Quantized trace rings

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Abstract

The general linear group acts on m-tuples of $N \times N$ matrices by simultaneous conjugation. Quantum deformations of the corresponding rings of invariants and the so-called trace rings are investigated.

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1 Introduction

Write M_N^m for the space $M_N(\mathbb{C}) \oplus \cdots \oplus M_N(\mathbb{C})$ of m-tuples of $N \times N$ complex matrices. Denote by $U(k) = \left(u(k)^i_j\right)_{i,j=1}^N$ the function mapping an m-tuple to its kth component ($k = 1, \ldots, m$). The N^2m-variable commutative polynomial algebra $\mathcal{A} = \mathcal{A}(N,m)$ generated by the $u(k)^i_j$ is the coordinate ring of M_N^m. Let us recall various subalgebras of $\mathcal{A}$ and $M_N(\mathcal{A})$. The $\mathbb{C}$-subalgebra of $M_N(\mathcal{A})$ generated by $U(1), \ldots, U(m)$ and the identity matrix, I, is known as the algebra $\mathcal{G}(N,m)$ of m generic $N \times N$ matrices. The complex special linear group SL_N acts on M_N^m by simultaneous conjugation. This induces an action on $\mathcal{A}$ by linear substitution of the variables: $g \in SL_N$ maps $u(k)^i_j$ to

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the (i,j)-entry of $g^{-1}U(k)g$. The ring $\mathcal{R}(N,m)$ of *matrix invariants* consists of

$$\mathcal{R}(N,m) = \{f \in \mathcal{A}(N,m) \mid \forall g \in SL_N : \ g \cdot f = f\}.$$

Let $\mathcal{T}(N,m)$ denote the algebra of SL_N-equivariant polynomial maps from M_N^m to $M_N(\mathbb{C})$. Identify polynomial maps $M_N^m \to M_N(\mathbb{C})$ with elements of $M_N(\mathcal{A})$ in the usual way. Then $\mathcal{T}(N,m)$ is the $\mathcal{R}(N,m)$-subalgebra of $M_N(\mathcal{A})$ generated by $I, U(1), \ldots, U(m)$. It is called the *trace ring* (or sometimes the *algebra of matrix concomitants*). The reason for the name is that $\mathcal{R}(N,m)$ is generated by elements of the form $\mathrm{Tr}(U(k_1)\cdots U(k_d))$, where $k_1, \ldots, k_d \in \{1, \ldots, m\}$, $d \leq N^2$, and Tr denotes the usual trace function (see [26], [24], [21]).

The algebra $\mathcal{G}(N,m)$ of generic matrices appeared in the theory of algebras satisfying polynomial identities, as a relatively free object. On the other hand, $\mathcal{R}(N,m)$ and $\mathcal{T}(N,m)$ are natural objects of invariant theory. By definition, $\mathcal{G}(N,m) \subset \mathcal{T}(N,m)$; although this is a proper inclusion, $\mathcal{G}(N,m)$ turns out to be closely related to $\mathcal{T}(N,m)$. As a remarkable consequence, certain questions in non-commutative algebra can be approached by methods of classical invariant theory, see [21] and [9] for details.

Our aim in the present paper is to find analogues of the algebras $\mathcal{R}(N,m)$, $\mathcal{T}(N,m)$, $\mathcal{G}(N,m)$ in the context of quantum groups. Write $\mathcal{F}_q(SL_N)$ for the quantized function algebra of SL_N, defined for example in [25]. Here $q \in \mathbb{C}^\times$ is a non-zero parameter. $\mathcal{F}_q(SL_N)$ is a Hopf algebra, which is a non-commutative deformation of the coordinate ring $\mathcal{F}(SL_N)$ of SL_N. (For unexplained terminology and facts relating to Hopf algebras, coactions, and quantum groups, we refer to the books [14], [19].) Works of Majid [19] and Kulish and Sasaki [15] provide us with an appropriate non-commutative deformation $\mathcal{A}_q(N,m)$ of $\mathcal{A}(N,m)$, together with a right coaction $\beta(m) : \mathcal{A}_q(N,m) \to \mathcal{A}_q(N,m) \otimes \mathcal{F}_q(SL_N)$, which is a deformation of the comorphism $\mathcal{A}(N,m) \to \mathcal{A}(N,m) \otimes \mathcal{F}(SL_N)$ of the simultaneous conjugation action of SL_N on M_N^m. To simplify notation, when $m = 1$ we write $\mathcal{A}_q(N)$ instead of $\mathcal{A}_q(N,1)$. We take for the algebra $\mathcal{A}_q(N)$ the *braided matrices* of [18] (called the *reflection equation algebra* in [16]), associated with the R-matrix belonging to the quantum SL_N. It is a right $\mathcal{F}_q(SL_N)$-comodule algebra, and for some problems relating the conjugation action of SL_N on $M_N(\mathbb{C})$, it appears to be a better quantum deformation of $\mathcal{F}(M_N)$, than the coordinate ring $\mathcal{F}_q(M_N)$ of quantum matrices, see for example [12], [6]. When $m=2$, we take for $\mathcal{A}_q(N,2)$ the *braided tensor product* (cf. [19]) $\mathcal{A}_q(N)\underline{\otimes}\mathcal{A}_q(N)$ of two copies of $\mathcal{A}_q(N)$. As a vector space, it is just the ordinary tensor product $\mathcal{A}_q(N)\otimes_{\mathbb{C}}\mathcal{A}_q(N)$, endowed with the tensor product $\mathcal{F}_q(SL_N)$-comodule structure. Using that $\mathcal{F}_q(SL_N)$ is coquasitriangular, one can define a multiplication on this tensor product space that makes the coaction multiplicative, see Corollary 9.2.14 in [19], and the sign $\underline{\otimes}$ indicates the vector space tensor product endowed with this multiplication. With an iterated use of braided tensor products we arrive at $\mathcal{A}_q(N,m)$ for arbitrary m. The same algebra was defined in [15] in terms of generators and relations.

Now we place the algebra $\mathcal{R}_q(N,m)$ of $\beta(m)$-coinvariants into the centre of investigation, as well as a subalgebra $\mathcal{T}_q(N,m)$ of $M_N(\mathcal{A}_q(N,m))$. Under certain assumptions on q we show that these algebras can be considered as quantum versions of the ring of matrix invariants and the trace ring, by investigating their Hilbert series, their generators, and ring theoretical properties. In the special case $N=2$ we present some explicit calculations, leading to the somewhat surprising conclusion that $\mathcal{T}_q(2,2)\cong\mathcal{T}(2,2)$, and $\mathcal{G}_q(2,2)\cong\mathcal{G}(2,2)$.

2 Some quadratic algebras

To define $\mathcal{F}_q(SL_N)$ one starts with the R-matrix $R=\left(R^{ik}_{jl}\right)^N_{i,j,k,l=1}$. This is an $N^2\times N^2$ matrix, whose rows and columns are indexed by pairs, and the entry in the crossing of the (i,k) row and the (j,l) column is

$$R^{ik}_{jl} = \begin{cases} q, & \text{if } i=j=k=l; \\ 1, & \text{if } i=j, k=l, i\neq k; \\ q-q^{-1}, & \text{if } i>j, i=l, j=k; \\ 0, & \text{otherwise.} \end{cases}$$

It is usual to think of R as the matrix of a linear endomorphism of $V\otimes V$ with respect to the basis $e_1\otimes e_1, e_1\otimes e_2, \ldots, e_1\otimes e_N, e_2\otimes e_1, \ldots, e_N\otimes e_N$. So identifying R and the endomorphism we have $R(e_j\otimes e_l) = R^{ik}_{jl} e_i\otimes e_k$, where the convention of summing over repeated indices is used (and will be used throughout the paper).

