PRIME IDEALS IN CERTAIN QUANTUM DETERMINANTAL RINGS

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Abstract. The ideal $\mathcal{I}_1$ generated by the 2×2 quantum minors in the coordinate algebra of quantum matrices, $\mathcal{O}_q(M_{m,n}(k))$, is investigated. Analogues of the First and Second Fundamental Theorems of Invariant Theory are proved. In particular, it is shown that $\mathcal{I}_1$ is a completely prime ideal, that is, $\mathcal{O}_q(M_{m,n}(k))/\mathcal{I}_1$ is an integral domain, and that $\mathcal{O}_q(M_{m,n}(k))/\mathcal{I}_1$ is the ring of coinvariants of a coaction of $k[x, x^{-1}]$ on $\mathcal{O}_q(k^m) \otimes \mathcal{O}_q(k^n)$, a tensor product of two quantum affine spaces. There is a natural torus action on $\mathcal{O}_q(M_{m,n}(k))/\mathcal{I}_1$ induced by an $(m+n)$-torus action on $\mathcal{O}_q(M_{m,n}(k))$. We identify the invariant prime ideals for this action and deduce consequences for the prime spectrum of $\mathcal{O}_q(M_{m,n}(k))/\mathcal{I}_1$.

Introduction

Let k be a field and let $q \in k^\times$. The *coordinate ring of quantum $m \times n$ matrices*, $\mathcal{A} := \mathcal{O}_q(M_{m,n}(k))$, is a deformation of the classical coordinate ring of $m \times n$ matrices, $\mathcal{O}(M_{m,n}(k))$. As such it is a k-algebra generated by mn indeterminates X_{ij}, for $1 \le i \le m$ and $1 \le j \le n$, subject to the relations

$$\begin{aligned}
X_{ij}X_{lj} &= qX_{lj}X_{ij} && \text{when } i < l;\\
X_{ij}X_{is} &= qX_{is}X_{ij} && \text{when } j < s;\\
X_{is}X_{lj} &= X_{lj}X_{is} && \text{when } i < l \text{ and } j < s;\\
X_{ij}X_{ls} - X_{ls}X_{ij} &= (q - q^{-1})X_{is}X_{lj} && \text{when } i < l \text{ and } j < s.
\end{aligned}$$

In some references (e.g., [**6**, §3.5]), q is replaced by q^{-1}. When $q = 1$ we recover $\mathcal{O}(M_{m,n}(k))$, which is the commutative polynomial algebra $k[X_{ij}]$.

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When $m = n$, the algebra $\mathcal{A}$ possesses a special element, the *quantum determinant*, D_q, defined by

$$D_q := \sum_{\sigma \in S_n} (-q)^{l(\sigma)} X_{1,\sigma(1)} X_{2,\sigma(2)} \cdots X_{n,\sigma(n)},$$

where $l(\sigma)$ denotes the number of inversions in the permutation σ. The quantum determinant D_q is a central element of $\mathcal{A}$ (see, for example, [**6**, Theorem 4.6.1]), and the localization $\mathcal{A}[D_q^{-1}]$ is the *coordinate ring of the quantum general linear group*, denoted $\mathcal{O}_q(GL_n(k))$.

If $I \subseteq \{1, \dots, m\}$ and $J \subseteq \{1, \dots, n\}$ with $|I| = |J| = t$, let $D(I, J)$ denote the $t \times t$ quantum minor obtained as the quantum determinant of the subalgebra of $\mathcal{A}$ obtained by deleting generators X_{ij} from the rows outside I and from the columns outside J. We write $\mathcal{I}_t$ for the ideal generated by the $(t+1) \times (t+1)$ quantum minors of $\mathcal{A}$. In [**3**] it is proved that $\mathcal{A}/\mathcal{I}_t$ is an integral domain, for each $1 \le t \le \min\{m, n\}$. Independently, Rigal [**7**] has shown that $\mathcal{A}/\mathcal{I}_1$ is a domain; he also shows that $\mathcal{A}/\mathcal{I}_1$ is a maximal order in its division ring of fractions.

There is an action of the torus $\mathcal{H} := (k^\times)^m \times (k^\times)^n$ by k-algebra automorphisms on $\mathcal{A}$ such that

$$(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n) \cdot X_{ij} := \alpha_i \beta_j X_{ij}$$

for all i, j. The ideals $\mathcal{I}_t$ are easily seen to be invariant under $\mathcal{H}$; so there is an induced action of $\mathcal{H}$ on the factor algebras $\mathcal{A}/\mathcal{I}_t$. In this paper, we study the prime ideal structure in the algebra $\mathcal{A}/\mathcal{I}_1$, paying particular attention to the $\mathcal{H}$-invariant prime ideals.

1. Complete primeness of $\mathcal{I}_1$

We give a direct derivation of the fact that $\mathcal{A}/\mathcal{I}_1$ is a domain. Although this is already established in both [**3**] and [**7**], the proof we give here is so much simpler and more transparent than either of the previous proofs that we think it will be useful to have it in a published form.

The *coordinate ring of quantum affine n-space*, denoted $\mathcal{O}_q(k^n)$, is defined to be the k-algebra generated by elements $y_1, \dots, y_n$ subject to the relations $y_i y_j = q y_j y_i$ for each $1 \le i < j \le n$. It is well known that $\mathcal{O}_q(k^n)$ is an iterated Ore extension, and thus, in particular, $\mathcal{O}_q(k^n)$ is a domain. Our strategy is to produce a homomorphism of $\mathcal{A}$ into $\mathcal{O}_q(k^m) \otimes \mathcal{O}_q(k^n)$. This latter algebra can also be presented as an iterated Ore extension and thus is a domain. We show that $\mathcal{I}_1$ is the kernel of this map and so $\mathcal{A}/\mathcal{I}_1$ is a domain.

