\}$. Let P_1 and P_2 denote the following ideals:

$$P_1=\big\langle\ \overline{[\widetilde{I}|J]}\ \big|\ J\subseteq\{1,\dots,n\}\text{ and }|J|=t\ \big\rangle\lhd\mathcal{O}_q(M_n^{\leq t})$$
$$P_2=\big\langle\ \overline{[I|\widetilde{J}]}\ \big|\ I\subseteq\{1,\dots,n\}\text{ and }|I|=t\ \big\rangle\lhd\mathcal{O}_q(M_n^{\leq t}).$$

These turn out to be completely prime ideals of $\mathcal{O}_q(M_n^{\leq t})$ – see Remark 6.4. In view of [**2**, Corollary 5.2], the quantum minors $[\widetilde{I}|J]$ can be arranged in a polynormal sequence, as can the $[I|\widetilde{J}]$. Hence,

$$P_1=\sum_{\substack{J\subseteq\{1,\dots,n\}\\|J|=t}}\overline{[\widetilde{I}|J]}\mathcal{O}_q(M_n^{\leq t})\qquad\text{and}\qquad P_2=\sum_{\substack{I\subseteq\{1,\dots,n\}\\|I|=t}}\overline{[I|\widetilde{J}]}\mathcal{O}_q(M_n^{\leq t}).$$

(ii) It can be proved that a K-basis for P_1 consists of all $\overline{[T|T']}$ where (T,T') is a preferred bitableaux with first row of the form $(\widetilde{I},?)$, and that a K-basis for P_2 consists of all $\overline{[T|T']}$ where (T,T') is a preferred bitableaux with first row of the form $(?,\widetilde{J})$. However, we shall not need these bases.

(iii) In view of point (i), powers of the P_i can be expressed as follows:

$$P_1^s=\sum_{\substack{J_1,\dots,J_s\subseteq\{1,\dots,n\}\\|J_1|=\cdots=|J_s|=t}}\overline{[\widetilde{I}|J_1][\widetilde{I}|J_2]\cdots[\widetilde{I}|J_s]}\mathcal{O}_q(M_n^{\leq t})$$
$$P_2^s=\sum_{\substack{I_1,\dots,I_s\subseteq\{1,\dots,n\}\\|I_1|=\cdots=|I_s|=t}}\overline{[I_1|\widetilde{J}][I_2|\widetilde{J}]\cdots[I_s|\widetilde{J}]}\mathcal{O}_q(M_n^{\leq t})$$

for $s=1,2,\dots$.

Lemma 6.2. *Let s be a nonnegative integer.*

(a) *A K-basis for $\langle d_Y^s\otimes 1\rangle$ consists of all pure tensors $[S|S']\otimes[T|T']$ in $\mathcal{O}_q(V)$ where (S,S') and (T,T') are preferred bitableaux and the first s rows of (S,S') equal $(\widetilde{I},\widetilde{J})$.*

(b) *A K-basis for $\langle 1\otimes d_Z^s\rangle$ consists of all pure tensors $[S|S']\otimes[T|T']$ in $\mathcal{O}_q(V)$ where (S,S') and (T,T') are preferred bitableaux and the first s rows of (T,T') equal $(\widetilde{J},\widetilde{J})$.*

Proof. Since $\langle d_Y^s\otimes 1\rangle=\langle d_Y^s\rangle\otimes\mathcal{O}_q(M_{t,n})$ and $\langle 1\otimes d_Z^s\rangle=\mathcal{O}_q(M_{n,t})\otimes\langle d_Z^s\rangle$, it suffices to check that the ideals $\langle d_Y^s\rangle$ and $\langle d_Z^s\rangle$ have bases of the appropriate forms. This can be done just as in Lemma 4.3(a)(b). $\square$

Proposition 6.3. *If s is any nonnegative integer, then*

$$(\mu_q^*)^{-1}\big(\langle d_Y^s \otimes 1\rangle\big) = P_1^s \qquad \textit{and} \qquad (\mu_q^*)^{-1}\big(\langle 1 \otimes d_Z^s\rangle\big) = P_2^s.$$

Proof. We prove the first equality; the proof of the second is analogous. The proof closely mimics that of Proposition 4.4.

In view of Remark 2.1(iii-v), we see that

$$\mu_q^* \overline{[\widetilde{I}|J]} = [\widetilde{I}|\widetilde{J}] \otimes [\widetilde{J}|J] = d_Y \otimes [\widetilde{J}|J]$$

for all t-element subsets $J \subseteq \{1, \dots, n\}$. Using Remark 6.1(iii) and the fact that μ_q^* is an algebra morphism, it follows that $P_1^s \subseteq (\mu_q^*)^{-1}\big(\langle d_Y^s \otimes 1\rangle\big)$.