The *quantized coordinate ring* $\mathcal{F}_q(M_N)$ is the $\mathbb{C}$-algebra generated by the entries of $T = \left(t^i_j\right)^N_{i,j=1}$, subject to the relations $RT_1T_2 = T_2T_1R$, with the matrix notation of [25]; that is, T_1, T_2 are the matrix Kronecker products $T_1 = T\otimes I$, $T_2 = I\otimes T$, so $(T_1)^{ik}_{jl} = t^i_j\delta^k_l$, $(T_2)^{ik}_{jl} = \delta^i_j t^k_l$, where $\delta^i_i = 1$, and $\delta^i_j = 0$ for $i\neq j$. The algebra $\mathcal{F}_q(M_N)$ is a bialgebra with comultiplication $\Delta(t^i_j) = t^i_k\otimes t^k_j$ and counit $\varepsilon(t^i_j) = \delta^i_j$. The quantum determinant $\det_q = \sum_{\pi\in S_N}(-q)^{\mathrm{length}(\pi)} t^1_{\pi(1)}\cdots t^N_{\pi(N)}$ is a central group-like element, and $\mathcal{F}_q(SL_N)$ is defined as the quotient of $\mathcal{F}_q(M_N)$ modulo the ideal $\langle \det_q -1\rangle$. We keep the notation $t^i_j\in\mathcal{F}_q(SL_N)$ for the images of the generators under the natural surjection. Then $\mathcal{F}_q(SL_N)$ is a Hopf algebra with antipode S satisfying $S(T)T = I = TS(T)$.

Denote by τ the flip endomorphism $\tau(v\otimes w) = w\otimes v$ of $V\otimes V$. Define $\widehat{R} = \tau\circ R$ and $R_{21} = \tau\circ R\circ\tau$. In coordinates, we have $\widehat{R}^{ik}_{jl} = R^{ki}_{jl}$ and $(R_{21})^{ik}_{jl} = R^{ki}_{lj}$. The *reflection equation algebra* $\mathcal{A}_q(N)$ is the $\mathbb{C}$-algebra generated by the entries of $X = \left(x^i_j\right)^N_{i,j=1}$ subject to the quadratic relations

$$X_2\widehat{R}X_2\widehat{R} = \widehat{R}X_2\widehat{R}X_2. \tag{1}$$

The relations can be written also in the equivalent form

$$X_2R_{21}X_1R = R_{21}X_1RX_2.$$

By Proposition 10.3.5 in [19], $\mathcal{A}_q(N)$ is a right $\mathcal{F}_q(SL_N)$-comodule algebra in a natural way. Namely, the map $X \mapsto S(T)XT$ extends to an algebra homomorphism $\beta : \mathcal{A}_q(N) \to \mathcal{A}_q(N) \otimes \mathcal{F}_q(SL_N)$, and β is a right coaction of the Hopf algebra $\mathcal{F}_q(SL_N)$ on $\mathcal{A}_q(N)$. Here we treat the entries of $S(T)XT$ (and more generally, products of the form txu with $t, u \in \mathcal{F}_q(SL_N)$, $x \in \mathcal{A}_q(N)$) as elements of $\mathcal{A}_q(N) \otimes \mathcal{F}_q(SL_N)$ by assuming that the generators of $\mathcal{F}_q(SL_N)$ commute with the generators of $\mathcal{A}_q(N)$, and suppressing the $\otimes$ sign from the notation; so $\beta(x^i_j) = x^k_l \otimes S(t^i_k)t^l_j$. Taking into account that $S(T) = T^{-1}$, this notational device will be very convenient later.

Next take two copies of $\mathcal{A}_q(N)$. The matrix generators of them will be denoted by $X = \left(x^i_j\right)$ and $Y = \left(y^i_j\right)$, respectively. Define $\mathcal{A}_q(N, 2)$ as the algebra generated by the $2N^2$ elements x^i_j, y^i_j, subject to the additional relations

$$R^{-1}Y_1RX_2 = X_2R^{-1}Y_1R. \tag{2}$$

By Theorem 10.3.1 of [19] this algebra can be identified with the braided tensor product $\mathcal{A}_q(N)\underline{\otimes}\mathcal{A}_q(N)$, by setting $x^i_j \mapsto x^i_j\underline{\otimes}1$, $y^i_j \mapsto 1\underline{\otimes}x^i_j$. The relations (2) are equivalent to

$$\forall i,j,k,l: \quad y^i_jx^k_l = (R^{-1})^{an}_{rc}R^{sc}_{bl}R^{id}_{am}(\widetilde{R})^{bk}_{jd}x^m_ny^r_s, \tag{3}$$

where $\widetilde{R}^{t_2} = (R^{t_2})^{-1}$, and for an $N^2 \times N^2$ matrix A, the matrix A^{t_2} is defined by $(A^{t_2})^{ik}_{jl} = A^{il}_{jk}$.

More generally, take m copies of $\mathcal{A}_q(N)$ with matrix generators

$$X(1) = \left(x(1)^i_j\right), \ldots, X(m) = \left(x(m)^i_j\right),$$

respectively, and define $\mathcal{A}_q(N, m)$ to be the algebra generated by $\{x(k)^i_j \mid k = 1, \ldots, m;\ i,j = 1, \ldots, N\}$, where in addition to the reflection equation (1) imposed for each $X(k) = X$, we impose the relations

$$\forall r < s: \quad R^{-1}X(s)_1RX(r)_2 = X(r)_2R^{-1}X(s)_1R. \tag{4}$$

In other words, for each $r < s$ the subalgebra of $\mathcal{A}_q(N, m)$ generated by the entries of $X(r)$ and $X(s)$ is isomorphic to $\mathcal{A}_q(N, 2)$ via an isomorphism

$x(r)^i_j \mapsto x^i_j$, $x(s)^i_j \mapsto y^i_j$. This algebra was defined in [15]. It is clearly the same as the braided tensor product $\mathcal{A}_q(N)\underline{\otimes}\cdots\underline{\otimes}\mathcal{A}_q(N)$ (m copies) in the sense of [19]. Hence we may define the coaction $\beta(m) : \mathcal{A}_q(N,m) \to \mathcal{A}_q(N,m) \otimes \mathcal{F}_q(SL_N)$ by taking the mth tensor power of β. The multiplication in the braided tensor product is designed to make $\beta(m)$ an algebra homomorphism. Therefore $\mathcal{A}_q(N,m)$ becomes an $\mathcal{F}_q(SL_N)$-comodule algebra. More explicitly, the coaction is given on the generators by $\beta(m) : X(r) \mapsto S(T)X(r)T$ for $r = 1,\dots,m$.

3 Polynormal sequences in the reflection equation algebra

It is standard that the $\mathbb{Z}$-graded algebra $\mathcal{F}_q(M_N)$ has the same Hilbert series as $\mathcal{F}(M_N)$. On the other hand, a vector space isomorphism between the degree d homogeneous components of $\mathcal{F}_q(M_N)$ and $\mathcal{A}_q(N)$ is established in (7.37) of [19]. Consequently, the graded algebra $\mathcal{A}_q(N)$ has the same Hilbert series as the N^2-variable commutative polynomial algebra. We refine this statement by having a closer look at the defining relations of $\mathcal{A}_q(N)$. The matrix equality (1) is a shorthand for the following:

$$\forall i,j,k,l: \quad \widehat{R}^{ib}_{st}\widehat{R}^{sa}_{jl}x^k_b x^t_a = \widehat{R}^{ik}_{sa}\widehat{R}^{st}_{jb}x^a_t x^b_l \tag{5}$$

Taking into account the exact form of the R-matrix, one gets the following list of relations.

$$\begin{aligned}
x^i_j x^i_l &= q x^i_l x^i_j \qquad \text{for } j<l, i\neq j, i\neq l;\\
q^{-1}x^i_j x^i_i - q x^i_i x^i_j &= (q-q^{-1})\sum_{s>i} x^i_s x^s_j \qquad \text{for } i>j;\\
x^i_l x^i_i - x^i_i x^i_l &= (1-q^{-2})\sum_{s>i} x^i_s x^s_l \qquad \text{for } i<l;\\
x^k_j x^i_j &= q x^i_j x^k_j \qquad \text{for } i<k, j\neq i, j\neq k;\\
x^i_i x^k_i - x^k_i x^i_i &= (1-q^{-2})\sum_{s>i} x^k_s x^s_i \qquad \text{for } i<k;
\end{aligned}$$