1.1. Theorem. *The algebra $\mathcal{O}_q(M_{m,n}(k))/\mathcal{I}_1$ is isomorphic to a subalgebra of the tensor product $\mathcal{O}_q(k^m) \otimes \mathcal{O}_q(k^n)$. In particular, $\mathcal{I}_1$ is a completely prime ideal of $\mathcal{O}_q(M_{m,n}(k))$.*

Proof. Let $\mathcal{O}_q(k^m) = k[y_1, \dots, y_m]$ and $\mathcal{O}_q(k^n) = k[z_1, \dots, z_n]$ be the coordinate rings of quantum affine m-space and n-space, respectively. We define an algebra homomorphism $\theta : \mathcal{A} \to \mathcal{O}_q(k^m) \otimes \mathcal{O}_q(k^n)$ such that $\theta(X_{ij}) = y_i \otimes z_j$ for all i, j. In order that this does extend to a well-defined algebra homomorphism, we must check that the elements $y_i \otimes z_j$ satisfy at least the relations defining $\mathcal{A}$. These are routine verifications; for example, if $i < l$ and $j < s$ then

$$(y_i \otimes z_j)(y_l \otimes z_s) = y_i y_l \otimes z_j z_s = y_i y_l \otimes q z_s z_j = q(y_i \otimes z_s)(y_l \otimes z_j),$$

while

$$(y_l \otimes z_s)(y_i \otimes z_j) = y_l y_i \otimes z_s z_j = q^{-1} y_i y_l \otimes z_s z_j = q^{-1}(y_i \otimes z_s)(y_l \otimes z_j).$$

Thus,

$$(y_i \otimes z_j)(y_l \otimes z_s) - (y_l \otimes z_s)(y_i \otimes z_j) = (q - q^{-1})(y_i \otimes z_s)(y_l \otimes z_j),$$

so that the fourth relation of the introduction holds. One can also obtain θ as the composition of the comultiplication $\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ with the tensor product of the quotient maps from $\mathcal{A}$ to $\mathcal{A}/\langle X_{ij} \mid i > 1\rangle$ and $\mathcal{A}/\langle X_{ij} \mid j > 1\rangle$. We shall pursue the latter point of view in the next section.

Thus, there exists a unique k-algebra homomorphism

$$\theta : \mathcal{A} \to \mathcal{O}_q(k^m) \otimes \mathcal{O}_q(k^n)$$

such that $\theta(X_{ij}) = y_i \otimes z_j$ for all i, j. If $i < l$ and $j < s$ then the above calculations also show that $\theta(X_{ij}X_{ls} - qX_{is}X_{lj}) = 0$; thus $\mathcal{I}_1 \subseteq \ker(\theta)$.

Now, $\mathcal{O}_q(k^m) \otimes \mathcal{O}_q(k^n)$ is a domain, since it can be viewed as a (multiparameter) quantum affine $(m+n)$-space with respect to the generators $y_1 \otimes 1, \dots, y_m \otimes 1, 1 \otimes z_1, \dots, 1 \otimes z_n$. Hence, $\ker(\theta)$ is a completely prime ideal of $\mathcal{A}$. We show that $\mathcal{I}_1 = \ker(\theta)$, so that $\mathcal{I}_1$ is completely prime. It remains to show that the induced map $\overline{\theta} : \mathcal{A}/\mathcal{I}_1 \to \mathcal{O}_q(k^m) \otimes \mathcal{O}_q(k^n)$ is injective. Let $\mathcal{S}$ denote the set of monomials $X_{i_1j_1}X_{i_2j_2}\dots X_{i_lj_l}$ in $\mathcal{A}$ such that $i_1 \geq i_2 \geq \cdots \geq i_l$ and $j_1 \leq j_2 \leq \cdots \leq j_l$. (We allow the monomial to be equal to 1 when $l = 0$.) We claim that the set $\overline{\mathcal{S}}$ of images forms a spanning set of $\mathcal{A}/I_1$.

It suffices to show that an arbitrary monomial C in $\mathcal{A}$ is congruent modulo $\mathcal{I}_1$ to a linear combination of monomials from $\mathcal{S}$. We proceed by induction on the index sets, where row index sequences $(i_1, i_2, \dots, i_l)$ are ordered lexicographically with respect to $\geq$, column index sequences $(j_1, j_2, \dots, j_l)$ are ordered lexicographically with respect to $\leq$, and pairs of sequences are ordered lexicographically.

If the claim fails, then it fails for a monomial $C = X_{i_1j_1}X_{i_2j_2}\dots X_{i_lj_l}$ whose index set is minimal with respect to the ordering given in the previous paragraph. In particular, $C \notin \mathcal{S}$. Let r be the first subindex such that either $i_r < i_{r+1}$ or $j_r > j_{r+1}$.

If $i_r < i_{r+1}$ and $j_r \geq j_{r+1}$ then $C = \lambda C'$, where λ is either 1 or q and C' is obtained from C by switching $X_{i_rj_r}$ and $X_{i_{r+1}j_{r+1}}$. However,

$$(i_1, , \dots, i_{r-1}, i_{r+1}, i_r, i_{r+2}, , \dots, i_l) < (i_1, i_2, \dots, i_l)$$

in our ordering, so C' is congruent modulo $\mathcal{I}_1$ to a linear combination of elements of $\mathcal{S}$. Then C is congruent to such a linear combination, contradicting our assumptions. A similar contradiction occurs if $i_r \leq i_{r+1}$ and $j_r > j_{r+1}$: this time, the row indices might not change, but

$$(j_1, \dots, j_{r-1}, j_{r+1}, j_r, j_{r+2}, \dots, j_l) < (j_1, \dots, j_l),$$

so again we have a contradiction. Therefore, we must either have $i_r < i_{r+1}$ and $j_r < j_{r+1}$ or $i_r > i_{r+1}$ and $j_r > j_{r+1}$.