If this inclusion is proper, choose an element

$$x = \sum_{i=1}^{r} \alpha_i \overline{[T_i|T_i']} \in (\mu_q^*)^{-1}\big(\langle d_Y^s \otimes 1\rangle\big) \setminus P_1^s$$

where the α_i are nonzero scalars, the (T_i, T_i') are distinct preferred bitableaux, and the T_i have at most t columns. We may assume that none of the $\overline{[T_i|T_i']}$ lie in P_1^s. Let $\overline{\rho}$ be the minimum of the $\overline{\rho}(T_i)$ under reverse lexicographic order.

After re-indexing, we may assume that there is some r' such that $\overline{\rho}(T_i) = \overline{\rho}$ for $i \le r'$ and $\overline{\rho}(T_i) >_{\mathrm{rlex}} \overline{\rho}$ for $i > r'$. Applying [**2**, Lemma 2.3] to each $\mu_q^*\overline{[T_i|T_i']}$ and collecting terms, we see that

$$\mu_q^*(x) = \sum_{i=1}^{r'} \alpha_i [T_i|\mu(T_i)] \otimes [\mu'(T_i)|T_i'] + \sum_j X_j \otimes Y_j$$

where the X_j and Y_j are homogeneous with $\overline{c}(X_j) = \overline{r}(Y_j) >_{\mathrm{rlex}} \overline{\rho}$. We then observe that all of the X_j belong to different homogeneous components than the $[T_i|\mu(T_i)]$ for $i \le r'$. Since

$$\mu_q^*(x) \in \langle d_Y^s \otimes 1\rangle = \langle d_Y^s\rangle \otimes \mathcal{O}_q(M_{t,n})$$

and the ideal $\langle d_Y^s\rangle$ is homogeneous, it follows that

$$\sum_{i=1}^{r'} \alpha_i [T_i|\mu(T_i)] \otimes [\mu'(T_i)|T_i'] \in \langle d_Y^s \otimes 1\rangle.$$

For $1 \le i < j \le r'$, either $(T_i, \mu(T_i)) \ne (T_j, \mu(T_j))$ or $(\mu'(T_i), T_i') \ne (\mu'(T_j), T_j')$. In view of Lemma 6.2(a), it follows that for $1 \le i \le r'$, the first s rows of T_i equal $\widetilde{I}$. But then $\overline{[T_i|T_i']} \in P_1^s$ for $1 \le i \le r'$, which contradicts our choices above and therefore establishes the desired equality. $\square$

Remark 6.4. Although we shall not need the fact here, we note that the case $s = 1$ of Proposition 6.3 implies that P_1 and P_2 are completely prime ideals of $\mathcal{O}_q(M_n^{\leq t})$. To see this, it obviously suffices to show that $\langle d_Y \otimes 1\rangle$ and $\langle 1 \otimes d_Z\rangle$ are completely prime ideals of $\mathcal{O}_q(V)$. Observe that $\mathcal{O}_q(M_{n,t})/\langle d_Y\rangle$ is an iterated skew polynomial ring over an isomorphic copy of $\mathcal{O}_q(M_t)$ modulo the ideal generated by its quantum determinant. Since the latter algebra is a domain ([**3**], [**4**, p. 182], or see [**2**, Theorem 2.5]), it follows that $\mathcal{O}_q(M_{n,t})/\langle d_Y\rangle$ is a domain. Now the algebra

$$\mathcal{O}_q(V)/\langle d_Y \otimes 1\rangle \cong \big(\mathcal{O}_q(M_{n,t})/\langle d_Y\rangle\big) \otimes \mathcal{O}_q(M_{t,n})$$

is an iterated skew polynomial ring over $\mathcal{O}_q(M_{n,t})/\langle d_Y\rangle$, and therefore $\mathcal{O}_q(V)/\langle d_Y \otimes 1\rangle$ is a domain. Thus $\langle d_Y \otimes 1\rangle$ is completely prime, and a similar argument shows that $\langle 1 \otimes d_Z\rangle$ is completely prime.

Lemma 6.5. *Let $w \in \mathcal{O}_q(V)_{i,j}$ be a Γ_q^*-coinvariant.*
(a) *If $i \geq j$, then $w \in (\mathcal{A}_1 \otimes 1) \cdot \operatorname{Im} \mu_q^*$.*
(b) *If $i \leq j$, then $w \in (1 \otimes \mathcal{A}_2) \cdot \operatorname{Im} \mu_q^*$.*

Proof. By Theorem 5.5(a), the ratio $s = (i - j)/t$ is an integer.