$$
\begin{aligned}
q^{-1}x_j^jx_j^i - qx_j^ix_j^j &= (q-q^{-1})\sum_{s>j} x_s^ix_j^s \qquad \text{for } i<j;\\
x_j^ix_l^k &= x_l^kx_j^i \qquad \text{for } i<k, j<l, i\neq l, j\neq k;\\
x_l^jx_j^i - qx_j^ix_l^j &= (q-q^{-1})\sum_{s>j} x_s^ix_l^s \qquad \text{for } i<j<l;\\
x_j^ix_i^k - qx_i^kx_j^i &= (q-q^{-1})\sum_{s>i} x_s^kx_j^s \qquad \text{for } j<i<k;\\
x_j^ix_l^k - x_l^kx_j^i &= (q-q^{-1})x_j^kx_l^i \qquad \text{for } i>k, j<l, i\neq l, j\neq k, i\neq j;\\
x_j^ix_l^k - x_l^kx_j^i &= (q-q^{-1})x_l^ix_j^k \qquad \text{for } i>k, j<l, i\neq l, j\neq k, k\neq l;\\
x_l^jx_j^i - qx_j^ix_l^j &= (q-q^{-1})(-x_j^jx_l^i + \sum_{s>j} x_s^ix_l^s) \qquad \text{for } j<l, j<i, i\neq l;\\
x_j^ix_i^k - qx_i^kx_j^i &= (q-q^{-1})(x_j^kx_i^i + \sum_{s>i} x_s^kx_j^s) \qquad \text{for } k<i, j<i, k\neq j;\\
x_j^ix_i^j - x_i^jx_j^i &= (1-q^{-2})(x_j^jx_i^i - \sum_{s>j} x_s^ix_i^s + \sum_{t>i} x_t^jx_j^t) \qquad \text{for } j<i.
\end{aligned}
$$

Proposition 3.1 *(i) The algebra $\mathcal{A}_q(N)$ is noetherian.*

(ii) Let $x_1,\ldots,x_{N^2}$ be an arbitrary permutation of $x_1^1,\ldots,x_N^N$. Then the monomials of the form $x_1^{a_1}\cdots x_{N^2}^{a_{N^2}}$ constitute a $\mathbb{C}$-vector space basis of $\mathcal{A}_q(N)$.

(iii) $\mathcal{A}_q(N)$ is a domain.

Proof. (i) Denote by I_j^i the ideal of $\mathcal{A}_q(N)$ generated by the x_t^s with $s\geq i$, $t\geq j$, $(s,t)\neq(i,j)$. The relation indexed by i,j,k,l in (5) says that for some q-power q_{jl}^{ik}, we have $x_j^ix_l^k - q_{jl}^{ik}x_l^kx_j^i \in I_j^i\cap I_l^k$. In particular, since $I_N^N=0$, the element x_N^N is normal in $\mathcal{A}_q(N)$; that is, $x_N^N\mathcal{A}_q(N)=\mathcal{A}_q(N)x_N^N$. Next, observe that $I_N^{N-1}=\langle x_N^N\rangle = I_{N-1}^N$. It follows that both the images of x_{N-1}^N and x_N^{N-1} are normal in the quotient $\mathcal{A}_q(N)\langle x_N^N\rangle$. Continuing this way it is easy to see that the generators can be arranged into a so-called polynormal sequence. To be more precise, let $x_1,\ldots,x_{N^2}$ be a permutation of $x_1^1,\ldots,x_N^N$. Define the function p by $x_{p(i,j)}=x_j^i$. Assume that

$$\text{for all } i,j,k,l \text{ with } i\leq k, j\leq l \text{ we have that } p(k,l)\leq p(i,j). \qquad (6)$$

Then $x_1, \ldots, x_{N^2}$ is a polynormal sequence: for all i, the image of x_i is normal in $\mathcal{A}_q(N)/\langle x_1, \ldots, x_{i-1}\rangle$. Condition (6) holds for example for the sequence $x_N^N, x_N^{N-1}, \ldots, x_N^1, x_{N-1}^N, x_{N-1}^{N-1}, \ldots, x_1^1$. So the graded algebra $\mathcal{A}_q(N)$ is generated by a polynormal sequence, hence is noetherian (both left and right) by Lemma 8.2 of [1]. (Alternatively, one may apply Proposition I.8.17 from [2] for a generating sequence satisfying (6) to conclude that $\mathcal{A}_q(N)$ is noetherian.)

(ii) Write W for the free semigroup generated by the variables x_j^i. Define the function h on W by

$$h(x_j^i \cdots x_l^k) = ij + \cdots + kl.$$

Note that if w' is obtained by permuting the factors of the monomial w, then $h(w') = h(w)$. Introduce a partial order $\prec$ on W: for $v, w \in W$, we set $v \prec w$ if $h(v) > h(w)$. Clearly, $\prec$ is compatible with multiplication in the sense that $v \prec w$ implies $avb \prec awb$ for all $a, b \in W \cup \{1\}$. In $\mathcal{A}_q(N)$, the relation indexed by i, j, k, l in (5) says that $x_j^i x_l^k - q_{jl}^{ik} x_l^k x_j^i$ is a linear combination of monomials w with $w \prec x_j^i x_l^k$ and $w \prec x_l^k x_j^i$ (where q_{jl}^{ik} is a q-power, hence non-zero). Restrict attention to the degree d homogeneous component $\mathcal{A}_q(N)^{(d)}$ of $\mathcal{A}_q(N)$. By an iterated use of the quadratic relations, we can rewrite $w = x_{j_1}^{i_1} \cdots x_{j_d}^{i_d}$ as a non-zero scalar multiple of some $w' = (x_1)^{a_1} \cdots (x_{N^2})^{a_{N^2}}$ (obtained by permuting the factors of w), modulo the span of monomials v with $v \prec w$, $v \prec w'$. This obviously implies that the monomials of the form $(x_1)^{a_1} \cdots (x_{N^2})^{a_{N^2}}$ span $\mathcal{A}_q(N)^{(d)}$. Their number coincides with the dimension of $\mathcal{A}_q(N)^{(d)}$ (which we know already), so they form a basis.

(iii) Given a homogeneous element f of $\mathcal{A}_q(N)$, express it in terms of the basis described in (ii). Among the monomials of f that are maximal with respect to the partial ordering $\prec$ choose the one which is largest with respect to the lexicographic ordering of the monomials of the form $(x_1)^{i_1} \cdots (x_{N^2})^{i_{N^2}}$, and call it the *leading term of* f. If f, g are non-zero homogeneous elements of $\mathcal{A}_q(N)$ with leading term $(x_1)^{i_1} \cdots (x_{N^2})^{i_{N^2}}$ and $(x_1)^{j_1} \cdots (x_{N^2})^{j_{N^2}}$, then by the rewriting process sketched in the proof of (ii), the leading term of fg

is $(x_1^{i_1+j_1})\cdots(x_{N^2})^{i_{N^2}+j_{N^2}}$, so fg is non-zero. Thus $\mathcal{A}_q(N)$ does not contain zero-divisors. $\square$

4 Quantum trace rings

After these preparations we are in position to define $\mathcal{R}_q(N,m)$ as the subalgebra of $\beta(m)$-coinvariants in $\mathcal{A}_q(N,m)$; that is,

$$\mathcal{R}_q(N,m) = \{f \in \mathcal{A}_q(N,m) \mid \beta(m)f = f \otimes 1\}.$$

Let $\mathcal{G}_q(N,m)$ denote the unitary subalgebra of $M_N(\mathcal{A}_q(N,m))$ (the algebra of $N \times N$ matrices over $\mathcal{A}_q(N,m)$) generated by $X(1),\ldots,X(m)$. And finally, consider

$$\mathcal{T}_q(N,m) = \{F \in M_N(\mathcal{A}_q(N,m)) \mid \beta(m)F = S(T)FT\},$$

where $\beta(m)F$ stands for the matrix obtained by applying $\beta(m)$ to each entry of F, and $S(T)XT$ is identified with an element of $M_N(\mathcal{A}_q(N,m)) \otimes \mathcal{F}_q(SL_N)$ in the way indicated in Section 2.

Proposition 4.1 *$\mathcal{T}_q(N,m)$ is a subalgebra of $M_N(\mathcal{A}_q(N,m))$.*

Proof. This follows from the formula $S(T) = T^{-1}$, together with the multiplicativity of the coaction $\beta(m) : \mathcal{A}_q(N,m) \to \mathcal{A}_q(N,m) \otimes \mathcal{F}_q(SL_N)$. $\square$

The generic matrices $X(k)$ belong to $\mathcal{T}_q(N,m)$ by definition of $\beta(m)$, hence all products $X(k_1)\cdots X(k_d)$ belong to $\mathcal{T}_q(N,m)$ by Proposition 4.1. Also, if $f \in \mathcal{R}_q(N,m)$, then the scalar matrix fI is contained in $\mathcal{T}_q(N,m)$.