Suppose that $i_r < i_{r+1}$ and $j_r < j_{r+1}$. In this case, we have

$$X_{i_rj_r}X_{i_{r+1}j_{r+1}} - qX_{i_{r+1}j_r}X_{i_rj_{r+1}} \in \mathcal{I}_1,$$

so that $C - qC' \in \mathcal{I}_1$, where

$$C' = X_{i_1j_1} \cdots X_{i_{r-1}j_{r-1}}X_{i_{r+1}j_r}X_{i_rj_{r+1}}X_{i_{r+2}j_{r+2}} \cdots X_{i_lj_l}.$$

We obtain a contradiction as above.

The final case is $i_r > i_{r+1}$ and $j_r > j_{r+1}$, where we have

$$X_{i_rj_r}X_{i_{r+1}j_{r+1}} - q^{-1}X_{i_rj_{r+1}}X_{i_{r+1}j_r} \in \mathcal{I}_1.$$

Thus, $C - q^{-1}C' \in \mathcal{I}_1$, where

$$C' = X_{i_1j_1} \cdots X_{i_{r-1}j_{r-1}}X_{i_rj_{r+1}}X_{i_{r+1}j_r}X_{i_{r+2}j_{r+2}} \cdots X_{i_lj_l},$$

and once again we reach a contradiction. This finishes the proof of the claim and establishes that $\overline{\mathcal{S}}$ spans $\mathcal{A}/\mathcal{I}_1$.

Now, observe that in $\mathcal{O}_q(k^m) \otimes \mathcal{O}_q(k^n)$ we have

$$\theta(X_{i_1j_1}X_{i_2j_2}\dots X_{i_lj_l}) = y_{i_1}y_{i_2}\dots y_{i_l} \otimes z_{j_1}z_{j_2}\dots z_{j_l}.$$

The monomials $y_{i_1}y_{i_2}\dots y_{i_l}$ with $i_1 \geq i_2 \geq \dots \geq i_l$ are linearly independent over k, and, likewise, the monomials $z_{j_1}z_{j_2}\dots z_{j_l}$ with $j_1 \leq j_2 \leq \dots \leq j_l$ are linearly independent over k. Hence, θ maps $\mathcal{S}$ bijectively to a linearly independent set in $\mathcal{O}_q(k^m) \otimes \mathcal{O}_q(k^n)$, so that $\overline{\mathcal{S}}$ is a linearly independent set in $\mathcal{A}/\mathcal{I}_1$. Therefore, the map $\overline{\theta} : \mathcal{A}/\mathcal{I}_1 \to \mathcal{O}_q(k^m) \otimes \mathcal{O}_q(k^n)$ maps the k-basis $\overline{\mathcal{S}}$ bijectively onto a linearly independent set, so that $\overline{\theta}$ is injective. $\square$

2. Coinvariants

Theorem 1.1 has an invariant theoretic interpretation, which we discuss in this section. First, we outline what happens in the classical ($q = 1$) case.

2.1. Let $M_{u,v}(k)$ denote the algebraic variety of $u \times v$ matrices over k. Fix positive integers m, n and $t \leq \min\{m, n\}$. The general linear group $GL_t(k)$ acts on $M_{m,t}(k) \times M_{t,n}(k)$ via

$$g \cdot (A, B) := (Ag^{-1}, gB).$$

Matrix multiplication yields a map

$$\mu : M_{m,t}(k) \times M_{t,n}(k) \to M_{m,n}(k),$$

the image of which is the variety of $m \times n$ matrices with rank at most t, and there is an induced map

$$\mu_* : \mathcal{O}(M_{m,n}(k)) \to \mathcal{O}\big(M_{m,t}(k) \times M_{t,n}(k)\big) = \mathcal{O}(M_{m,t}(k)) \otimes \mathcal{O}(M_{t,n}(k)).$$

The First Fundamental Theorem of invariant theory identifies the fixed ring of the coordinate ring $\mathcal{O}\big(M_{m,t}(k) \times M_{t,n}(k)\big)$ under the induced action of $GL_t(k)$ as precisely the image of μ_*. The Second Fundamental Theorem states that the kernel of μ_* is $\mathcal{I}_t$, the ideal generated by the $(t+1) \times (t+1)$ minors of $\mathcal{O}(M_{m,n}(k))$, so that the coordinate ring of the variety of $m \times n$ matrices of rank at most t is $\mathcal{O}(M_{m,n}(k))/\mathcal{I}_t$. As a consequence, since this variety is irreducible, the ideal $\mathcal{I}_t$ is a prime ideal of $\mathcal{O}(M_{m,n}(k))$.

2.2. We now proceed to explain the connection between Theorem 1.1 and the above invariant theoretic point of view.

The analog of μ_* is the k-algebra homomorphism

$$\theta_t : \mathcal{O}_q(M_{m,n}(k)) \to \mathcal{O}_q(M_{m,t}(k)) \otimes \mathcal{O}_q(M_{t,n}(k))$$

induced from the comultiplication on $\mathcal{O}_q(M_{m,n}(k))$, that is,

$$\theta_t(X_{ij}) = \sum_{l=1}^{t} X_{il} \otimes X_{lj}$$

for $1 \le i \le m$ and $1 \le j \le n$. The comultiplications on $\mathcal{O}_q(M_{m,t}(k))$ and $\mathcal{O}_q(M_{t,n}(k))$ yield k-algebra homomorphisms