(a) In this case, Theorem 5.5(b) shows that $(1 \otimes d_Z^s)w$ is a γ_q^*-coinvariant. Thus, by Theorem 4.5, $(1 \otimes d_Z^s)w = \mu_q^*(x)$ for some $x \in \mathcal{O}_q(M_n^{\leq t})$. Now $\mu_q^*(x) \in \langle 1 \otimes d_Z^s\rangle$, and hence $x \in P_2^s$, by Proposition 6.3. In view of Remark 6.1(iii), we thus have $x = \sum_{i=1}^r \overline{p}_i x_i$ for some $x_i \in \mathcal{O}_q(M_n^{\leq t})$ and some $p_i \in \mathcal{O}_q(M_n)$ of the form

$$p_i = [I_{i,1}|\widetilde{J}][I_{i,2}|\widetilde{J}] \cdots [I_{i,s}|\widetilde{J}].$$

Now by Remark 2.1(iii-v), each

$$\mu_q^*\overline{[I_{i,l}|\widetilde{J}]} = [I_{i,l}|\widetilde{J}] \otimes [\widetilde{J}|\widetilde{J}] = [I_{i,l}|\widetilde{J}] \otimes d_Z.$$

Since μ_q^* is an algebra morphism, it follows that each

$$\mu_q^*(\overline{p}_i x_i) = (1 \otimes d_Z^s)\big([I_{i,1}|\widetilde{J}] \otimes 1\big)\big([I_{i,2}|\widetilde{J}] \otimes 1\big) \cdots \big([I_{i,s}|\widetilde{J}] \otimes 1\big)\mu_q^*(x_i).$$

Consequently, $(1 \otimes d_Z^s)w = \mu_q^*(x) = (1 \otimes d_Z^s)v$ for some $v \in (\mathcal{A}_1 \otimes 1) \cdot \operatorname{Im} \mu_q^*$. Since $1 \otimes d_Z$ is a nonzero element of the domain $\mathcal{O}_q(V)$, we conclude that $w = v$, and the proof of part (a) is complete.

(b) This is proved in a similar fashion. $\square$

Theorem 6.6. *Let m, n, t be arbitrary positive integers, and let $\mathcal{A}_1$ and $\mathcal{A}_2$ denote the respective subalgebras of $\mathcal{O}_q(M_{m,t})$ and $\mathcal{O}_q(M_{t,n})$ generated by all the $t \times t$ quantum minors. The set of Γ_q^*-coinvariants in $\mathcal{O}_q(V) = \mathcal{O}_q(M_{m,t}) \otimes \mathcal{O}_q(M_{t,n})$ is the subalgebra generated by $\mathcal{A}_1 \otimes \mathcal{A}_2$ and $\operatorname{Im} \mu_q^*$. More precisely,*

$$\big(\mathcal{O}_q(M_{m,t}) \otimes \mathcal{O}_q(M_{t,n})\big)^{\operatorname{co} \mathcal{O}_q(SL_t)} = \big(\mathcal{A}_1 \otimes \mathcal{A}_2\big) \cdot \mu_q^*\big(\mathcal{O}_q(M_{m,n}^{\leq t})\big).$$

Proof. As in Remark 2.4, it is enough to consider the case that $m = n > t$; we leave the details of the reduction to the reader. Let $\mathcal{C}$ denote the set of Γ_q^*-coinvariants in $\mathcal{O}_q(V)$, and recall from Proposition 5.1 that $\mathcal{C}$ is a subalgebra of $\mathcal{O}_q(V)$ containing $\operatorname{Im}\mu_q^*$.

If $[I|J]$ is a $t \times t$ quantum minor in $\mathcal{O}_q(M_{n,t})$ (necessarily, $J = \widetilde{J}$), then as in Remark 2.1(iii)(iv), we see that $\rho_q^*[I|J] = [I|J] \otimes d_T$. It follows that $\gamma_q^*([I|J] \otimes 1) = d_T^{-1} \otimes [I|J] \otimes 1$, whence

$$\Gamma_q^*([I|J] \otimes 1) = (\pi \otimes \mathrm{id})(d_T^{-1} \otimes [I|J] \otimes 1) = 1 \otimes [I|J] \otimes 1$$

and so $[I|J] \otimes 1 \in \mathcal{C}$. Similarly, $1 \otimes [I|J] \in \mathcal{C}$ for all $t \times t$ quantum minors $[I|J]$ in $\mathcal{O}_q(M_{t,n})$. Therefore $\mathcal{C}$ contains the subalgebra of $\mathcal{O}_q(V)$ generated by $\mathcal{A}_1 \otimes \mathcal{A}_2$ and $\operatorname{Im}\mu_q^*$.

On the other hand, it is immediate from Lemmas 5.2(b) and 6.5 that $\mathcal{C}$ is contained in $(\mathcal{A}_1 \otimes \mathcal{A}_2) \cdot \operatorname{Im}\mu_q^*$. The theorem follows. □

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