One can produce $\beta(m)$-coinvariants using the so-called *quantum trace* operation. For an arbitrary $N \times N$ matrix F with entries from a $\mathbb{C}$-algebra, define $\mathrm{Tr}_q(F) = \mathrm{Tr}(QF)$, where Q is the diagonal matrix with diagonal entries $q^{N-1}, q^{N-3}, q^{N-5}, \ldots, q^{-N+1}$. In particular, $\mathrm{Tr}_q(I) = [N]_q$, the Nth q-integer. Now $\mathrm{Tr}_q(X)$ is a β-coinvariant in $\mathcal{A}_q(N)$; more generally, if $F \in \mathcal{T}_q(N,m)$, then $\mathrm{Tr}_q(F)$ is contained in $\mathcal{R}_q(N,m)$ (see the proof of Theorem 4.2). In particular, the elements $\mathrm{Tr}_q(X(k_1)\cdots X(k_d))$ are all contained in $\mathcal{R}_q(N,m)$.

Note that there is a natural $\mathbb{Z}^m$-grading on the algebra $\mathcal{A}_q(N,m)$: the multidegree of the entries of $X(k)$ is the kth standard basis vector of $\mathbb{Z}^m$. This induces a $\mathbb{Z}^m$-grading on $M_N(\mathcal{A}_q(N,m))$ as well. Since the multihomogeneous components of $\mathcal{A}_q(N,m)$ are subcomodules with respect to $\beta(m)$, we have that $\mathcal{R}_q(N,m)$ is a $\mathbb{Z}^m$-graded subalgebra of $\mathcal{A}_q(N,m)$, and similarly, $\mathcal{T}_q(N,m)$ is a $\mathbb{Z}^m$-graded subalgebra of $M_N(\mathcal{A}_q(N,m))$.

Theorem 4.2 *Assume that $q\in\mathbb{C}^\times$ is not a root of unity. Then the $\mathbb{Z}^m$-graded algebra $\mathcal{R}_q(N,m)$ has the same Hilbert series as its classical counterpart $\mathcal{R}(N,m)$, and $\mathcal{T}_q(N,m)$ has the same Hilbert series as $\mathcal{T}(N,m)$.*

Proof. Consider the multihomogeneous component $\mathcal{A}_q(N,m)^{(d_1,\dots,d_m)}$, it is a subcomodule of $\mathcal{A}_q(N,m)$ with respect to β. Since q is assumed to be not a root of unity, $\mathcal{F}_q(SL_N)$ is cosemisimple (as in the case that $q=1$), and this finite dimensional comodule decomposes as the direct sum of simple subcomodules. By definition, the coefficient of $t_1^{d_1}\cdots t_m^{d_m}$ in the Hilbert series $H(\mathcal{R}_q(N,m);t_1,\dots,t_m)$ is the multiplicity of the trivial comodule as a direct summand. This multiplicity can be computed by "restricting the coaction to the diagonal subgroup" (see for example [4] for a more detailed explanation). To be more precise, denote by $\mathcal{F}(K)$ the coordinate Hopf algebra of the diagonal subgroup of $SL_N(\mathbb{C})$, so $\mathcal{F}(K)=\mathbb{C}[z_1,\dots,z_N\mid z_1\cdots z_N=1]$. There is a surjective homomorphism $\pi:\mathcal{F}_q(SL_N)\to\mathcal{F}(K)$, $\pi(t_i^i)=z_i$, and $\pi(t_j^i)=0$ for $i\neq j$. Consider the coaction $(\mathrm{id}\otimes\pi)\beta(m):\mathcal{A}_q(N,m)\to\mathcal{A}_q(N,m)\otimes\mathcal{F}(K)$ of $\mathcal{F}(K)$. It follows from Proposition 3.1 (ii) and the interpretation of $\mathcal{A}_q(N,m)$ as a braided tensor product of copies of $\mathcal{A}_q(N)$ that the $\mathcal{F}(K)$-comodule $\mathcal{A}_q(N,m)^{(d_1,\dots,d_m)}$ is isomorphic to its counterpart in the classical case $q=1$. That is, formally the same computation gives the coefficients of the Hilbert series of $\mathcal{R}_q(N,m)$ as that of $\mathcal{R}(N,m)$.

The proof of the statement about the Hilbert series of $\mathcal{T}_q(N,m)$ is similar, one just needs an appropriate interpretation of the elements of $\mathcal{T}_q(N,m)$ in terms of corepresentation theory. Denote by φ the restriction of the coaction β to the N^2-dimensional space $\mathcal{X} = \mathrm{Span}_{\mathbb{C}}\{x^i_j\}_{i,j=1}^N$. As in the classical case $q = 1$, this corepresentation decomposes as $\varphi \cong \mathrm{triv} \oplus \varphi_0$, where triv is the trivial corepresentation, and φ_0 is an irreducible (N^2-1)-dimensional corepresentation. It is well known (and easy to show; we shall comment on it later in this Section) that triv is realized on the subspace of $\mathcal{X}$ spanned by $\mathrm{Tr}_q(X)$. Thus the decomposition of $\mathcal{X}$ as a sum of simple subcomodules is $\mathcal{X} = \mathbb{C}\mathrm{Tr}_q(X) \oplus \mathcal{X}_0$, where $\mathcal{X}_0$ is spanned by the entries of $X - \frac{\mathrm{Tr}_q(X)}{[N]_q} I$, with $[N]_q$ standing for the q-integer $q^{N-1} + q^{N-3} + \cdots + q^{-N+1}$.

By definition, a matrix $F = (f^i_j)$ belongs to $\mathcal{T}_q(N,m)$ if and only if its entries span a $\beta(m)$-subcomodule of $\mathcal{A}_q(N,m)$, such that $\mathcal{X} \to \mathrm{Span}_{\mathbb{C}}\{f^i_j\}$, $x^i_j \mapsto f^i_j$ is a homomorphism of $\mathcal{F}_q(SL_N)$-comodules. It follows in particular that if $F \in \mathcal{T}_q(N,m)$, then $\mathrm{Tr}_q(F) \in \mathcal{R}_q(N,m)$. Therefore $G = F - \frac{\mathrm{Tr}_q(F)}{[N_q]} I \in \mathcal{T}_q(N,m)$ with $\mathrm{Tr}_q(G) = 0$, and the entries of G span a subcomodule of $\mathcal{A}_q(N,m)$ which is a homomorphic image of the simple comodule $\mathcal{X}_0$. Conversely, given an element $f \in \mathcal{R}_q(N,m)$, the scalar matrix fI obviously belongs to $\mathcal{T}_q(N,m)$. Now take a subcomodule $\mathcal{Y}$ of $\mathcal{A}_q(N,m)$ which is isomorphic to $\mathcal{X}_0$. Fix a comodule surjection $\mathcal{X} \to \mathcal{Y}$, $x^i_j \mapsto f^i_j$. (Note that by Schur's Lemma, this homomorphism is determined up to a scalar multiple.) Then the matrix $F = (f^i_j)$ belongs to $\mathcal{T}_q(N,m)$ by construction, and has the property $\mathrm{Tr}_q(F) = 0$.