$$\begin{aligned}
\rho_t : \mathcal{O}_q(M_{m,t}(k)) &\to \mathcal{O}_q(M_{m,t}(k)) \otimes \mathcal{O}_q(M_t(k)) \\
&\qquad \to \mathcal{O}_q(M_{m,t}(k)) \otimes \mathcal{O}_q(GL_t(k)) \\
\lambda_t : \mathcal{O}_q(M_{t,n}(k)) &\to \mathcal{O}_q(M_t(k)) \otimes \mathcal{O}_q(M_{t,n}(k)) \\
&\qquad \to \mathcal{O}_q(GL_t(k)) \otimes \mathcal{O}_q(M_{t,n}(k))
\end{aligned}$$

which make $\mathcal{O}_q(M_{m,t}(k))$ into a right $\mathcal{O}_q(GL_t(k))$-comodule and $\mathcal{O}_q(M_{t,n}(k))$ into a left $\mathcal{O}_q(GL_t(k))$-comodule. Since $\mathcal{O}_q(GL_t(k))$ is a Hopf algebra, the right comodule $\mathcal{O}_q(M_{m,t}(k))$ becomes a <u>left</u> $\mathcal{O}_q(GL_t(k))$-comodule on composing ρ_t with $1 \otimes S$ followed by the flip (where S denotes the antipode). Finally, the tensor product of the two left $\mathcal{O}_q(GL_t(k))$-comodules $\mathcal{O}_q(M_{m,t}(k))$ and $\mathcal{O}_q(M_{t,n}(k))$ becomes a left $\mathcal{O}_q(GL_t(k))$-comodule via the multiplication map on $\mathcal{O}_q(GL_t(k))$. This comodule structure map,

$$\gamma_t : \mathcal{O}_q(M_{m,t}(k)) \otimes \mathcal{O}_q(M_{t,n}(k)) \to \mathcal{O}_q(GL_t(k)) \otimes \mathcal{O}_q(M_{m,t}(k)) \otimes \mathcal{O}_q(M_{t,n}(k)),$$

can be described (using the Sweedler summation notation) as follows:

$$\gamma_t(a \otimes b) = \sum_{(a)} \sum_{(b)} S(a_1) b_{-1} \otimes a_0 \otimes b_0$$

where $\rho_t(a) = \sum_{(a)} a_0 \otimes a_1$ and $\lambda_t(b) = \sum_{(b)} b_{-1} \otimes b_0$ for $a \in \mathcal{O}_q(M_{m,t}(k))$ and $b \in \mathcal{O}_q(M_{t,n}(k))$. Note that for $t > 1$, the map γ_t is not an algebra homomorphism, since neither the antipode nor the multiplication map on $\mathcal{O}_q(GL_t(k))$ is an algebra homomorphism. On the other hand, γ_1 is a k-algebra homomorphism.

Recall that the *coinvariants* of the coaction γ_t are the elements x in the tensor product $\mathcal{O}_q(M_{m,t}(k)) \otimes \mathcal{O}_q(M_{t,n}(k))$ such that $\gamma_t(x) = 1 \otimes x$. Quantum analogs of the First and Second Fundamental Theorems would be the following:

Conjecture 1. The set of coinvariants of γ_t equals the image of θ_t.

Conjecture 2. The kernel of θ_t is the ideal $\mathcal{I}_t$.

We have proved Conjecture 2 in [**3**, Proposition 2.4] (essentially; the cited result covers the case $m = n$, and the general case follows easily by the method of [**3**, Corollary 2.6]). However, Conjecture 1 is open at present. Here we shall establish it in the case $t = 1$.

2.3. Note that $\mathcal{O}_q(M_{m,1}(k))$ and $\mathcal{O}_q(M_{1,n}(k))$ are quantum affine spaces on generators $X_{11}, X_{21}, \dots, X_{m1}$ and $X_{11}, X_{12}, \dots, X_{1n}$, respectively. In studying the case $t = 1$, it is convenient to replace $\mathcal{O}_q(M_{m,1}(k))$ and $\mathcal{O}_q(M_{1,n}(k))$ by $\mathcal{O}_q(k^m) = k[y_1, \dots, y_m]$ and $\mathcal{O}_q(k^n) = k[z_1, \dots, z_n]$, respectively. Then θ_1 becomes the k-algebra homomorphism

$$\theta : \mathcal{O}_q(M_{m,n}(k)) \to \mathcal{O}_q(k^m) \otimes \mathcal{O}_q(k^n), \qquad X_{ij} \mapsto y_i \otimes z_j$$

used in the proof of Theorem 1.1. Next, the (quantum) coordinate ring of 1×1 matrices is just a polynomial ring $k[x]$, and the (quantum) coordinate ring of the 1×1 general linear group is the localization $k[x, x^{-1}]$. Thus, in the present case the coaction γ_1 becomes the k-algebra homomorphism

$$\begin{gathered} \gamma : \mathcal{O}_q(k^m) \otimes \mathcal{O}_q(k^n) \to k[x^{\pm 1}] \otimes \mathcal{O}_q(k^m) \otimes \mathcal{O}_q(k^n), \\ y_i \otimes 1 \mapsto x^{-1} \otimes y_i \otimes 1, \qquad 1 \otimes z_j \mapsto x \otimes 1 \otimes z_j. \end{gathered}$$

2.4. Theorem. *The set of coinvariants of γ is exactly the image of the algebra $\mathcal{O}_q(M_{m,n}(k))$ in $\mathcal{O}_q(k^m) \otimes \mathcal{O}_q(k^n)$ under θ.*

Proof. Clearly $\gamma\theta(X_{ij}) = 1 \otimes y_i \otimes z_j = 1 \otimes \theta(X_{ij})$ for all i, j. Since θ and γ are k-algebra homomorphisms, it follows that the image of θ is contained in the coinvariants of γ.