Putting this all together, we reach the conclusion that the dimension of $\mathcal{T}_q(N,m)^{(d_1,\dots,d_m)}$ is the sum of the dimension of $\mathcal{R}_q(N,m)^{(d_1,\dots,d_m)}$ and the multiplicity of the irreducible corepresentation φ_0 as a direct summand of $\beta(m)$ on $\mathcal{A}_q(N,m)^{(d_1,\dots,d_m)}$. This latter multiplicity can be computed by restriction to the corresponding $\mathcal{F}(K)$-coaction, which by Proposition 3.1 (ii) is isomorphic to its classical counterpart. Finally, note that the dimension of $\mathcal{T}(N,m)^{(d_1,\dots,d_m)}$ in the classical case clearly can be expressed in the same way in terms of multiplicities of two types of irreducible summands in $\mathcal{A}(N,m)$. $\square$

Remark 4.3 Let us extract from the above proof explicitly that the $\mathbb{C}$-vector space $\mathcal{T}_q(N,m)$ can be identified with the space of $\mathcal{F}_q(SL_N)$-comodule homomorphisms $\mathrm{Hom}_{\mathcal{F}_q(SL_N)}(\mathcal{X},\mathcal{A}_q(N,m))$. There is a natural vector space isomorphism $\mathrm{Hom}_{\mathbb{C}}(\mathcal{X},\mathcal{A}_q(N,m)) \cong \mathcal{A}_q(N,m)\otimes\mathcal{X}^*$. The dual space $\mathcal{X}^*$ of $\mathcal{X}$ is an $\mathcal{F}_q(SL_N)$-comodule via the contragredient corepresentation of φ (cf. section 11.1.3, page 398 in [14]), and consider the tensor product comodule structure on $\mathcal{A}_q(N,m)\otimes\mathcal{X}^*$. It is straightforward to check that under the isomorphism $\mathrm{Hom}_{\mathbb{C}}(\mathcal{X},\mathcal{A}_q(N,m)) \cong \mathcal{A}_q(N,m)\otimes\mathcal{X}^*$ the space of coinvariants $(\mathcal{A}_q(N,m)\otimes\mathcal{X}^*)^{\mathcal{F}_q(SL_N)}$ is mapped onto $\mathrm{Hom}_{\mathcal{F}_q(SL_N)}(\mathcal{X},\mathcal{A}_q(N,m))$. In particular, we have $\mathcal{T}_q(N,m)\cong(\mathcal{A}_q(N,m)\otimes\mathcal{X}^*)^{\mathcal{F}_q(SL_N)}$ as vector spaces.

Under a stronger restriction on q we are able to present explicit generators of $\mathcal{R}_q(N,m)$ and $\mathcal{T}_q(N,m)$.

Theorem 4.4 *Assume that q is transcendental over the rationals. Then $\mathcal{R}_q(N,m)$ is the $\mathbb{C}$-subalgebra of $\mathcal{A}_q(N,m)$ generated by the elements of the form $\mathrm{Tr}_q(X(i_1)\cdots X(i_d))$, where $i_1,\dots,i_d\in\{1,\dots,m\}$, and $d\le N^2$. Furthermore, $\mathcal{T}_q(N,m)$ is the left (respectively, right) $\mathcal{R}_q(N,m)$-submodule of $M_N(\mathcal{A}_q(N,m))$ generated by the elements $X(i_1)\cdots X(i_d)$ with $d\le N^2-1$.*

Proof. These statements are known to hold when $q=1$. Take a set of multihomogeneous monomials whose traces form a basis of $\mathcal{R}(N,m)$ when $q=1$. Take the q-traces of formally the same monomials (but interpret them as matrices with entries from $\mathcal{A}_q(N,m)$). We obtain some multihomogeneous elements in $\mathcal{R}_q(N,m)$. Since q is assumed to be transcendental, we may conclude by a standard argument that these elements in $\mathcal{R}_q(N,m)$ are linearly independent (because of the linear independency of the corresponding elements for $q=1$). Therefore they constitute a basis of $\mathcal{R}_q(N,m)$ by Theorem 4.2. The proof of the statement about $\mathcal{T}_q(N,m)$ is the same. $\square$

When q is not a root of unity, the multilinear component $\mathcal{A}_q(N,m)^{(1,\dots,1)}$ can be naturally identified with a homomorphic image of the Hecke algebra of the symmetric group S_m. This follows from some natural linear algebra isomorphisms (mentioned already in Remark 4.3). Indeed, consider the fundamental corepresentation $\omega : V \to V \otimes \mathcal{F}_q(SL_N)$, $e_j \mapsto e_i \otimes t^i_j$, where V is an N-dimensional vector space with basis $e_1, \dots, e_N$. Write $\varepsilon^1, \dots, \varepsilon^N$ for the corresponding dual basis in V^*. Then using the notation of the proof of Theorem 4.2, $\mathcal{X} \to V^* \otimes V$, $x^i_j \mapsto \varepsilon^i \otimes e_j$ intertwines φ and $\omega^* \otimes \omega$. Therefore as a comodule, the multilinear component $\mathcal{A}_q(N,m)^{(1,\dots,1)}$ can be identified with $V^* \otimes V \otimes \cdots \otimes V^* \otimes V$. With a repeated use of the comodule isomorphism $V \otimes V^* \to V^* \otimes V$, $e_i \otimes \varepsilon^j \mapsto \widetilde{R}^{kj}_{il} \varepsilon^l \otimes e_k$, and the other natural isomorphisms mentioned in Remark 4.3 (note also that $(A \otimes B)^* \cong B^* \otimes A^*$ for any finite dimensional $\mathcal{F}_q(SL_N)$-comodules A, B) we obtain an isomorphism $\mathcal{A}_q(N,m)^{(1,\dots,1)} \cong \mathrm{End}_{\mathbb{C}}(V \otimes \cdots \otimes V)$, which restricts to $\mathcal{R}_q(N,m)^{(1,\dots,1)} \cong \mathrm{End}_{\mathcal{F}_q(SL_N)}(V^{\otimes m})$. The latter is known to be generated as an algebra by $\widehat{R}_{i,i+1}$ (operating as $\widehat{R}$ on the ith and $(i+1)$th tensor component, and operating as the identity on the other components), $i = 1, \dots, m-1$. That way we get an explicit vector space homomorphism from the Hecke algebra of the symmetric group S_m onto $\mathcal{R}_q(N,m)^{(1,\dots,1)}$. The kernel of this homomorphism is a known two-sided ideal of the Hecke algebra. For example, in the special case $m = 1$, the element $\mathrm{id} \in \mathrm{End}_{\mathcal{F}_q(SL_N)}(V)$ is mapped to $q^{-2}\mathrm{Tr}_q(X)$. However, it does not seem to be easy to exploit this connection for explicit calculations, because of the excessive use of the R-matrix in some of the above identifications. For example, in the case $N = m = 2$, we have

$$\mathrm{id}_{V \otimes V} \mapsto e_i \otimes e_j \otimes \varepsilon^j \otimes \varepsilon^i \mapsto \widetilde{R}^{si}_{kt} \widetilde{R}^{ml}_{in} \widetilde{R}^{kj}_{jl} x^n_m y^s_t,$$

which is $q^{-2}\mathrm{Tr}_q(XY) + (q^{-5} - q^{-3})\mathrm{Tr}_q(X)\mathrm{Tr}_q(Y) \in \mathcal{A}_q(2,2)$.

Another interesting fact about $\mathcal{A}_q(N,m)$ is the centrality of certain $\beta(m)$-coinvariants. The β-coinvariants are central in $\mathcal{A}_q(N)$ (see e.g. [19]), and $\mathrm{Tr}_q(X(k)^n)$ is central in $\mathcal{A}_q(N,m)$, as it was observed in [15]. Using the approach of [19] it is easy to show that if $f \in \mathcal{R}_q(N,m)$ belongs to the subalgebra generated by the entries of $X(k)$ for some k, then f is central in $\mathcal{A}_q(N,m)$. It is sufficient to show that f commutes with the entries of $X(l)$ for $l \neq k$ (since we know already that f commutes with the entries of $X(k)$). So one can reduce the question to the case $m = 2$. Let x, y be elements of $\mathcal{A}_q(N,2)$, x depending only on the entries of X, and y depending only on the entries of Y. With Sweedler's notation, $\beta(x) = \sum x_0 \otimes x_1$, and $\beta(y) = \sum y_0 \otimes y_1$. By definition of the multiplication in the braided tensor product $\mathcal{A}_q(N)\underline{\otimes}\mathcal{A}_q(N)$, we have

$$yx = \sum r(y_1, x_1) x_0 y_0, \tag{7}$$

where r is the universal r-form in the coquasitriangular Hopf algebra $\mathcal{F}_q(SL_N)$ (see e.g. Proposition 35, 10.3.2 in [14]). If $\beta(x) = x \otimes 1$, then we get $yx = \sum r(y_1, 1) x y_0 = x \sum \varepsilon(y_1) y_0 = xy$. Similarly, $\beta(y) = y \otimes 1$ implies $yx = xy$.

We mention also that the entries of XY satisfy the defining relations of $\mathcal{A}_q(N)$; this is stated in [15]. Indeed,

$$\begin{aligned}
(XY)_2 R_{21} (XY)_1 R &= X_2 Y_2 \tau R \tau X_1 Y_1 R = X_2 \tau (Y_1 R X_2) \tau Y_1 R \\
&= X_2 \tau R X_2 R^{-1} Y_1 R \tau Y_1 R = (X_2 R_{21} X_1) \tau R^{-1} \tau (Y_2 R_{21} Y_1 R) \\
&= R_{21} X_1 R X_2 R^{-1} R_{21}^{-1} R_{21} Y_1 R Y_2 = R_{21} X_1 R (X_2 R^{-1} Y_1 R) Y_2 \\
&= R_{21} X_1 R R^{-1} Y_1 R X_2 Y_2 = R_{21} X_1 Y_1 R X_2 Y_2 \\
&= R_{21} (XY)_1 R (XY)_2,
\end{aligned}$$

so XY satisfies the matrix equality (1). Furthermore, the map $X \mapsto X(1)X(2)$, $Y \mapsto X(3)$ extends to an algebra homomorphism $\mathcal{A}_q(N,2) \to \mathcal{A}_q(N,3)$. All one needs to check is that the matrices $X(1)X(2)$ and $X(3)$ satisfy the equality (2). This can be done as in the calculation above. Alternatively, think of $\mathcal{A}_q(N,3)$ as $\mathcal{A}_q(N,2)\underline{\otimes}\mathcal{A}_q(N)$; the multiplication in the braided tensor product is expressed in terms of the $\mathcal{F}_q(SL_N)$-comodule algebra structures of the factors (see (7)). Since the linear isomorphisms $X \mapsto X(1)X(2)$, $Y \mapsto X(3)$ intertwine the corresponding coactions, the cross relations (3) hold between the entries of $X(1)X(2)$ and $X(3)$.

5 The 2×2 case

Throughout this section we assume $N = 2$. Recall from Section 3 that $\mathcal{A}_q(2)$ is generated by $x_1^1, x_2^1, x_1^2, x_2^2$, subject to the relations

$$\begin{array}{ll} x_2^2 x_2^1 = q^2 x_2^1 x_2^2; & x_1^1 x_2^1 = x_2^1 x_1^1 + (q^{-2} - 1) x_2^1 x_2^2; \\ x_1^1 x_2^2 = x_2^2 x_1^1; & x_1^2 x_2^1 = x_2^1 x_1^2 + (q^{-2} - 1) x_2^2 (x_2^2 - x_1^1); \\ x_1^2 x_2^2 = q^2 x_2^2 x_1^2; & x_1^2 x_1^1 = x_1^1 x_1^2 + (q^{-2} - 1) x_2^2 x_1^2. \end{array} \tag{8}$$

The cross relations in $\mathcal{A}_q(2,2)$ between the entries of X and Y are the following:

$$\begin{aligned}
y_1^1 x_1^1 &= x_1^1 y_1^1 + (q^{-4} - q^{-2}) x_1^2 y_2^1; \qquad (9)\\
y_2^1 x_1^1 &= x_1^1 y_2^1; \\
y_1^1 x_2^1 &= x_2^1 y_1^1 + (1 - q^{-2}) x_1^1 y_2^1 + (q^{-2} - 1) x_2^2 y_2^1; \\
y_2^1 x_2^1 &= q^2 x_2^1 y_2^1; \\
y_1^2 x_1^1 &= x_1^1 y_1^2 + (q^{-2} - 1) x_1^2 y_2^2 + (1 - q^{-2}) x_1^2 y_1^1; \\
y_2^2 x_1^1 &= x_1^1 y_2^2 + (1 - q^{-2}) x_1^2 y_2^1; \\
y_1^2 x_2^1 &= q^{-2} x_2^1 y_1^2 + (q^{-2} - 1) x_1^1 y_1^1 + (1 - q^{-2}) x_1^1 y_2^2 + (1 - q^{-2}) x_2^2 y_1^1 + \\
&\quad + (q^{-2} - 1) x_2^2 y_2^2 + (q^{-4} - q^{-2} - 1 + q^2) x_1^2 y_2^1; \\
y_2^2 x_2^1 &= x_2^1 y_2^2 + (1 - q^2) x_1^1 y_2^1 + (q^2 - 1) x_2^2 y_2^1; \\
y_1^1 x_1^2 &= x_1^2 y_1^1; \\
y_2^1 x_1^2 &= q^{-2} x_1^2 y_2^1; \\
y_1^1 x_2^2 &= x_2^2 y_1^1 + (1 - q^{-2}) x_1^2 y_2^1; \\
y_2^1 x_2^2 &= x_2^2 y_2^1; \\
y_1^2 x_1^2 &= q^2 x_1^2 y_1^2; \\
y_2^2 x_1^2 &= x_1^2 y_2^2; \\
y_1^2 x_2^2 &= x_2^2 y_1^2 + (q^2 - 1) x_1^2 y_2^2 + (1 - q^2) x_1^2 y_1^1; \\
y_2^2 x_2^2 &= x_2^2 y_2^2 + (1 - q^2) x_1^2 y_2^1.
\end{aligned}$$

The algebra $\mathcal{A}_q(2,m)$ is generated by the entries of the 2×2 matrices $X(1), \ldots, X(m)$. For each k, the entries of $X(k)$ satisfy (8) with $X \mapsto X(k)$, and for each $k < l$, the entries of $X(k)$ and $X(l)$ satisfy (9) with $X \mapsto X(k)$, $Y \mapsto X(l)$. The monomials

$$w(1)\ldots w(m) \quad \text{with} \quad w(k) = (x(k)^1_2)^{a_k}(x(k)^2_2)^{b_k}(x(k)^1_1)^{c_k}(x(k)^2_1)^{d_k} \tag{10}$$

form a basis of $\mathcal{A}_q(2,m)$. Above we have presented the relations of $\mathcal{A}_q(2,m)$ in such a form that makes obvious an algorithm to rewrite an arbitrary monomial in the generators as a linear combination of these basis elements. This allows us to perform calculations in $\mathcal{A}_q(2,m)$.

The quantum version of the Cayley-Hamilton identity for the matrix X of the generators of $\mathcal{A}_q(N)$ was found in [23] (see also [13]). In the special case $N = 2$ it takes the form

$$X^2 - q^{-1}\mathrm{Tr}_q(X)X + \frac{1}{[2]_q}(q^{-1}\mathrm{Tr}_q(X)^2 - \mathrm{Tr}_q(X^2))I = 0. \tag{11}$$

We note that $\frac{1}{q+q^{-1}}(q^{-1}\mathrm{Tr}_q(X)^2 - \mathrm{Tr}_q(X^2)) = q^{-2}x^2_2x^1_1 - x^1_2x^2_1$.

Classically one can polarize the 2×2 Cayley-Hamilton identity to obtain an equivalent bilinear identity depending on two matrix variables. This bilinear identity has an analogues in the quantum setting:

Proposition 5.1 *The following equality holds in $\mathcal{T}_q(2,2)$:*

$$\begin{aligned} &XY + q^2YX - q\mathrm{Tr}_q(X)Y - q\mathrm{Tr}_q(Y)X \\ &+(\mathrm{Tr}_q(X)\mathrm{Tr}_q(Y) - q^{-1}\mathrm{Tr}_q(XY))I = 0. \end{aligned} \tag{12}$$

Proof. Direct computation, using the above mentioned basis and rewriting algorithm in $\mathcal{A}_q(2,2)$. $\square$

We take a digression and for illustrative purposes we list a couple of other equalities in $\mathcal{A}_q(2,2)$ and in $\mathcal{A}_q(2,3)$. We have

$$q^2\mathrm{Tr}_q(YX) - q^{-2}\mathrm{Tr}_q(XY) = (q - q^{-1})\mathrm{Tr}_q(X)\mathrm{Tr}_q(Y) \text{ in } \mathcal{A}_q(2,2) \tag{13}$$