The algebra $\mathcal{O}_q(k^m) \otimes \mathcal{O}_q(k^n)$ has a basis consisting of pure tensors $Y \otimes Z$ where Y is an ordered monomial in the y_i and Z is an ordered monomial in the z_j. Note that $\gamma(Y \otimes Z) = x^{s-r} \otimes Y \otimes Z$ where r and s are the total degrees of Y and Z, respectively. Hence, the images $\gamma(Y \otimes Z)$ are k-linearly independent, and a linear combination $\sum_{l=1}^{d} \alpha_l Y_l \otimes Z_l$ of distinct monomial tensors is a coinvariant for γ if and only if each $Y_l \otimes Z_l$ is a coinvariant.

Thus, we need only consider a single term

$$Y \otimes Z = y_{i_1} y_{i_2} \cdots y_{i_r} \otimes z_{j_1} z_{j_2} \cdots z_{j_s}.$$

If $Y \otimes Z$ is a coinvariant, then because $\gamma(Y \otimes Z) = x^{s-r} \otimes Y \otimes Z$ we must have $r = s$. Therefore

$$Y \otimes Z = \theta(X_{i_1 j_1} X_{i_2 j_2} \cdots X_{i_r j_r}),$$

which shows that $Y \otimes Z$ is in the image of θ, as desired. $\square$

3. $\mathcal{H}$-invariant prime ideals of $\mathcal{O}_q(M_{m,n}(k))/\mathcal{I}_1$

Under the mild assumption that our ground field k is infinite, we identify the $\mathcal{H}$-invariant prime ideals of the domain $\mathcal{A}/\mathcal{I}_1 = \mathcal{O}_q(M_{m,n}(k))/\mathcal{I}_1$. (Recall that $\mathcal{H}$ denotes the torus $(k^\times)^m \times (k^\times)^n$, acting on $\mathcal{A}$ as described in the introduction.) This identifies the minimal elements in a stratification of $\operatorname{spec} \mathcal{A}/I_1$, and yields a description of this prime spectrum as a finite disjoint union of commutative schemes corresponding to Laurent polynomial rings.

3.1. Let H be a group acting as automorphisms on a ring A. We refer the reader to [**1**] for the definition of the *H-stratification of* $\operatorname{spec} A$, and here recall only that the H-stratum of $\operatorname{spec} A$ corresponding to an H-prime ideal J is the set

$$\operatorname{spec}_J A := \{P \in \operatorname{spec} A \mid (P : H) = J\}.$$

In the case of the algebra $\mathcal{A}/\mathcal{I}_1$, we shall (assuming k infinite) identify the $\mathcal{H}$-prime ideals – they turn out to be the same as the $\mathcal{H}$-invariant primes – and thus pin down the minimum elements of the $\mathcal{H}$-strata. Further, we shall see that each $\mathcal{H}$-stratum of $\operatorname{spec} \mathcal{A}/\mathcal{I}_1$ is homeomorphic to the spectrum of a Laurent polynomial ring over an algebraic extension of k. This pattern is also known to hold for $\operatorname{spec} \mathcal{A}$ itself (at least when q is not a root of unity), but there the $\mathcal{H}$-prime ideals have not yet been completely identified.

3.2. It turns out that if a generator X_{ij} lies in an $\mathcal{H}$-prime ideal P of $\mathcal{A}$ containing $\mathcal{I}_1$, then either all the generators from the same row, or all the generators from the same column must also lie in P. This leads us to make the following definition.

For subsets $I \subseteq \{1, \dots, m\}$ and $J \subseteq \{1, \dots, n\}$, set

$$P(I, J) := \mathcal{I}_1 + \langle X_{ij} \mid i \in I \rangle + \langle X_{ij} \mid j \in J \rangle.$$

Obviously, $P(I, J)$ is an $\mathcal{H}$-invariant ideal of $\mathcal{A}$. We shall show that $P(I, J)$ is (completely) prime, and hence $\mathcal{H}$-prime.

Lemma. *The factor algebra $\mathcal{A}/P(I,J)$ is isomorphic to $\mathcal{O}_q(M_{m',n'}(k))/\mathcal{I}_1'$, where $m' = m - |I|$ and $n' = n - |J|$, and $\mathcal{I}_1'$ is the ideal generated by the 2×2 quantum minors of $\mathcal{O}_q(M_{m',n'}(k))$. Hence, $P(I,J)$ is a completely prime ideal of $\mathcal{A}$.*

Proof. The second statement follows immediately from the first statement and Theorem 1.1.

Set $I' := \{1,\dots,m\}\backslash I$, and $J' := \{1,\dots,n\}\backslash J$, and let $\mathcal{A}'$ be the k-subalgebra of $\mathcal{A}$ generated by the X_{ij} for $i \in I'$ and $j \in J'$. Note that $\mathcal{A}'$ is isomorphic to $\mathcal{O}_q(M_{m',n'}(k))$. Let $\mathcal{I}_1'$ be the ideal of $\mathcal{A}'$ generated by the 2×2 quantum minors of $\mathcal{A}'$; that is, those for which both row indices are in I' and both column indices are in J'. Obviously, $\mathcal{I}_1' \subseteq \mathcal{A}' \cap \mathcal{I}_1$, so that the inclusion $\mathcal{A}' \hookrightarrow \mathcal{A}$ induces a k-algebra homomorphism $f : \mathcal{A}'/\mathcal{I}_1' \to \mathcal{A}/P(I,J)$. It suffices to show that f is an isomorphism.

The factor $\mathcal{A}/P(I,J)$ is generated by the cosets of those X_{ij} with $i \in I'$ and $j \in J'$, since $X_{ij} \in P(I,J)$ whenever $i \in I$ or $j \in J$. These cosets are all in the image of f; so f is surjective.