(this can be checked by direct computation again). To simplify notation, write X, Y, Z (instead of $X(1), X(2), X(3)$) for the matrices of the generators of $\mathcal{A}_q(2,3)$. Multiplying (12) by Z from the right and taking q-trace one obtains

$$\begin{aligned} &\mathrm{Tr}_q(X)\mathrm{Tr}_q(Y)\mathrm{Tr}_q(Z) - q^{-1}\mathrm{Tr}_q(XY)\mathrm{Tr}_q(Z) - q\mathrm{Tr}_q(X)\mathrm{Tr}_q(YZ) \\ &-q\mathrm{Tr}_q(Y)\mathrm{Tr}_q(XZ) + q^2\mathrm{Tr}_q(YXZ) + \mathrm{Tr}_q(XYZ) = 0 \text{ in } \mathcal{A}_q(2,3), \end{aligned} \tag{14}$$

whereas making the substitution $X \mapsto Y$, $Y \mapsto Z$ in (12), multiplying by X from the left, and taking q-trace one gets

$$\begin{aligned} &\mathrm{Tr}_q(X)\mathrm{Tr}_q(Y)\mathrm{Tr}_q(Z) - q\mathrm{Tr}_q(XY)\mathrm{Tr}_q(Z) - q^{-1}\mathrm{Tr}_q(X)\mathrm{Tr}_q(YZ) \\ &-q\mathrm{Tr}_q(XZ)\mathrm{Tr}_q(Y) + q^2\mathrm{Tr}_q(XZY) + \mathrm{Tr}_q(XYZ) = 0 \text{ in } \mathcal{A}_q(2,3). \end{aligned} \tag{15}$$

From either of the above two equalities one recovers the so-called fundamental trace identity of 2×2 matrices as the special case $q = 1$. However, we have no canonical choice to single out one of them as "the fundamental q-trace identity", because as is indicated by (13), the number of the different trilinear products of q-traces is larger than 6, whereas $\dim_{\mathbb{C}}(\mathcal{A}_q(2,3)^{(1,1,1)}) = 5$, as in the classical case. Furthermore, make the substitutions $X \mapsto XY$, $Y \mapsto Z$ (respectively $X \mapsto X$, $Y \mapsto YZ$) in the identity (13); by the discussion at the end of Section 4, we obtain the equalities

$$q^2\mathrm{Tr}_q(ZXY) - q^{-2}\mathrm{Tr}_q(XYZ) = (q - q^{-1})\mathrm{Tr}_q(XY)\mathrm{Tr}_q(Z), \tag{16}$$

$$q^2\mathrm{Tr}_q(YZX) - q^{-2}\mathrm{Tr}_q(XYZ) = (q - q^{-1})\mathrm{Tr}_q(X)\mathrm{Tr}_q(YZ). \tag{17}$$

Finally, multiply (12) by Z from the left and take its q-trace; from this, (15), (14), (16), and (17) one gets that modulo the subalgebra generated by q-traces of monomials of degree ≤ 2, we have $\mathrm{Tr}_q(XYZ) \equiv -q^2\mathrm{Tr}_q(YXZ) \equiv -q^2\mathrm{Tr}_q(XZY) \equiv q^4\mathrm{Tr}_q(YZX) \equiv q^4\mathrm{Tr}_q(ZXY) \equiv -q^6\mathrm{Tr}_q(ZYX)$.

Next we investigate $\mathcal{T}_q(2,2)$. In order to simplify notation, set

$$\begin{aligned} A = q^{-1}\mathrm{Tr}_q(X)I; \quad B = q^{-1}\mathrm{Tr}_q(Y)I; \\ C = (q^{-2}x_2^2x_1^1 - x_2^1x_1^2)I; \quad D = (q^{-2}y_2^2y_1^1 - y_2^1y_1^2)I; \\ E = AB - q^{-3}\mathrm{Tr}_q(XY)I. \end{aligned}$$

Theorem 5.2 *Suppose that q is transcendental over the rationals. Then we have the following.*

(i) $\mathcal{R}_q(2,2)$ *is a 5-variable commutative polynomial algebra generated by* $\mathrm{Tr}_q(X), \mathrm{Tr}_q(X^2), \mathrm{Tr}_q(Y), \mathrm{Tr}_q(Y^2), \mathrm{Tr}_q(XY)$.

(ii) $\mathcal{T}_q(2,2)$ *is a free left* $\mathcal{R}_q(2,2)$*-module generated by* I, X, Y, XY.

(iii) As a $\mathbb{C}$*-algebra,* $\mathcal{T}_q(2,2)$ *is generated by* X, Y, A, B, C, D, E*, and a complete list of relations among these generators is the following:*

$$\begin{aligned} &A, B, C, D \text{ are central;} \\ X^2 = AX - C; &\quad Y^2 = BY - D; \\ YX = &-q^{-2}XY + AY + BX - E; \\ XE = q^2EX + (1-q^2)&ABX + (q^{-2}-q^2)CY + \\ &+(1-q^{-2})AXY + (q^2-1)CB; \\ YE = q^{-2}EY + (1-q^{-2})&ABY + (1-q^{-4})DX + \\ &+(q^{-4}-q^{-2})BXY + (q^{-2}-1)AD. \end{aligned}$$

Proof. (i) As we noted in Section 4, $\mathrm{Tr}_q(X), \mathrm{Tr}_q(X^2), \mathrm{Tr}_q(Y), \mathrm{Tr}_q(Y^2)$ are central in $\mathcal{A}_q(2,2)$, hence the given five elements pairwise commute. They are algebraically independent because of the algebraic independence of the corresponding elements in the case $q = 1$ (here we use the assumption on q, in the same way as in the proof of Theorem 4.4). We know from Theorem 4.2 that $\mathcal{R}_q(2,2)$ has the same Hilbert series as a polynomial algebra generated by two degree 1 elements and three degree 2 elements. This implies that $\mathcal{R}_q(2,2)$ coincides with its subalgebra $\mathbb{C}[\mathrm{Tr}_q(X), \mathrm{Tr}_q(X^2), \mathrm{Tr}_q(Y), \mathrm{Tr}_q(Y^2), \mathrm{Tr}_q(XY)]$.

(ii) $\mathcal{T}_q(2,2)$ contains I, X, Y, XY, hence the left $\mathcal{R}_q(2,2)$-module generated by them. This is a free module, because the corresponding module is free when $q = 1$ (again the transcendentality of q is used). Consequently, it has the same Hilbert series as $\mathcal{T}(2,2)$ (computed in [11]). Hence by Theorem 4.2, the left $\mathcal{R}_q(2,2)$-module generated by I, X, Y, XY coincides with $\mathcal{T}_q(2,2)$.

(iii) It follows from (ii) that X, Y, A, B, C, D, E generate $\mathcal{T}_q(2,2)$ as an algebra. By the remarks in Section 4 about the centrality of certain $\beta(m)$-coinvariants, we have that A, B, C, D are central in $\mathcal{T}_q(2,2)$. The relations $X^2 = AX - C$, $Y^2 = BY - D$ are the Cayley-Hamilton identities of [23], see (11). The next relation in the list is the bilinear Cayley-Hamilton identity of Proposition 5.1. The remaining two relations are verified by direct calculation, using the basis and rewriting algorithm in $\mathcal{A}_q(2,2)$ explained at the beginning of this Section. Thus the relations we listed all hold. It is straightforward that modulo these relations an arbitrary product of the elements X, Y, A, B, C, D, E can be rewritten as an element of the left $\mathbb{C}[A, B, C, D, E]$-module generated by I, X, Y, XY. Hence by (ii), the given list of relations is complete. $\square$

Remark 5.3 A rather lengthy calculation using the relations in (iii) yields that $XYE = EXY$. Knowing that A, B are central, this is equivalent to the assertion that $\mathrm{Tr}_q(XY)$ commutes with XY. However, this latter statement immediately follows (for arbitrary N) from the existence of the algebra homomorphism $\mathcal{A}_q(N) \to \mathcal{A}_q(N,2)$, $X \mapsto XY$ (discussed in Section 4), and the centrality of $\mathrm{Tr}_q(X)$ in $\mathcal{A}_q(N)$.