Observe that there exists a k-algebra homomorphism $g : \mathcal{A} \to \mathcal{A}'$ such that $g(X_{ij}) = X_{ij}$ when $i \in I'$ and $j \in J'$, and $g(X_{ij}) = 0$ otherwise. To see this, note that the only problematic relations are those of the form $X_{ij}X_{ls} - X_{ls}X_{ij} = (q - q^{-1})X_{is}X_{lj}$ for $i < l$ and $j < s$. However, if $i \notin I'$ then X_{ij} and X_{is} both map to zero, and the relation maps to $0 = 0$. Likewise, this happens in all cases except when $i, l \in I'$ and $j, s \in J'$: in this case, the relation above maps to a relation in $\mathcal{A}'$.

Consider a 2×2 quantum minor in $\mathcal{A}$ of the form $D = X_{ij}X_{ls} - qX_{is}X_{lj}$ where $i < l$ and $j < s$. If $i \notin I'$ then both X_{ij} and X_{is} are in $\ker(g)$, so that $D \in \ker(g)$. Likewise, $g(D) = 0$ when $l \notin I'$, or $j \notin J'$, or $s \notin J'$. On the other hand, $g(D) = D$ when $i, l \in I'$ and $j, s \in J'$. Further, $g(X_{ij}) = 0$ when $i \in I$ or $j \in J$. Hence, $g(P(I,J)) \subseteq \mathcal{I}_1'$.

Therefore, g induces a k-algebra homomorphism $\overline{g} : \mathcal{A}/P(I,J) \to \mathcal{A}'/\mathcal{I}_1'$. Both of these algebras are generated by the cosets corresponding to those X_{ij} such that $i \in I'$ and $j \in J'$. It follows that both $f\overline{g}$ and $\overline{g}f$ are identity maps, since both f and $\overline{g}$ preserve these cosets. Hence, f is an isomorphism. □

Somewhat suprisingly, the $P(I,J)$ turn out to be the only $\mathcal{H}$-prime ideals of $\mathcal{A}$ that contain $\mathcal{I}_1$. The following lemma will be helpful in establishing this fact.

3.3. Lemma. *Let $i, s \in \{1,\dots,m\}$ and $j, t \in \{1,\dots,n\}$. Then there exist scalars $\alpha \in \{1, q^{\pm 1}, q^{\pm 2}\}$ and $\beta \in \{1, q^{\pm 1}\}$ such that $X_{ij}X_{st} - \alpha X_{st}X_{ij}$ and*

$X_{ij}X_{st} - \beta X_{it}X_{sj}$ lie in $\mathcal{I}_1$. In particular, the cosets $X_{ij} + \mathcal{I}_1$ are all normal elements of $\mathcal{A}/\mathcal{I}_1$.

Proof. If $i = s$, then in view of the relations in $\mathcal{A}$ we can take $\alpha = \beta$ to be q, 1, or q^{-1} (depending on whether $j < t$ or $j = t$ or $j > t$). Similarly, if $j = t$, we can take $\alpha \in \{1, q^{\pm 1}\}$ and $\beta = 1$.

If $i < s$ and $j > t$, or if $i > s$ and $j < t$, then X_{ij} and X_{st} commute, and we can take $\alpha = 1$. On the other hand, one of $X_{it}X_{sj} - q^{\pm 1}X_{ij}X_{st}$ is a 2×2 quantum minor, and so we can take β to be q or q^{-1}.

Now suppose that $i < s$ and $j < t$. Then $X_{ij}X_{st} - qX_{it}X_{sj}$ is a quantum minor, and we can take $\beta = q$. But $X_{st}X_{ij} - q^{-1}X_{it}X_{sj}$ is also a quantum minor, so we have $X_{ij}X_{st} \equiv qX_{it}X_{sj} \equiv q^2 X_{st}X_{ij} \pmod{\mathcal{I}_1}$, and hence we can take $\alpha = q^2$.

The remaining case follows from the previous one by exchanging (i, j) and (s, t), and then the final statement of the lemma is clear. $\square$

3.4. Proposition. *Assume that k is an infinite field. Then the $\mathcal{H}$-prime ideals of $\mathcal{O}_q(M_{m,n}(k))$ that contain $\mathcal{I}_1$ are precisely the ideals $P(I, J)$.*

Proof. By Lemma 3.2, we know that the ideals $P(I, J)$ are $\mathcal{H}$-prime. Consider an arbitrary $\mathcal{H}$-prime ideal P of $\mathcal{A}$ that contains $\mathcal{I}_1$. If all of the X_{ij} are in P then P must be the maximal ideal generated by the X_{ij}. In that case, $P = P(I, J)$, where $I = \{1, \dots, m\}$ and $J = \{1, \dots, n\}$. Hence, we may assume that not all X_{ij} are in P. Set

$$I = \{i \in \{1, \dots, m\} \mid X_{ij} \in P \text{ for all } j\}$$
$$J = \{j \in \{1, \dots, n\} \mid X_{ij} \in P \text{ for all } i\}.$$

We first show that $X_{ij} \in P$ if and only if $i \in I$ or $j \in J$. Certainly, if $i \in I$ or $j \in J$ then $X_{ij} \in P$, by the definition of I and J. Suppose that there exists an $X_{ij} \in P$ such that $i \notin I$ and $j \notin J$. Then there exists an index $s \neq i$ such that $X_{sj} \notin P$ and also there exists an index $t \neq j$ such that $X_{it} \notin P$. By Lemma 3.3, there is a nonzero scalar $\beta \in k$ such that $X_{ij}X_{st} - \beta X_{it}X_{sj} \in P$. Thus, $X_{ij} \in P$ would imply that $X_{it}X_{sj} \in P$. However, X_{it} and X_{sj} are $\mathcal{H}$-eigenvectors which, by Lemma 3.3, are normal modulo P. Hence, because P is $\mathcal{H}$-prime, $X_{it}X_{sj} \in P$ would imply $X_{it} \in P$ or $X_{sj} \in P$, contradicting the choices of s and t. Thus, we have established that $X_{ij} \in P$ if and only if $i \in I$ or $j \in J$. Now $P(I, J) \subseteq P$, and we need to establish equality.

Set $B := \mathcal{A}/P(I, J)$ and $\overline{P} = P/P(I, J)$, and note that B is a domain by Lemma 3.2. Write Y_{ij} for the image of X_{ij} in B. The claim just established implies that $Y_{ij} \notin \overline{P}$ if $i \notin I$ or $j \notin J$. Recall from Lemma 3.3 that the Y_{ij} scalar-commute among themselves.

Now, $I \neq \{1, \dots, m\}$ and $J \neq \{1, \dots, n\}$, since not all of the X_{ij} are in P. Let $s \in \{1, \dots, m\} \setminus I$ and $t \in \{1, \dots, n\} \setminus J$ be minimal, and consider the localization $C := B[Y_{st}^{-1}]$. Since $Y_{st} \notin \overline{P}$ there is an embedding of B into C, and $\overline{P}C$ is an $\mathcal{H}$-prime ideal of C such that $\overline{P}C \cap B = \overline{P}$.

Note that $Y_{ij} = 0$ if $i < s$ or $j < t$. If $i > s$ and $j > t$, then we have $Y_{st}Y_{ij} - qY_{sj}Y_{it} = 0$, so that $Y_{ij} = qY_{st}^{-1}Y_{sj}Y_{it}$ in C. Hence, C is generated as an algebra by $Y_{st}^{\pm 1}$ together with Y_{sj} for $j > t$ and Y_{it} for $i > s$. Thus, C is a homomorphic image of a localized multiparameter quantum affine space $\mathcal{O}_{\boldsymbol{\lambda}}(k^r)[z_1^{-1}]$, for $r = m - s + n - t + 1$ and for a suitable parameter matrix $\boldsymbol{\lambda}$.

The standard action of the torus $\mathcal{H}_r := (k^\times)^r$ on $\mathcal{O}_{\boldsymbol{\lambda}}(k^r)$ has 1-dimensional eigenspaces generated by individual monomials (here, we use the fact that k is infinite). Therefore, the same holds for C. Hence, any nonzero $\mathcal{H}_r$-invariant ideal of C contains a monomial, and so any nonzero $\mathcal{H}_r$-prime ideal of C must contain one of $Y_{s+1,t}, \dots, Y_{mt}$, $Y_{s,t+1}, \dots, Y_{sn}$. Since $\overline{P}C$ contains none of these elements, to show that $\overline{P}C = 0$ it suffices to establish that $\overline{P}C$ is $\mathcal{H}_r$-prime. But $\overline{P}C$ is already $\mathcal{H}$-prime, so it is enough to see that the $\mathcal{H}_r$-invariant ideals of C are the same as the $\mathcal{H}$-invariant ideals. This will follow from showing that the images of $\mathcal{H}$ and $\mathcal{H}_r$ in $\operatorname{aut} C$ coincide.

Since the Y_{ij} are $\mathcal{H}$-eigenvectors, it is clear that the image of $\mathcal{H}$ is contained in the image of $\mathcal{H}_r$. The reverse inclusion amounts to the following statement:

(*) Given any $\alpha_s, \dots, \alpha_m, \beta_{t+1}, \dots, \beta_n \in k^\times$, there exists $h \in \mathcal{H}$ such that $h(Y_{it}) = \alpha_i Y_{it}$ for $i = s, \dots, m$ and $h(Y_{sj}) = \beta_j Y_{sj}$ for $j = t+1, \dots, n$.

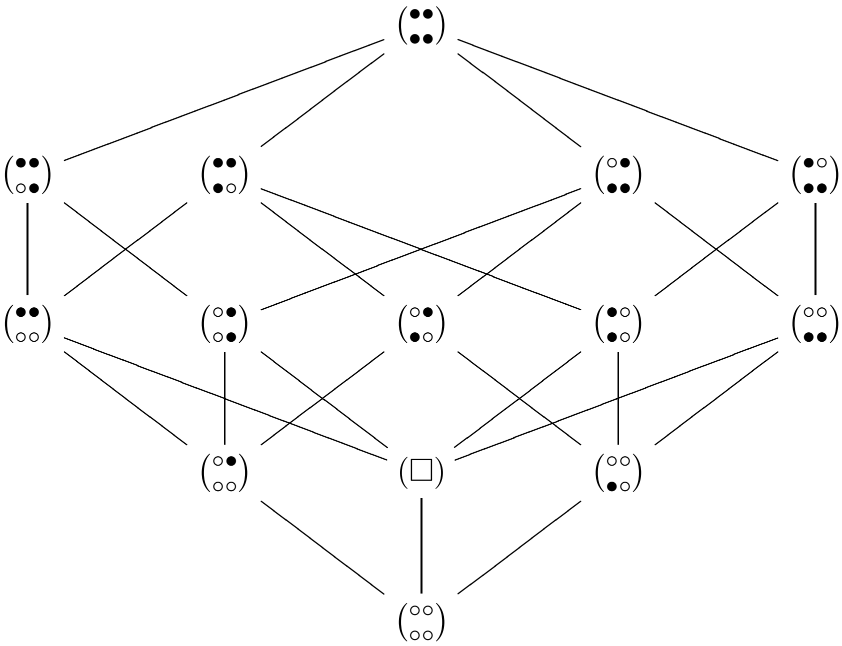
Now, there exists $h_1 \in \mathcal{H}$ such that $h_1(X_{ij}) = X_{ij}$ for all i, j with $i < s$, and $h_1(X_{ij}) = \alpha_i X_{ij}$ for all i, j with $i \geq s$. Also, there exists $h_2 \in \mathcal{H}$ such that $h_2(X_{ij}) = X_{ij}$ for all i, j with $j \leq t$ and $h_2(X_{ij}) = \alpha_s^{-1}\beta_j X_{ij}$ for all i, j with $j > t$. Setting $h = h_1 h_2$ gives the desired element of $\mathcal{H}$, establishing (*).