Now consider the ordinary trace ring $\mathcal{T}(2,2)$, and write x, y for the matrix generators, so x, y are 2×2 generic matrices with pairwise commuting algebraically independent entries. The algebra $\mathcal{T}(2,2)$ probably appeared first in [27]; see [10] for a recent survey. We set

$$a = \mathrm{Tr}(x)I; \quad b = \mathrm{Tr}(y)I; \quad c = \det(x)I; \quad d = \det(y)I;$$
$$e = ab - \mathrm{Tr}(xy)I.$$

Theorem 5.4 *Suppose that q is transcendental over the rationals. Then the map*

$$X \mapsto x,\ Y \mapsto y,\ A \mapsto a, B \mapsto b,\ C \mapsto c,\ D \mapsto d,$$
$$E \mapsto e + (1 - q^{-2})xy$$

extends to an algebra isomorphism $\mathcal{T}_q(2,2) \to \mathcal{T}(2,2)$*. In particular, we have* $\mathcal{G}_q(2,2) \cong \mathcal{G}(2,2)$ *are isomorphic algebras, hence the Hilbert series of* $\mathcal{G}_q(2,2)$ *is*

$$\frac{(1-s)(1-t)(1-st)+st}{(1-s)^2(1-t)^2(1-st)}.$$

Proof. The elements a, b, c, d, e are obviously central in $\mathcal{T}(2,2)$, the Cayley-Hamilton Theorem asserts $x^2 = ax - c$, $y^2 = by - d$, and its bilinearization says $yx = -xy + ay + bx - e$. With these equalities at hand, direct computation verifies that the elements $x, y, a, b, c, d, e + (1 - q^{-2})xy$ satisfy the relations of X, Y, A, B, C, D, E. Therefore there exists an algebra homomorphism $\phi : \mathcal{T}_q(2,2) \to \mathcal{T}(2,2)$ mapping the generators to the prescribed images. These images clearly generate the whole $\mathcal{T}(2,2)$, hence ϕ is surjective. On the other hand, ϕ is a grading preserving map between two spaces with the same Hilbert series. So ϕ must be an isomorphism. Finally, note that under this isomorphism, $\mathcal{G}_q(2,2)$ is mapped onto $\mathcal{G}(2,2)$. The Hilbert series of $\mathcal{G}(2,2)$ was determined in [11]. $\square$

Remark 5.5 Note that the above isomorphism $\mathcal{T}_q(2,2) \to \mathcal{T}(2,2)$ is non-trivial in the sense that the subalgebra $\mathcal{R}_q(2,2)I$ is not mapped into $\mathcal{R}(2,2)I$: the element E is mapped outside $\mathcal{R}(2,2)I$.

Remark 5.6 The algebras $\mathcal{G}(N,m)$ are highly non-trivial objects from the combinatorial point of view. For example, they are not finitely presented algebras: the ideal of relations among the generators is not finitely generated (although it is finitely generated as a T-ideal); see the papers [7], [8], [22], [17] for some computations in the 2×2 case. Therefore the isomorphism $\mathcal{G}_q(2,2) \cong \mathcal{G}(2,2)$ looks a bit surprising to us.

Not all $\beta(m)$-coinvariants are central in $\mathcal{A}_q(N,m)$, and the algebra $\mathcal{R}_q(N,m)$ is not commutative in general, as the following Proposition shows.

Proposition 5.7 *Suppose that* $q^2 \neq 1$.

(i) The elements $\mathrm{Tr}_q(XY)$ *and* $\mathrm{Tr}_q(XZ)$ *do not commute in* $\mathcal{A}_q(2,3)$.

(ii) The element $\mathrm{Tr}_q(XY)$ *is not central in* $\mathcal{A}_q(2,2)$*.*

Proof. (i) The expansion of the commutator $[\mathrm{Tr}_q(XY), \mathrm{Tr}_q(XZ)]$ in the basis (10) is $\sum_{i,j,k,l} f^{ik}_{jl} y^i_j z^k_l$, where f^{ik}_{jl} is a homogeneous quadratic expression of the variables x^s_t. We claim that f^{22}_{11}, the coefficient of $y^2_1 z^2_1$, is non-zero. An inspection of (8), (9) shows that y^2_1 (respectively, z^2_1) appears on the right hand side of a relation if and only if y^2_1 (respectively, z^2_1) appears on the left hand side of the relation. Therefore the coefficient of $y^2_1 z^2_1$ is the same as in $[qx^1_2 y^2_1, qx^1_2 z^2_1]$, hence $f^{22}_{11} = (1-q^2)x^1_2 x^1_2$.

(ii) This follows from the relations in Theorem 5.2 (iii), but it is easy to check directly, similarly to (i). Write $[\mathrm{Tr}_q(XY), x^1_1]$ as $\sum f^i_j y^i_j$, where f^i_j is a linear combination of the entries of X. We claim that $f^2_1 \neq 0$. By the same observation on the relations we used in (i), we see that f^2_1 is the same as the coefficient of y^2_1 in $[qx^1_2 y^2_1, x^1_1]$, so $f^2_1 = q[x^1_2, x^1_1] \neq 0$. □

Remark 5.8 If we endow the coordinate ring of pairs of 2×2 matrices with the Poisson bracket that is the classical limit of the tensor product of two copies of the quantum coordinate ring of 2×2 matrices, then a calculation similar to the above proof yields that $\mathcal{R}(2,2)$ is not a Poisson subalgebra. So roughly speaking, there is no subalgebra in the tensor product of two copies of the quantum coordinate ring of 2×2 matrices that would lie above $\mathcal{R}(2,2)$. This indicates that we are forced to move to the reflection equation algebra and braided tensor products to find quantum analogues of the rings of matrix invariants.

We continue with the ring theoretic study of our algebras.

Theorem 5.9 *(i) The algebra* $\mathcal{A}_q(2,m)$ *is noetherian.*

(ii) Suppose that q *is not a root of unity. Then* $\mathcal{R}_q(2,m)$ *is noetherian and is finitely generated as a* $\mathbb{C}$*-algebra.*

(iii) Suppose that q *is not a root of unity. Then* $\mathcal{T}_q(2,m)$ *is finitely generated as a left (respectively right)* $\mathcal{R}_q(2,m)$*-module. In particular, it is noetherian and is finitely generated as a* $\mathbb{C}$*-algebra.*

Proof. (i) One may build up $\mathcal{A}_q(2,m)$ by adjoining step-by-step the variables $x(k)^i_j$ in the following order:

$$x(1)^1_2, x(1)^2_2, x(1)^1_1, x(1)^2_1, x(2)^1_2, x(2)^2_2, x(2)^1_1, x(2)^2_1, \dots .$$

For notational simplicity, rename the members of this ordered sequence as $x_1, x_2, \dots, x_{4m}$. Denote by A_i the subalgebra generated by $x_1, \dots, x_i$. Then it is easy to see from the defining relations of $\mathcal{A}_q(2,m)$ that for all i, we have that $A_i x_{i+1}$ is contained in $A_i + x_{i+1}A_i$, and $x_{i+1}A_i$ is contained in $A_i + A_i x_{i+1}$. This implies successively that the algebras A_i are noetherian (see 1.2.10 in [20]), and thus $\mathcal{A}_q(2,m)$ is noetherian.

(ii) The Haar functional of the cosemisimple Hopf algebra $\mathcal{F}_q(SL_N)$ can be used to construct a projection (Reynolds operator) $\mathcal{A}_q(N,m) \to \mathcal{R}_q(N,m)$ with certain good properties (see for example [5] for details). Then both statements can be derived from (i) by the well known argument of Hilbert.

(iii) In order to show that $\mathcal{T}_q(2,m)$ is a finitely generated left (right) module over $\mathcal{R}_q(2,m)$, we recall from Remark 4.3 the $\mathbb{C}$-vector space isomorphism $\mathcal{T}_q(2,m) \cong (\mathcal{A}_q(2,m) \otimes \mathcal{X}^*)^{\mathcal{F}_q(SL_2)}$. This is clearly an isomorphism of left (right) $\mathcal{R}_q(2,m)$-modules (note that $\mathcal{A}_q(2,m) \otimes \mathcal{X}^*$ is naturally a left and right $\mathcal{A}_q(2,m)$-module). Now $\mathcal{A}_q(2,m) \otimes \mathcal{X}^*$ is a finitely generated module over $\mathcal{A}_q(2,m)$, hence is noetherian by (i). Hence $(\mathcal{A}_q(2,m) \otimes \mathcal{X}^*)^{\mathcal{F}_q(SL_2)}$ is a finitely generated module over $\mathcal{A}_q(2,m)^{\mathcal{F}_q(SL_2)} = \mathcal{R}_q(2,m)$ by a well known argument using the Reynolds operator $\mathcal{A}_q(2,m) \to \mathcal{R}_q(2,m)$ again; see for example page 71 in [3] for this argument. $\square$

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