Therefore, $\overline{P}C = 0$, and so $\overline{P} = 0$. This means that $P = P(I, J)$. $\square$

3.5. Corollary. *If the field k is infinite, then $\mathcal{O}_q(M_{m,n}(k))/\mathcal{I}_1$ has precisely $(2^m - 1)(2^n - 1) + 1$ distinct $\mathcal{H}$-prime ideals, all of which are completely prime. Further, each $\mathcal{H}$-stratum of* $\operatorname{spec} \mathcal{O}_q(M_{m,n}(k))/\mathcal{I}_1$ *is homeomorphic to the prime spectrum of a Laurent polynomial ring over an algebraic field extension of k.*

Proof. The first statement is clear from Proposition 3.4. The second statement is not actually a corollary of the Proposition, but is included to fill in the picture. It may be obtained from [**1**, Theorems 5.3, 5.5] (all but the algebraicity of the coefficient fields also follows from [**4**, Theorem 6.6]). $\square$

3.6. In particular, the corollary above explains why in the algebra $\mathcal{O}_q(M_2(k))$ there are precisely $10 = (2^2-1)^2+1$ distinct $\mathcal{H}$-primes which contain the quantum determinant. This fact was known previously by direct enumeration of these primes. The remaining $\mathcal{H}$-primes correspond to $\mathcal{H}$-primes of $\mathcal{O}_q(GL_2(k))$; there are 4 of these, as has long been known. We can display the lattice of $\mathcal{H}$-prime ideals of $\mathcal{O}_q(M_2(k))$ as in the diagram below, where the symbols $\bullet$ and $\circ$ stand for generators X_{ij} which are or are not included in a given prime, and $\square$ stands for the 2×2 quantum determinant. For example, $\left(\begin{smallmatrix}\circ&\bullet\\ \bullet&\circ\end{smallmatrix}\right)$ stands for the ideal $\langle X_{12}, X_{21}\rangle$, and $(\square)$ stands for the ideal $\langle X_{11}X_{22} - qX_{12}X_{21}\rangle$.



$\mathcal{H}$-spec $\mathcal{O}_q(M_2(k))$

The corresponding $\mathcal{H}$-strata in spec $\mathcal{O}_q(M_2(k))$ can be easily calculated. For instance, if q is not a root of unity, the strata corresponding to $\left(\begin{smallmatrix}\circ&\circ\\ \circ&\circ\end{smallmatrix}\right)$ and

$\left(\begin{smallmatrix}\circ & \bullet \\ \bullet & \circ\end{smallmatrix}\right)$ are 2-dimensional, the strata corresponding to $\left(\begin{smallmatrix}\bullet & \bullet \\ \circ & \bullet\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}\bullet & \bullet \\ \bullet & \circ\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}\circ & \bullet \\ \bullet & \bullet\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix}\bullet & \circ \\ \bullet & \bullet\end{smallmatrix}\right)$ are all 1-dimensional, and the remaining 8 strata are singletons.

3.7. We close with some remarks concerning catenarity. (Recall that the prime spectrum of a ring A is *catenary* provided that for any comparable primes $P \subset Q$ in $\operatorname{spec} A$, all saturated chains of primes from P to Q have the same length.) It is conjectured that $\operatorname{spec} \mathcal{O}_q(M_{m,n}(k))$ is catenary. In [**2**, Theorem 1.6], we showed that catenarity holds for any affine, noetherian, Auslander-Gorenstein, Cohen-Macaulay algebra A with finite Gelfand-Kirillov dimension, provided $\operatorname{spec} A$ has normal separation. All hypotheses but the last are known to hold for the algebra $\mathcal{A} = \mathcal{O}_q(M_{m,n}(k))$. We can, at least, say that the portion of $\operatorname{spec} \mathcal{A}$ above $\mathcal{I}_1$ – that is, $\operatorname{spec} \mathcal{A}/\mathcal{I}_1$ – is catenary: In view of Lemma 3.3, $\mathcal{A}/\mathcal{I}_1$ is a homomorphic image of a multiparameter quantum affine space $\mathcal{O}_{\boldsymbol{\lambda}}(k^{n^2})$, and $\operatorname{spec} \mathcal{O}_{\boldsymbol{\lambda}}(k^{n^2})$ is catenary by [**2**, Theorem 2.6].

References

1. K. R. Goodearl, *Prime spectra of quantized coordinate rings*, This volume.

2. K. R. Goodearl and T. H. Lenagan, *Catenarity in quantum algebras*, J. Pure and Applied Algebra **111** (1996), 123-142.

3. K. R. Goodearl and T. H. Lenagan, *Quantum determinantal ideals*, Preprint, 1997.

4. K. R. Goodearl and E. S. Letzter, *The Dixmier-Moeglin equivalence in quantum coordinate rings and quantized Weyl algebras*, Trans. Amer. Math. Soc. (to appear).

5. J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings*, Wiley-Interscience, Chichester-New York, 1987.

6. B. Parshall and J.-p. Wang, *Quantum linear groups*, Memoirs Amer. Math. Soc. **89** (1991).

7. L. Rigal, *Normalité de certains anneaux déterminantiels quantiques*, Proc. Edinburgh Math. Soc. (to appear).

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